

Advanced Control Systems
Prof. Somanath Majhi
Department of Electronics and Electrical Engineering
Indian Institute of Technology, Guwahati

Module No. # 04
Design of Controllers
Lecture No. # 04
Model based PID Controller Design

Welcome to the lecture titled model based PID controller design; controller designed based on the phase and gain margins are found to be satisfactory; attempt will be made in this lecture to design simple PID controllers based on phase and gain margins. And phase and gain margins are to be user defined, therefore one has to choose judiciously the phase and gain margins to design proper PID controllers.

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A relay test is conducted to identify the transfer function model for the dynamics of a real time process.

$$G_m(s) = \frac{K e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)}$$

The controller t.f. be

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s}\right) \left(1 + \frac{T_d s}{1 + \frac{T_d s}{T_f}}\right)$$

$\alpha = 0.1 T_d$ $G_c(s) = K_p \left(1 + \frac{1}{T_i s}\right) (1 + T_d s)$



Now, I will go to the analysis first; initially a relay test is conducted to identify the transfer function model for the dynamics of a system, for the dynamics of a real time process or plant. So, how do we find that? The detail has been explained in some lecture; now you please refer to the identification techniques we have discussed for identification of plant dynamics. And also we have found the plant dynamics given by the transfer

function models $G_m(s)$ is equal to $K e^{-\theta s}$ divided by $T_1 s + 1$ times $T_2 s + 1$, so this is the transfer function model for a second order plus dead time dynamics or second order plus dead time transfer function model, which has got, how many parameters? The steady state gain K , the time delay θ , the time constants T_1 and T_2 . So, we have got four parameters associated with the transfer function model.

Now, we shall attempt to design PID controller for the system, where the controller will assume the form $G_c(s)$ is equal to $K_p \frac{1 + 1/T_i s}{1 + T_d s}$ divided by $1 + \alpha T_d s$, so this is the real realistic PID controller in series form will be used in our study. Now as you know α will be some percentage of T_d only; so, α is assumed to be 0.1 of T_d or less than that. So that way, this α will be neglected in the analysis; thus giving us finally, the form of $G_c(s)$ as $G_c(s)$ is equal to $K_p \frac{1 + 1/T_i s}{1 + T_d s}$.

So, assuming that we have with us the transfer function model of a plant given by this and a series form of PID controller of this form will proceed with our analysis the way we can find explicit expressions for K_p , T_i and T_d in terms of the model parameter of the plant dynamics. The model parameters means in terms of K , θ , T_1 and T_2 , now one more thing I would like to say we will begin our analysis with a second order plus dead time stable process model then we will go for unstable process model later on, because in one go if you apply then you will mess up you will end up with horrible expressions.

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SOPDT
stable
transfer function
model

$$G_m(s) = \frac{K e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)}$$

$$G_c(s) = K_p \left(1 + \frac{1}{T_i s}\right) (1 + T_d s)$$

$$= K_p \left(\frac{T_i s + 1}{T_i s}\right) (1 + T_d s)$$

Let $T_i = T_1$ and $T_d = T_2$

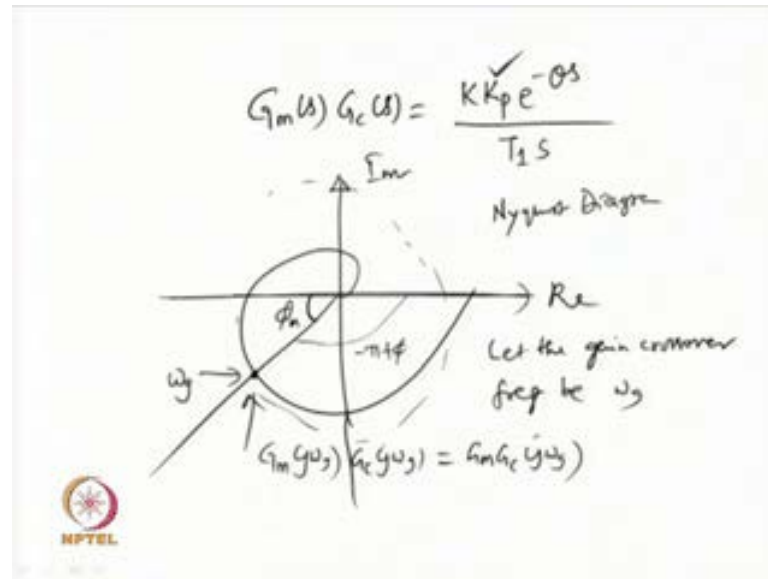
$$G_m(s) G_c(s) = \frac{K e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)} \times \frac{K_p (T_i s + 1)(1 + T_d s)}{T_i s}$$

