Advanced Topics in Probability and Random Processes
Department of Electrical Engineering
Indian Institute of Technology, Guwahati

Lecture - 20
Continuous Time Markov Chain – 1

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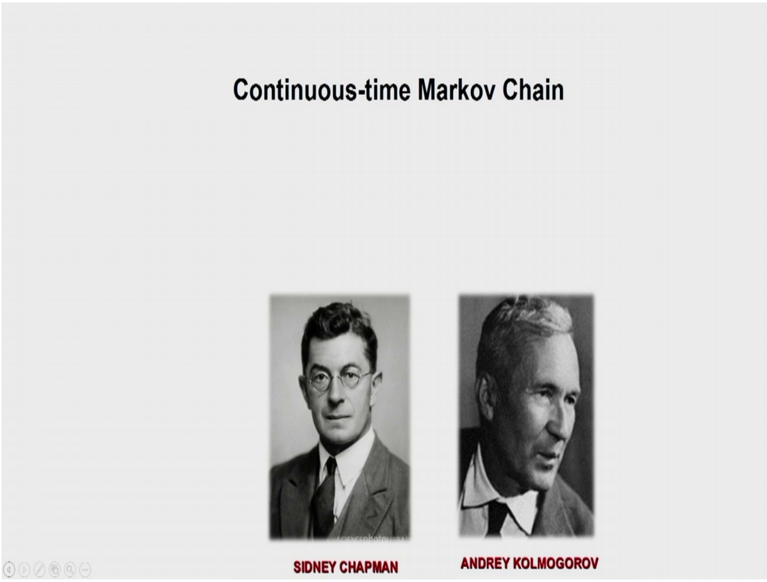

Continuous-time Markov Chain; Sidney's Chapman and Andrey kolmogorov, these are two great mathematicians, who have contributed to Markov chains. Sidney's Chapman is a British mathematician and Kolmogorov, we know that he was a famous Russian mathematician. Now we discussed discrete time Markov Chain were both dead space and the time is discrete earlier we discussed DTCM.

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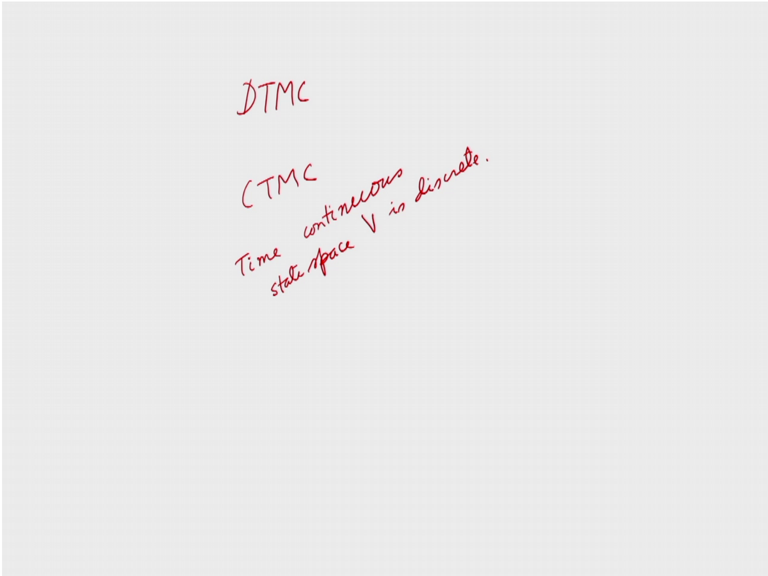

So, we will discuss now CTMC, Continuous Time Markov Chain where a time is continuous. Time is continuous and state space V is discrete ok. And continuous time Markov chain has many applications for example, that the queuing system, population studies, etcetera where CTMC different type of CTMC gets applications.

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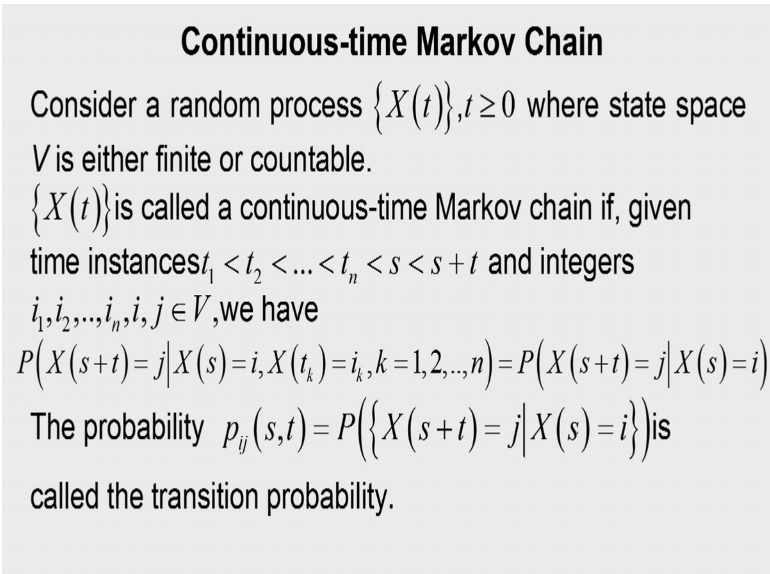

We will start with the definition of continuous time Markov chain. Consider a random process X t, t greater than or equal to 0 where state space V is either finite or countable basically it is discrete. X t is called a continuous time Markov chain if given time instances t 1 less than t 2 like that t t n less than s less than s plus t. So, we have different instances of time starting with t 1 to s, then s plus t. And integers i 1, i 2 up to i n then i j these are the state values actually belonging to V we have. Now what is the condition for CTMC same as a definition of Markov process that is probability of X of s plus t is equal to j.

Given that X s is equal to i and remaining instances X of t k is equal to i k k is equal to one to up to n. So, this is the conditional probability we want to find out this is same as probability of X s plus t is equal to j given that X s is equal to i. So, the conditional probability given that all the previous instances depend only on the current instance. So, that way probability of X s plus t is equal to j given that X s is equal to i.

Now, this probability is important for continuous time Markov chain and this is called this Transition Probability. Step transition probability p ij s, t we can write p i comma j or p i j for simplicity p ij s,t is equal to probability that X of s plus t is equal to j given that X s is equal to i is the transition probability.

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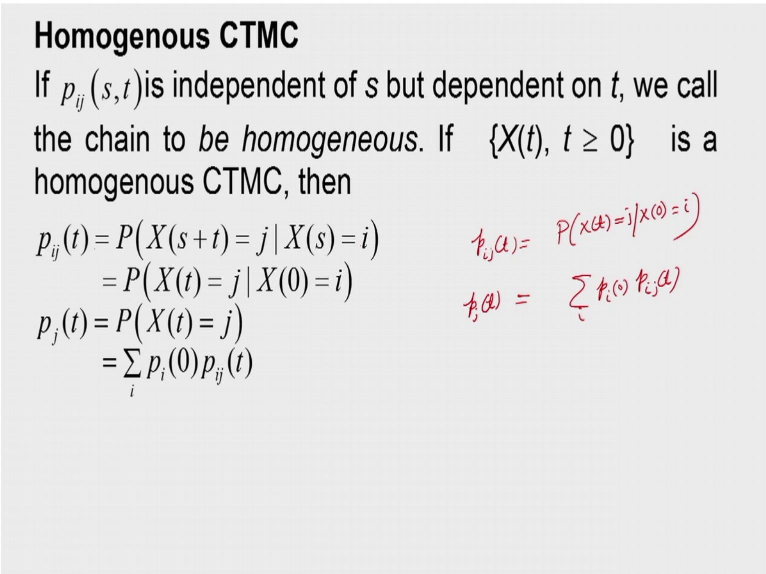

Now, we will discuss about homogeneous CTMC suppose p ij s, t is independent of s, but dependent on t. We call the same to be homogeneous in that situation we will call the same to be homogeneous. So, that means, P ij s, t is a function of t, but not a function of s. If X t, t greater than 0is a homogenous CTMC then P i j t is equal to probability of X s plus t is equal to j given that X s is equal to i is same as now because it is homogeneous so what we will get is probability of X t is equal to j given that X 0 is equal to i.

So, that is one important relationship because it is independent of starting point s. So, we can write it as p i j t is equal to probability of X t is equal to j given that X 0 is equal to i. Now suppose we have to find out any state probability suppose p j t that is the probability of state j at instant t. So, probability of X t is equal to j. So, how we can find it? Summation over i p i 0 into p i j t because this will give the joint probability and from the joint probability if we sum over i, then we will get the marginal probability so that way we can find out p j t. So, p j t is equal to that is first joint probability p i 0 into p i j t and summation over all i. So, this is the their probability at time t.

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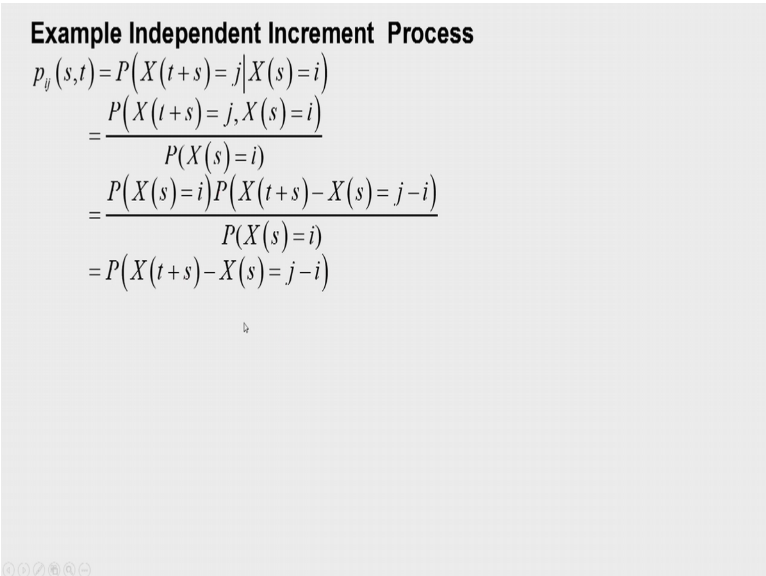

We will see the transition probability for independent increment process. So, p i j s t is equal to probability of s t plus equal to j even that X s is equal to i. So, that by definition of conditional probability probability of X of t plus s is equal to j and X s is equal to i divided by probability of X s is equal to i. Now this joint probability now can be written as probability of X s is equal to i probability into probability of X of t plus s minus X s is equal to j minus i given that X of s is equal to i.