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So, I will begin with the transfer function model $G_m(s)$ is equal to $K e^{-\theta s}$ to the power minus theta s upon $T_1 s + 1$ times $T_2 s + 1$. So, we do consider a second order plus dead time stable transfer function model, now the controller dynamics is again repeated here $G_c(s)$ is equal to K $G_c(s)$ is equal to K not K_p the static gain steady state gain of the process is given by K . So the controller is now represented by $G_c(s)$ is equal to K_p times $1 + 1$ upon $T_i s$ times $1 + T_d s$ which is equal to K_p times $T_i s + 1$ divided by $T_i s$ times $1 + T_d s$, now if I carefully look at the form of the controller and that of the transfer function model, why not to make use of cancellation of poles and zeroes? And find the remaining parameter of the PID controller so one will advise one will go for simple technique, why to make our life miserable? Why not to assume that $K_p T_i$ is equal to T_1 and $T_2 T_d$ is equal to T_2 .

So, let T_i is equal to T_1 and T_d is equal to T_2 that will allow me to write the loop transfer function or loop gain as $G_m(s) G_c(s)$ as $K e^{-\theta s}$ to the power minus theta s divided by $T_1 s + 1$ times $T_2 s + 1$ into K_p times $T_i s + 1$ times $T_d s + 1$ divided by $T_i s$, now as we have assumed T_i is equal to T_1 and T_d is equal to T_2 . So, we will have pole 0 cancellations and cancel those poles, now T_i is equal to T_1 now. So, I can cancel this term with this one similarly $T_d s + 1$ will cancel out with $T_2 s + 1$ thus giving us a loop transfer function of much simpler form.

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So, the loop transfer function now $G_m(s) G_c(s)$ becomes $G_m(s) G_c(s)$ is equal to $K K_p e^{-\theta s}$ divided by $T_1 s$ because T_i is equal to T_1 .

So, since you are getting a simpler loop transfer function now the phase and gain margin criterion can be used to design the gain parameters associated with the PID controller, already we have designed two parameters of the PID controller, and how you have designed? as you know T_d is equal to T_2 and T_i is equal to T_1 thus the 2 parameters of the controller T_i T_d and T_i already been designed or chosen based on the time constants of the process dynamic model. So, I will now try to find explicit expression for the proportional gain associated with the controller K_p , now we know that from the nyquist diagram when the dynamic system has got a plot of this form the phase and gain margins can be easily obtain.

So, let the phase margin be ϕ_m at this time, what is the angle of the loop gain? That will be minus π plus ϕ this is known to you this is very simple, but the loop gain at that gain crossover frequency so this point is having a gain crossover frequency ω_g . So, let the gain crossover frequency be ω_g then the loop at this gain crossover frequency has a gain of $G_m(j\omega_g)$ times $G_c(j\omega_g)$ which can also be written as $G_m G_c(j\omega_g)$. So, $G_m G_c(j\omega_g)$ magnitude will be one and phase angle will be, how much minus π plus ϕ ? when the process is subjected to or the plant is subjected to a phase margin of angle ϕ_m .

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$$\begin{aligned}
 & |G_m G_c(j\omega_g)| = 1 \checkmark \\
 & \text{and } \pi + \angle G_m G_c(j\omega_g) = \phi_m \checkmark \\
 & \Rightarrow \left| \frac{K K_p e^{-j\omega_g \theta}}{j\omega_g T_1} \right| = 1 \\
 & \Rightarrow \frac{K K_p}{\omega_g T_1} = 1 \\
 & \Rightarrow \omega_g = \frac{K K_p}{T_1}
 \end{aligned}$$

So, by definition $G_m G_c(j\omega_g)$ magnitude is equal to 1 and $\pi + \angle G_m G_c(j\omega_g)$ is equal to ϕ_m . So, this is how based on the gain margin criterion when ω_g becomes the gain crossover frequency we have got the two equations related to the phase margin of a system. So, I shall make use of the expressions now we have already found. So, $G_m G_c(j\omega_g)$ can be written as $K K_p e^{-j\omega_g \theta} / (j\omega_g T_1)$ this magnitude is equal to 1, implies $K K_p / (\omega_g T_1)$ is equal to 1 next ω_g will be equal to $K K_p / T_1$.

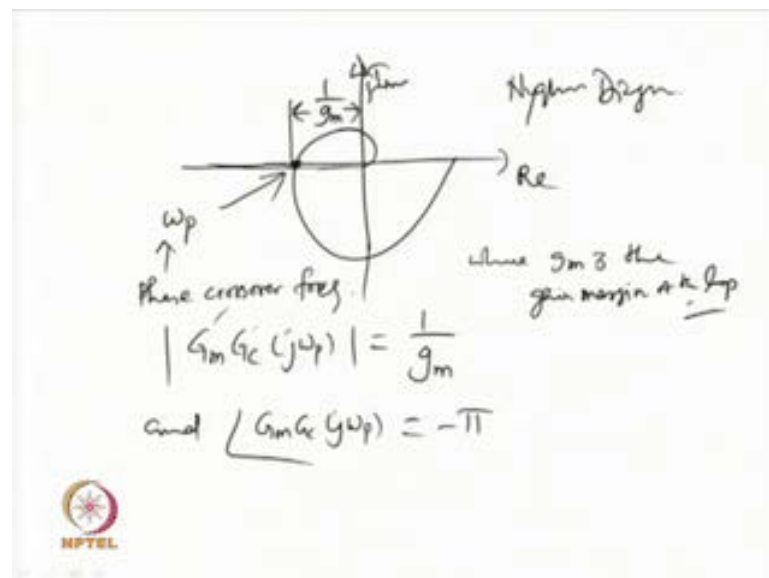
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$$\begin{aligned}
 & \pi + \angle G_m G_c(j\omega_g) = \phi_m \\
 & \Rightarrow \pi - \omega_g \theta - \pi/2 = \phi_m \\
 & \Rightarrow \omega_g \theta = \pi/2 - \phi_m \\
 & \text{But } \omega_g = \frac{K K_p}{T_1} \\
 & \Rightarrow \frac{K K_p \theta}{T_1} = \pi/2 - \phi_m \\
 & \Rightarrow K_p = \frac{(\pi/2 - \phi_m) T_1}{K \theta}
 \end{aligned}$$

So, ω_g is equal to $K K_p$ by T 1 this is what you gain get from the gain condition coming to the phase margin or phase condition then I will get similar analytical expression given by π plus angle of $G_m G_c$ at angle at frequency ω_g as angle $G_m G_c(j\omega_g)$ is equal to the phase margin, implies π minus ω_g theta please recall the loop gain is $G_m G_c(j\omega_g)$ which can be given as $K K_p e$ to the power minus $j\omega_g$ theta divided by $j\omega_g T$ 1 so please recall this one because I am directly straight forward writing the expression for the phase angles. So, that way this will give us π minus ω_g theta minus π by 2 is equal to ϕ_m implies ω_g theta is equal to π by 2 minus ϕ_m , but ω_g is found to be $K K_p$ by T 1 please see already we have find g is equal to $K K_p$ by T 1, therefore $K K_p$ divided by T 1 theta is equal to π by 2 minus ϕ_m implies K_p is equal to π by 2 minus ϕ_m times T 1 by K theta, thus the proportional gain of the PID controller is obtained in terms of the plant model parameters and the phase margin.

So, define certain phase margin and as you know $K T$ 1 and theta are known to you therefore, you will be able to estimate the proportional gain of the PID controller using this formula, also the proportional gain can be found using the gain margin criteria. So, let us try to find the proportional gain expression for the proportional gain using gain margin criteria.

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So, what is that gain margin criteria? We know that, again I will go back to the nyquist plot or nyquist diagram real and imaginary, then nyquist diagram when you have got a nyquist plot of this form then this gives you inverse of the gain margin. So, let the gain margin be G_m then this will be one upon g_m this span will be one upon g_m , where g_m is the gain margin of the control loop gain margin of the loop.