But because of the independent increment property we write simply probability of X of t plus s minus X s is equal to j minus i. So, this and this get cancelled what we will get is that p i j s t is equal to probability of the increment probability of X of t plus s minus X s is equal to j minus i. That is the transition probability in the case of independent increment process.

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State-holding Time

When the CTMC enters a state i, the time it spends there before it leaves the state i is called the holding time in the state i. The holding time T_i of the state i is a continuous random variable.

T_i

i

Theorem (a) T_i is memory-less. In other words

$$P(T_i > t+s / T_i > s) = P(T_i > t)$$

(b) $f_{T_i}(t) = v_i e^{-v_i t}$ where $v_i > 0$ is a constant

Proof:(a)

$$P(T_i > s+t / T_i > s) = P(X(u)=i, 0 \le u \le s+t / X(u)=i, 0 \le u \le s)$$

$$= P(X(u)=i, s < u \le s+t / X(u)=i, 0 \le u \le s)$$

$$= P(X(u)=i, s < u \le s+t / X(s)=i)$$ (Using Markov property

$$= P(X(u)=i, 0 < u \le t / X(0)=i)$$ homogenity property

$$= P(T_i > t)$$

Thus T_i is memory-less.

We want to study the behaviour of the continuous time Markov chain so for that state holding time is an important concept. When the CTMC enters a state i, the time it spends there before it leaves is called is State Holding Time. So, for a CTMC suppose it will enter a state i and it will remain in this state i for a some random time T i that time is known as the State Holding Time and it is a continuous random variable because T i can be any any time.

Now, we have one important theorem which has two parts. Part a says thatT i is memory less, what does it mean that probability thatT i greater than plus s given that T i greater than s is same as probability ofT i greater than t. So, probability of T i is greater than t plus s, but given that T i is greater than s. So, this probability, conditional probabilities same as unconditional probability that T i is greater than t.

So, that way that it is already T i is greater than s, that does not have any implication on T i greater than t plus s so this is the memory less property. And second thing is that T i is exponentially distributed with parameter nu i where nu i greater than 0 is a constant. We will prove both parts first proof of first part a probability of T i greater than s plus t given that T i greater than s. What does it means? Probability that X u is equal to i up to in during the interval 0 to s plus t, because there is no change in state.

Given that X u is equal to i during the time 0 to s given that there is no change in state during time 0 to s.; the probability that there will be no change in state during time 0 to s plus t. Now this probability because here u lies between 0 and s plus t and here u lies between 0 and s so this probability will be same as already given that X u is equal to i for u lying between 0 and s. Therefore, this probability will be same as probability of X u is equal to i, u lies between s and s plus t given that X u is equal to i what u lying between 0 and s.

Now, we can use the Markov property. So, that way we will get probability of X u is equal to i, u lies between s and s plus t; given that X s is equal to i so this will get using Markov property ok. So, then what we will get is that probability of X u is equal to i. Now because of the homogeneity property we can consider u lies between 0 and t given that X 0 is equal to i so this is same as probability of T i greater than t. So, that way we have used the Markov property once then we used the homogeneity property and proof depth probability of T i greater than s plus t given that T i greater than s is same as probability of T i greater than t, thus T i is memory less.

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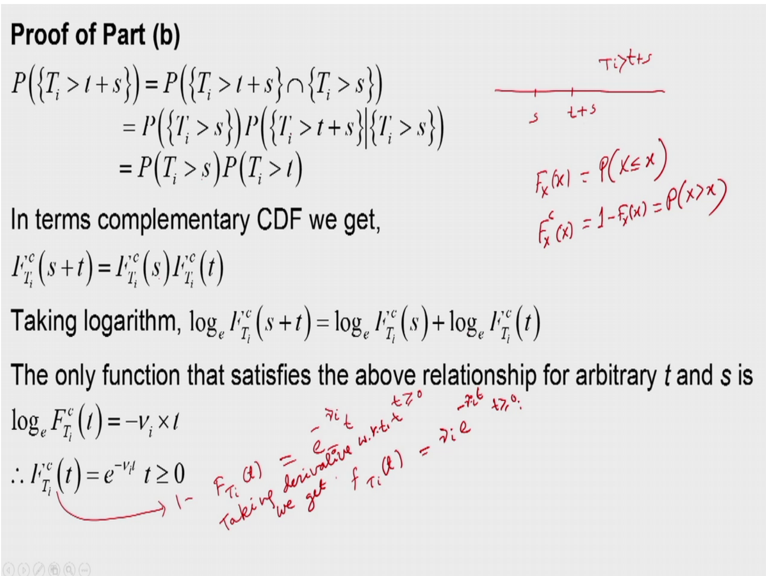