So, I will write two analytical expressions corresponding to this operating point which is giving a gain margin of g_m now the magnitude of the loop function at this phase crossover frequency because, what is the frequency during at this operating point? we have got the phase crossover frequency because the phase angle is minus 180 degree now negative real axis keep in mind this is the negative real axis and it is crossing the plot is crossing the negative real axis therefore, the phase crossover frequency the phase crossover frequency ω_p will give analytical expressions $G_m G_c(j\omega_p)$ magnitude will be one upon g_m , where g_m is the gain margin of the loop and angle $G_m G_c(j\omega_p)$ this will be equal to minus pi. Now, substitute the expression for G_m and G_c .

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$$\frac{K K_p}{\omega_p T_1} = \frac{1}{g_m}$$

$$\Rightarrow \omega_p = \frac{K K_p \cdot g_m}{T_1}$$

further, $\angle G_m G_c(j\omega_p) = -\pi$

$$\Rightarrow -\omega_p \theta - \pi/2 = -\pi$$

$$\Rightarrow \omega_p \theta = \pi/2 \Rightarrow \omega_p = \frac{\pi}{2\theta}$$

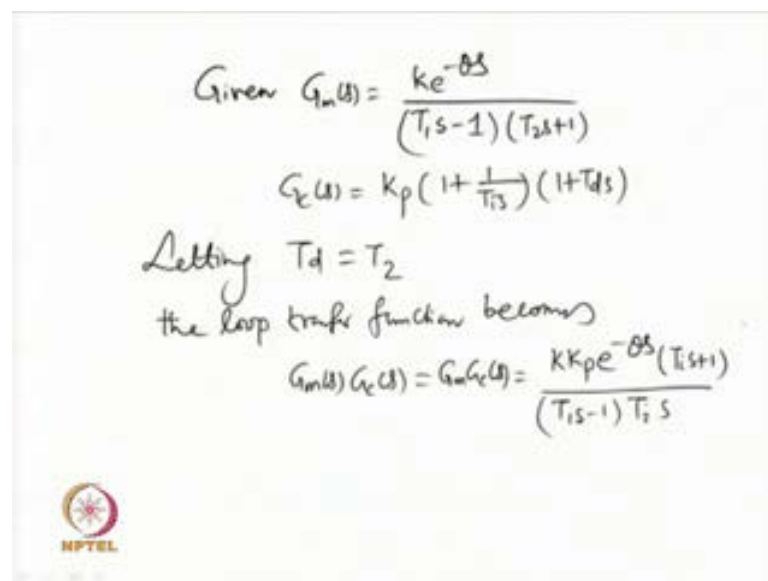
$$\Rightarrow \frac{K K_p g_m}{T_1} = \frac{\pi}{2\theta} \Rightarrow K_p = \frac{\pi T_1}{2\theta K g_m}$$

So, we will get $K K_p$ divided by $\omega_p T_1$ this the magnitude you are getting $K K_p$ upon $\omega_p T_1$ is equal to one upon g_m implies ω_p is equal to $K K_p g_m$ divided by T_1 , further considering the phase angle criterion we know that angle $G_m G_c$ at the phase crossover frequency of ω_p is equal to minus pi implies minus ω_p

$\theta - \pi/2 - \omega_p \theta - \pi/2$ is equal to $-\pi$ implies $\omega_p \theta$ is equal to $\pi/2$ or ω_p is equal to $\pi/2\theta$ substitute the ω_p here giving us $K K_p g_m$ divided by T_1 is equal to $\pi/2\theta$ implies K_p is equal to πT_1 divided by $2\theta K g_m$.

So, this is the final expression we have got for the proportional gain of the PID controller, K_p is equal to πT_1 upon $2\theta K g_m$. So, choose some gain margin for the closed loop and as you know T_1 and θ and π thus you will be able to estimate the proportional gain of the PID controller either using this formula or using this formula using either using phase margin or using gain margin values it is possible to estimate the proportional gain for the PID controller, using this technique now effort will be made to design controller for a second order unstable process.