We will now prove part b; now let us consider the probability that T i is greater than t plus s. And that is suppose this is the time axis this is the point t plus s and this is the point suppose s. Now probability that T i greater than t plus s now T i greater than t plus s this side is T i greater than t plus s. So, this right hand side interval is this interval is T i greater than t plus s. Now this interval if I consider that is T i greater than s so you observe that T i greater than t plus s is the intersection of two interval T i greater than s and T i greater than t plus s.

So, that is why probability of T i greater than t plus s it is same as probability that T i greater than t plus s intersection T i greater than s. So, now, we can write suppose if we take this as the first probability, that is probability of T i greater than s into probability of T i greater than t plus s given that T i is greater than s. Now, using the memory less property; so, this probability will be same as probability of T i greater than t. Therefore, what we get is that probability of T i greater than t plus s in the case of this state holding time is same as probability of T i greater than s multiplied by probability of T i greater than t.

So, this quantity is the complementary CDF therefore, in terms of because we know that suppose F X of X is probability that X less than equal to small x. Therefore, complementary CDF F X C X complementary CDF is equal to 1 minus F X of X. That is equal to probability that X is greater than small x. So, this is the definition of complementary CDF. And therefore, in terms of complementary CDF we get that F T i c at point s plus t is equal to F T i c at point s into F T i c at point t. So, complementary CDF at point s plus t is the product of complementary CDF at instant s into complementary CDF at instant t. Taking the logarithm both sides we will get that log F T i c s plus t base e is same as log of F T i c s base e plus log of F T i c t base e.

So, this is the remarkable result that is the logarithm of a function at the sum of the two arguments is same as logarithm of the function of at one argument plus logarithm of the function at the other argument. So, logarithm of this sum is equal to sum of the individual logarithm. The only function that satisfy this relationship for arbitrary t and s is that is log of F T i c t must be equal to minus nu i into t; where nu i is a positive number. So, that what we will get F T i c t is equal to e to the power nu i t and the this is equal to 1 minus F T i t. And therefore if we take the derivative this quantity is 1 minus F T i t is equal to e to the power minus nu i t for t greater than 0 greater than equal to 0.

Now taking the derivative taking derivative with respect to t; we get what we get what we will get derivative of this is 0 derivative of this is the CDF and right hand side. Similarly we can take the derivative and we will get F T i of t is equal to nu i into e to the power minus nu i t t greater than equal to 0. So, this is the CDF of the state holding time. So, that way we have established their state holding time is memory less and it is distributed at the exponential random variable.

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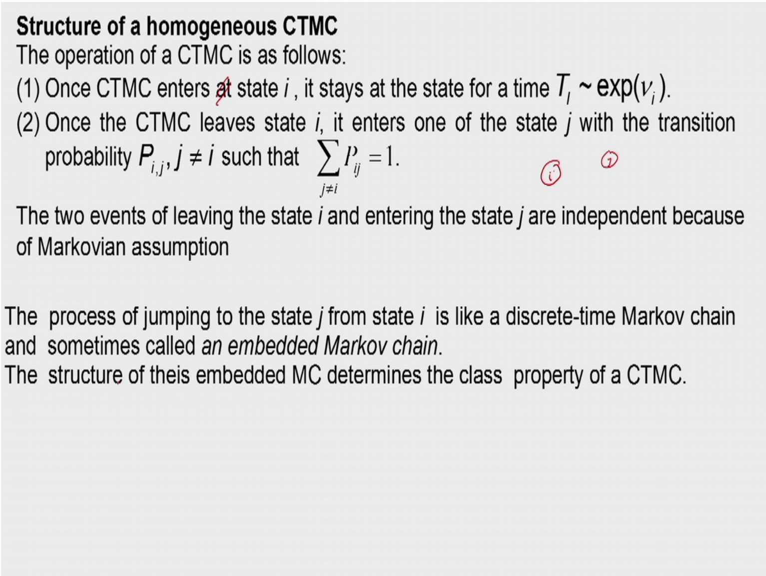

After defining the state holding time we can get the structure of a homogeneous CTMC. The operation of a CTMC is as follows, once CTMC enters at state i, enter state i it states at the state for a time T i which is exponentially distributed that we have established that it will remain in state i for a random time with pdf exponential nu i. Once the CTMC leaves state i, it enters one of the state j with the transition probability P i j j not equal to i such that summation P i j is equal to one suppose it is in state i now after leaving the state i it will go into other states, but not this state. So, that way it will suppose there is a state j. So, it will go to state j with probability P i j such that sum of all these probabilities where j not equal to i must be equal to 1.

So, after leaving this state i it has to go to one of these states j with probability P i j such that summation of P i j is equal to 1. Also due to events of leaving the state i and entering the state j are independent because of the Markovian assumption. Now, this process of jumping to state j from state i is like a discrete time Markov chain and is sometimes called an embedded Markov chain. So, along with a continuous time Markov chain there is always a always an embedded Markov chain. And this structure of the embedded

Markov chain determines the nature of the CTMC. This structure of this embedded Markov chain determines the class property of a CTMC.

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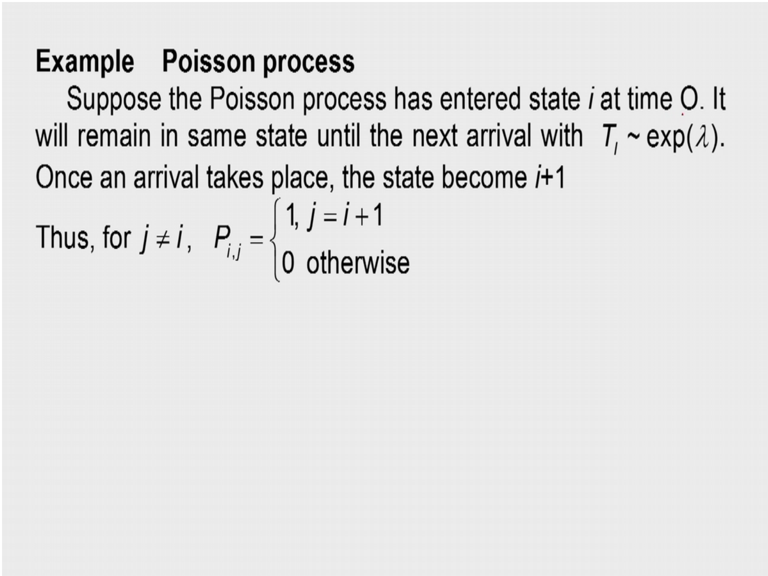

Example Poisson process, suppose the Poisson process has entered state i at time 0. It will remain in same state until the next arrival which is exponentially distributed. Once an arrival takes place the number of count in the Poisson process will go up by 1 that is this state will become i plus 1. Therefore, for j not equal to i P i, j will be always equal to 1, for j is equal to i plus 1 is equal to 0 otherwise. So, once it leaves state i, it will go into state i plus one with probability 1 and remaining state with probability 0.

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Short-time Behaviour of the chain at a time interval $(t, t+\Delta t)$

$$p_{ii}\left(\Delta t\right) = p(T_i > \Delta t) + o\left(\Delta t\right)$$

$$= e^{-v_i \Delta t} + o\left(\Delta t\right)$$

$$= 1 - v_i \Delta t + o\left(\Delta t\right)$$

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = v_i$$

Let us see the Short-Time behaviour of the chain at the time interval t, t plus delta t. This behaviour will determine the dynamics of the CTMC so, that we can get the corresponding equations which govern the behaviour of the CTMC. Now Pp ii delta t that is the probability that state holding time T i is greater than delta t and the; that means, there is no transition during the time t and t plus delta t plus o delta t.

Suppose this will take account of if there is multiple transition from state i, it has go to state j and it has again come back to state i. So, that probability will be less so that we have been denoted by o delta t. And the therefore, this time we know e to the power nu i delta t plus o delta t. and if i expand e to the power nu i delta t then i can write it as 1 minus nu by delta t plus remaining term I will club with this o delta t it will be o delta t only. So, this is the probability that this state will be i after delta t given that it is at state i at time instant 0. So, that way we can interpret this probability. So, from this what we will get 1 minus p ii delta t, 1 minus p ii delta t, 1 e minus p ii delta t divided by delta t e equal to nu i; that is one important result.

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For $j \neq i$

$$p_{ij}(\Delta t) = P(T_i < \Delta t) \times P_{ij}$$
$$= (1 - e^{-\nu_i \Delta t}) \times P_{ij}$$
$$= (\nu_i \Delta t + o(\Delta t)) \times P_{ij}$$
$$= (\nu_i \Delta t) P_{ij} + o(\Delta t) \quad = (\nu_i P_{ij}) \Delta t + o(\Delta t)$$
$$= q_{ij} \Delta t + o(\Delta t)$$

where $q_{ij} = \nu_i P_{ij}$ is the probability rate function.