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Given $G_m(s) = \frac{K e^{-\theta s}}{(T_1 s - 1)(T_2 s + 1)}$

$G_c(s) = K_p \left(1 + \frac{1}{T_i s} + T_d s\right)$

Letting $T_d = T_2$
the loop transfer function becomes

$G_m(s) G_c(s) = G_{m,c}(s) = \frac{K K_p e^{-\theta s} (T_i s + 1)}{(T_1 s - 1) T_i s}$



So, given the second order unstable process transfer function model in the form of $G_m(s)$ is equal to $K e^{-\theta s} / (T_1 s - 1)(T_2 s + 1)$ and a controller of the form $G_c(s)$ is equal to $K_p (1 + 1/T_i s + T_d s)$, now letting T_d is equal to T_2 the loop transfer function becomes $G_m(s) G_c(s)$ is equal to $G_{m,c}(s)$ is equal to $K K_p e^{-\theta s} (T_i s + 1) / (T_1 s - 1) T_i s$.

So, this is the loop transfer function we will have now the loop transfer function has got one pole located at $-1/T_1$ and one pole located at $-1/T_i$ and one pole located at

1 upon T 1 because you have got an unstable pole for the second order system please keep in mind. So, you cannot make cancellation of the pole zero that is not allowed. So, that way the analysis for this loop transfer function will be a bit involved unlike the earlier case we may not be able to design all the parameters all right using simple formulae or technique.

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At the gain crossover frequency ω_g

$$|G_m G_c(j\omega_g)| = 1$$

$$\pi + \angle G_m G_c(j\omega_g) = \phi_m$$

$$\frac{K K_p \sqrt{(\omega_g T_i)^2 + 1}}{\omega_g T_i \sqrt{(\omega_g T_i)^2 + 1}} = 1$$

$\omega_g T_i \gg 1$ and $\omega_g T_i \gg 1$

$$\frac{K K_p \omega_g T_i}{\omega_g T_i \times \omega_g T_i} = 1 \Rightarrow \omega_g = \frac{K K_p}{T_i}$$

So, we will proceed with the analysis of this one using the same concept of phase margin and gain margin, I will use the phase margin condition, phase margin condition gives us that at the gain crossover frequency ω_g is $G_m G_c(j\omega_g)$ magnitude becomes 1 and $\pi + \angle G_m G_c(j\omega_g)$ will be equal to ϕ_m , where ϕ_m is the phase margin, so these are the two conditions we know as far as the phase margin of the system is concerned. So, we shall make use of the analysis of the analytical expressions you we have obtained substitution of G_m and G_c will result in the first expression as $K K_p \sqrt{(\omega_g T_i)^2 + 1} / \omega_g T_i \sqrt{(\omega_g T_i)^2 + 1}$ is equal to 1 as far as magnitude is concerned the exponential term $e^{-j\omega_g T_i}$ is not contributing that is equal to 1 therefore, the magnitude condition gives us an expression of this form.

Now, to simplify this expression I have to make some assumptions and the assumptions are not arbitrary based by the intuition and theory also it is possible to assume make assumptions and with the assumption of $\omega_g T_i$ is larger than 1 and $\omega_g T_1$ is

greater than 1 it is possible to get the same expression explicitly expressed in the form of $K K_p \omega_g T_i$ in the numerator divided by $\omega_g T_i$ into $\omega_g T_i$ in the denominator which is equal to 1, now so what I have done with the assumption of $\omega_g T_i$ is greater than 1 or greater than 1 and $\omega_g T_i$ is very large compared to one then the amplitude expression or the magnitude of the loop gain at the gain crossover frequency gives us an expression of the form $K K_p \omega_g T_i$ divided by $\omega_g T_i$ $\omega_g T_i$ is equal to 1. So, cancel these two terms and then we will get ω_g equal to $K K_p$ by T_i $K K_p$ by T_i so please keep in mind this will be used further in subsequent analytical expression.

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$$\pi + \angle G_m G_c(j\omega_g) = \phi_m \quad \leftarrow \checkmark$$

$$\pi - \omega_g \theta + \tan^{-1}(\omega_g T_i) + \tan^{-1}(\omega_g T_l) = \phi_m$$

$$\phi_m = \pi - \omega_g \theta + \tan^{-1}(\omega_g T_i) + \tan^{-1}(\omega_g T_l)$$

$$\frac{d\phi_m}{d\omega_g} = 0 = -\theta + \frac{T_i}{1 + (\omega_g T_i)^2} + \frac{T_l}{1 + (\omega_g T_l)^2}$$

$$d(\tan^{-1} x) = \frac{dx}{1 + x^2}$$

$$d(\tan^{-1}(\omega_g T_i)) = \frac{T_i}{1 + (\omega_g T_i)^2}$$

So, now coming to the condition on the phase angles we know that π plus angle of $G_m G_c(j\omega_g)$ is equal to the phase margin ϕ_m this is by definition of the phase margin we get this analytical expression, now substitution of the angles will give us π minus $\omega_g \theta$ plus \tan inverse of $\omega_g T_i$ plus \tan inverse of $\omega_g T_l$ is equal to ϕ_m , how I have got these angles? Because, in the numerator of the loop gain we have got terms like $1 + j\omega_g T_i$ and in the denominator you have got the term $j\omega_g T_l$ $j\omega_g T_l$ minus 1 so the phase angle of this one can be written as \tan inverse of $\omega_g T_i$ minus of \tan inverse of $\omega_g T_l$. So, this minus of minus will give you ultimately \tan inverse $\omega_g T_i$ minus will be plus \tan inverse $\omega_g T_l$. So, that is how I have obtained the phase angle from the loop gain.

Now, we will simplify this expression, but before going to the simplification let us attempt to find optimum value of the phase margin is it possible to find optimum value for the phase margin for the loop? yes. So, to find the optimum value what you have to do write the expression for ϕ_m , ϕ_m is equal to $\pi - \omega_g \theta + \tan^{-1} \omega_g T_i + \tan^{-1} \omega_g T_1$, now differentiate ϕ_m with respect to the gain crossover frequency and equate that to 0 and the gain frequency that you will find will ensure maximum phase margin because using this condition first order derivative of a function equated to zero results in optimum value for that function at some variable.

So, that way using that I will find the first order derivative now that will give me $-\theta + T_i \omega_g / (1 + \omega_g^2 T_i^2) + T_1 \omega_g / (1 + \omega_g^2 T_1^2)$, how I have found the first order differentiation of arc tan functions? we know the rule that differentiation of arc tan function $d(\tan^{-1} x) / dx = 1 / (1 + x^2)$ I will take the d to that side. So, differentiation of $\tan^{-1} x$ is equal to $dx / (1 + x^2)$ the formula is very simple so differentiation of $\tan^{-1} \omega_g T_i$ will be how much differentiation of $\omega_g T_i$ with respect to ω_g will give you T_i in the numerator and in the denominator $1 + \omega_g^2 T_i^2$. So, when you differentiate with respect to ω_g you are getting like this. So, using that differentiation we get this term.

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$$\begin{aligned} \pi + \angle G_m h_c (j\omega_g) &= \phi_m \quad \leftarrow \checkmark \\ \pi - \omega_g \theta + \tan^{-1}(\omega_g T_i) + \tan^{-1}(\omega_g T_1) &= \phi_m \\ \phi_m &= \pi - \omega_g \theta + \tan^{-1}(\omega_g T_i) + \tan^{-1}(\omega_g T_1) \\ \frac{d\phi_m}{d\omega_g} &= 0 = -\theta + \frac{T_i}{1+(\omega_g T_i)^2} + \frac{T_1}{1+(\omega_g T_1)^2} \\ \Rightarrow \frac{T_i}{\omega_g^2 T_i^2} + \frac{T_1}{\omega_g^2 T_1^2} &= \theta \\ \Rightarrow \frac{1}{\omega_g^2} \left[\frac{1}{T_i} + \frac{1}{T_1} \right] &= \theta \end{aligned}$$

So, this when simplified further gives us T_i divided by $1 + \omega_g^2 T_i^2$ plus T_1 divided by $1 + \omega_g^2 T_1^2$ is equal to θ , but we have assumed $\omega_g T_i$ is greater than 1 and $\omega_g T_1$ is greater than 1 therefore, the same expression can be having no one in the denominators giving you expression like this now you simplify this one take one upon ω_g^2 as common leaving you 1 upon T_i plus 1 upon T_1 equal to θ . So, this is one expression we have got for the maximum phase margin of the loop.