Note that $\sum_{j \neq i} q_{ij} = \nu_i \sum_{j \neq i} P_{ij} = \cancel{1}\nu_i$ $\quad \sum_{j \neq i} q_{ij} = \nu_i \sum_{j \neq i} P_{ij} = \nu_i$

Denoting $\nu_i = -q_{ii}$, we get $\sum_j q_{ij} = 0$ $\quad \sum_j q_{ij} = 0$

Now, let us see what happened for j not equal to i in that case p ij delta t. So, what will happen that random time is less than delta t, state holding time is less than delta t that is the probability that t i is less than delta t, multiplied by. Now there will be a transition from state i to state j so multiplied by P ij this is also capital P. So, this will be equal to again 1 minus e to the power minus nu i delta t. That is the probability that T i greater than less than delta t. So, greater than delta t e to the power nu i delta t that we have derived; so, that will list will be 1 minus e to the power minus nu i delta t multiplied by P ij. And this quantity again if I expand in terms of that is exponential series the e to the power minus X suppose if I expand then I will get this term, 1 minus e to the power minus nu i delta t is nu i delta t plus o delta t and whole multiplied by P i j.

So, that way we will get nu i delta t multiplied by P i j plus o delta t. Now this quantity actually this nu i into P i j delta t plus o delta t; now this quantity nu i into P ij that is known as the probability rate function so and we denote it by q ij. So, that way q i j into delta t that will become probability so q ij is a probability rate function. And now let us see this sum of the, this q ij values sum of q ij j not equal to i, because there will be no transition to state itself. So, that way this is j not equal to i that is equal to nu i into P i j summation over j not equal to i is equal to that will be now this i know this quantity is equal to 1. So, this will be equal to therefore, this will be equal to nu i because this quantity is equal to 1.

Now, if i denote that nu i is equal to minus q ii because here I do not consider q ii and sum of these quantity is nu i. Therefore, if I denote q i i is equal to minus nu i then i will get summation of q i j over all j is equal to 0. Here we got this summation of q ij j not equal to i that is equal to nu i. Therefore, if I let q ii is equal to minus nu i then I will get this sum a is equal to 0. So, that way this sum of the probability rate function q ij over all state if I consider that must be equal to 0.

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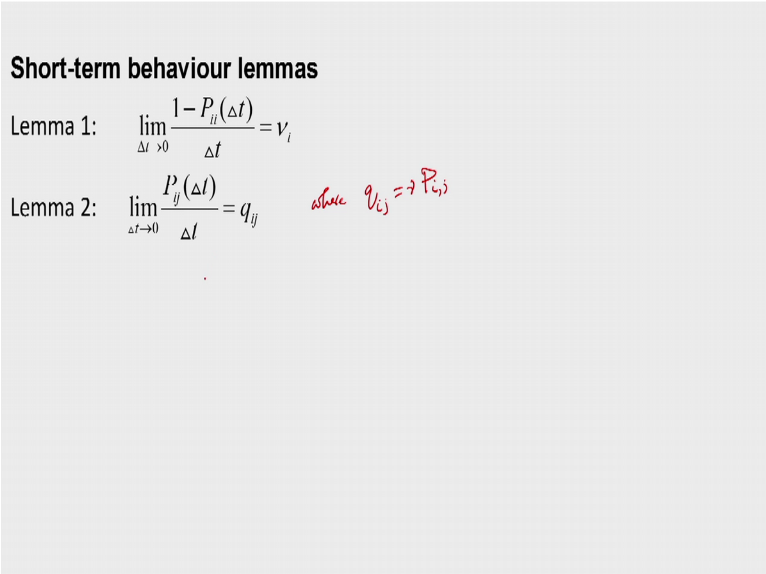

On the basis of the above discussion we present two important lemmas about this short term behaviour of continuous time Markova chain. Lemma 1 states that limit 1 minus P ij delta t divided by delta t as delta t tends to 0 is equal to nu i. So, 1 minus P ii delta t is the probability that the chain leaves state i during delta t so this is 1 minus P ii delta t. And now according to this lemma therefore, the rate of the probability of leaving the state i is equal to nu i.

Similarly lemma 2 states that limit P i j delta t divided by delta t as delta t tends to 0 is equal to q ij where q ij is equal to nu i into P ij. So, this P ij is the state transition probability of the embedded Markov chain. So, according to this lemma the rate of the probability of entering this state j from state i is equal to q ij. So, these two lemmas will be very useful in deriving the differential equations governing the evolution of continuous time Markov chains.

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Kolmogorov equations

Chapman Kolmogorov Equation: $p_{ij}(s+t) = \sum_k p_{ik}(s)p_{kj}(t)$

To characterize the transition probabilities dynamically, Kolmogorov backward and forward differential equations are used.

Kolmogorov Backward Equation

0 Δt $t+\Delta t$

$$\begin{aligned} p_{ij}(t+\Delta t) &= P(X(t+\Delta t)=j \mid X(0)=i) \\ &= \sum_k p_{ik}(\Delta t)p_{kj}(t) \\ &= p_{ii}(\Delta t)p_{ij}(t) + \sum_{k \neq j} p_{ik}(\Delta t)p_{kj}(t) \\ &= (1 - v_i\Delta t + o(\Delta t))p_{ij}(t) + \sum_{k \neq j} q_{ik}\Delta t p_{kj}(t) \end{aligned}$$

$$\therefore \lim_{\Delta t \to 0} \frac{p_{ij}(t+\Delta t) - p_{ij}(t)}{\Delta t} = -v_i p_{ij}(t) + \sum_{k \neq j} q_{ik} p_{kj}(t)$$

$$\therefore p_{ij}'(t) = -v_i p_{ij}(t) + \sum_{k \neq j} q_{ik} p_{kj}(t)$$

Kolmogorov equations for a CTMC we know this Chapman Kolmogorov equation that is p ij s plus t that is very obvious. We can write it as summation over p ik s over k into P k j t. So, P ij s plus t we can write as summation p i k s into p k j t over k so this is the Chapman Kolmogorov equation. So, first we consider transition to state s and from state s to kolmogorov equations for CTMC. We can get this Chapman Kolmogorov equation for continuous time Markova chain that is p i j s plus t is same as summation p i k s into p k j t. So, this is the transition to state k during s and from state k to state j during t.

So, that way we can get the Chapman Kolmogorov equation, but for describing the chain dynamically Kolmogorov forward and backward differential equations are used. First we will derive the Kolmogorov backward differential equation. So, here what we assume is that suppose to this state j at t plus delta t there is transition from state i to some state during delta t and then from those states it will there will be transition during t. So, first this is the time instant delta t we consider and remaining time we will consider later. So, that way it is backward delta t is before t plus delta t. So, that way backward tenure come and now p i j t plus delta t that is equal to probability that X of t plus delta t is equal to j given that X 0 is equal to i. So, p i j at t plus delta t we are interested to find then we can find out the derivative.

So, this is same as so from 0 to delta t we can consider and from delta t to t. So, that way this is same as summation p i k delta t into p k j t summation over k. Now here we can separate out p i i delta t into p i j t. First term we have taken out k is equal to i term we have taken out and remaining term suppose k not equal to i. So, this will be summation p i k delta t into p k j t. Now I know that p i i delta t is equal to one minus nu i delta t plus o delta t into p i j t plus remaining term. Now i know that p i k delta t using the lemma two that is equal to q i k into delta t multiplied by p k j t.

So, that way we have p i j t plus delta t given by this expression. Now if i take this p i j t to the left hand side and divided by delta t and consider when delta t tends to 0, then we will get this limit is equal to minus nu i p i j t. Because this delta t will come here to the denominator so that way minus nu i into p i j t plus q i k into p k t because delta t will come to this denominator. And as delta t tends to 0 this will be the derivative of p i j t. So, p i j dash t will be equal to nu i p i j t plus q i k q i k p k j t k not equal to i.

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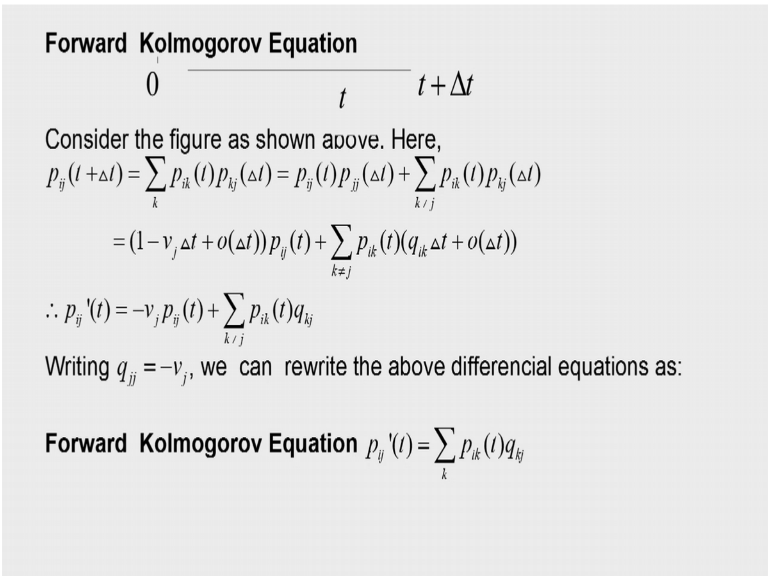

So, this is the Kolmogorov acquired equation, similarly in the forward equation we consider first time t 0 to t then t to t plus delta t. So, here p i j t plus delta t will be p i k t because there will be transition to state k during time t p i k t into p k j delta t. So, we can consider here one k is a. So, that will be p i j t into p j delta t plus summation p i k t into p k j delta t where k not equal to j. Now, this quantity I know and this quantity also I know. So, that way using lemma 1 and lemma 2 I can write this as this quantity as 1 minus nu j delta t plus o delta t multiplied by p i j t. So, this this term I am writing using lemma 1.

Similarly, in the here also i use the lemma 2 to get this term p i p k j delta t is equal to q i k q k j q k j delta t p k p k j delta t is equal to q k j delta t plus o delta t so that we have written using lemma 2. So, now, what we will do is we will take be p j p i j t left hand side and 1 into p i j t that is equal to p i j t. So, this will take to the left hand side and divided by delta t. So, what we will be left with as delta t tends to 0. What we will get is that? p i j dash t that is the derivative is equal to minus nu j into p i j t plus summation p i k t q k j k not equal to j.

So, this is the forward Kolmogorov equation and we can also we earlier defined that q j j is equal to minus nu j. Then we can write this term also we can club along with this summation we can write p i j dash t is equal to summation p i k t into q k j over k this is the Kolmogorov forward equation because here that incremental part we have considered after t. So, that way it is forward equation earlier with derived backward equation.

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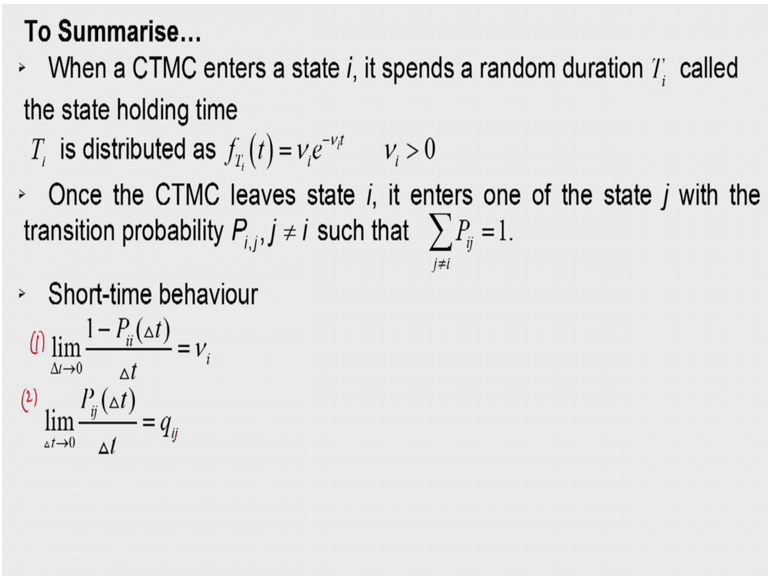

Let us summarize the lecture when a CTMC enters a state i, it spins a random duration t i this is the state holding time and t i is distributed as exponential distribution that is f t i of t is equal to nu i into e to the power minus nu i t where nu i is greater than 0. Once the CTMC leave state i, it enters one of the state j with the transition probability P i j where j not equal to i such that summation P i j over j not equal to i is equal to 1.

So, once the CTMC leaves the state i, it will enter into one of the state j with probability P i j. We also stated two important lemmas regarding the short term behaviour of a

Markov chain these two lemmas are lemma 1 this was the lemma 1limit 1 minus P i i delta t by delta t as delta t tends to 0 is equal to nu i. And the lemma 2 was limit P i j delta t by delta t as delta t tends to 0 is equal to q i j.

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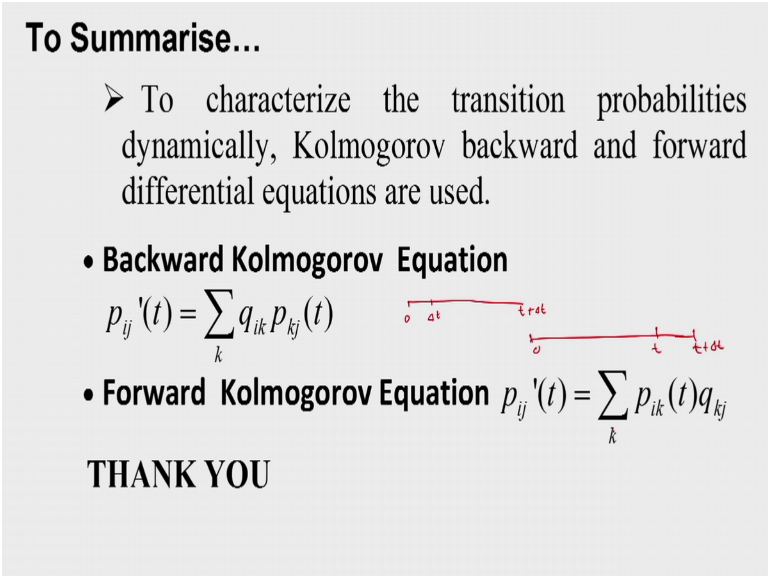

Now, to characterize the transition probabilities dynamically Kolmogorov backward and forward differential equations are use. So, what we got Kolmogorov backward equation that is P i j dash t is equal to summation over k q i k. So, initially in delta interval of time it will go to state in state k. So, that is taken care of by q i k into p k j t. So, that is the backward Kolmogorov equation here from 0 then delta t then t plus delta t.

So, the first transition will take place two different states during duration delta t. And after that from those states to it will go to state j during interval t. so that way this is backward kolmogorov equation similarly forward kolmogorov equation where we have this is 0 then t and t plus delta t. So, that first that there will be a transition to state k at during time t and then from those states there will be transition to state j during the incremental time delta t. So, that way we derived the Kolmogorov forward equation and it is given by p i j dash t is equal to summation p i k t into q k j over k.

Thank you.