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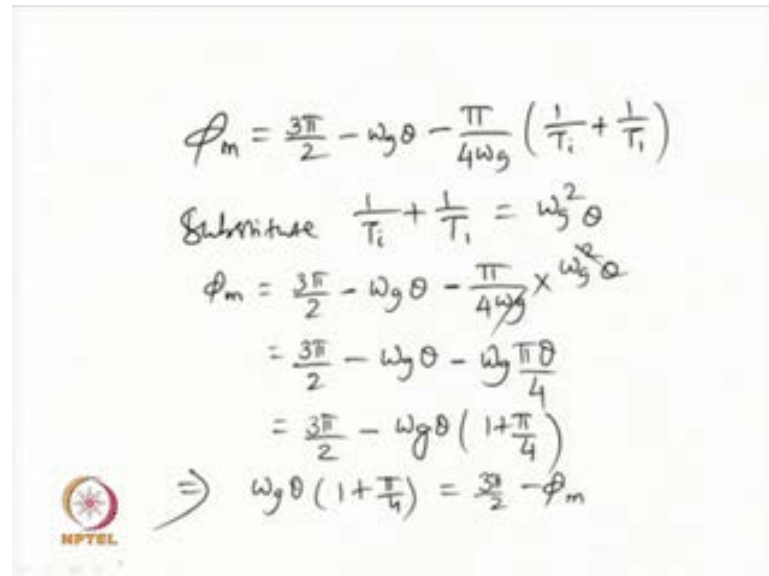
$$\begin{aligned} \boxed{\frac{1}{T_i} + \frac{1}{T_1} = \omega_g^2 \theta} \\ \pi + \angle G_m h_c (j\omega_g) &= \phi_m \\ \boxed{\frac{\pi}{2} - \omega_g \theta + \tan^{-1}(\omega_g T_i) + \tan^{-1}(\omega_g T_1) = \phi_m} \\ \tan^{-1} x &= \frac{\pi}{2} - \frac{\pi}{4x} \quad x \geq 1 \\ \left(\frac{\pi}{2} - \omega_g \theta + \left(\frac{\pi}{2} - \frac{\pi}{4(\omega_g T_i)} \right) + \left(\frac{\pi}{2} - \frac{\pi}{4(\omega_g T_1)} \right) \right) &= \phi_m \end{aligned}$$

So, I can finally, write $1 + \frac{1}{T_i}$ is equal to $\omega_g^2 \theta$. Now, we will go back to the expression for the phase margin which will give us π plus angle of $G_m G_c(j\omega_g)$ is equal to ϕ_m this is what we know and we have found the expression as $\pi + \pi - \omega_g \theta - \pi/2$, have I forgotten the $\pi/2$ because $\omega_g T_i$ is there this $j\omega_g T_i$ this is a magnitude coming to the phase angle $j\omega_g T_i$.

So, minus $\pi/2$ term will come minus $\pi/2$ that I have forgotten because if you write this the expression for this one angle $j\omega_g$ the loop gain will involve $j\omega_g T_i$ due to this j minus $\pi/2$ angle will come. So, I have made one mistake over here, it does not matter as far as differentiation is concerned. So, it will be getting added with minus $\pi/2$ terms minus $\pi/2$. So, I will push that term over here minus $\pi/2$ is equal to ϕ_m . So, ϕ_m is equal to $\pi/2 - \omega_g \theta + \tan^{-1} \omega_g T_i$ as far as differentiation is concerned this term has no role therefore, whatever expression we have got is correct.

Now, this is now finally given as $\pi/2 - \omega_g \theta + \tan^{-1} \omega_g T_i$ is equal to ϕ_m , this the expression for the phase margin now it is very difficult to solve this analytical expression unless we make some assumption of the arc tan function as you know we are dealing with non-linear equation and it is not possible to solve the non-linear equation by hand and for therefore, please allow me to make the assumption using the assumption that $\tan^{-1} x$ is equal to $\pi/2 - \pi/4 x$ for x is greater than 1, please keep in mind this is the assumption I am going to make further non-linear function arc tan function. So, $\tan^{-1} x$ is equal to $\pi/2 - \pi/4 x$, for all x greater than equal to 1 that will give us the expression for the phase margin as $\pi/2 - \omega_g \theta + \pi/2 - \pi/4 \omega_g T_i$ is equal to ϕ_m , now let me simplify we have got $3\pi/2$ so $3\pi/2$ will go to this side if I take the other terms to this side then it will give as ϕ_m is equal to.

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The image shows a handwritten derivation for the phase margin ϕ_m . It starts with the equation $\phi_m = \frac{3\pi}{2} - \omega_g \theta - \frac{\pi}{4\omega_g} \left(\frac{1}{T_i} + \frac{1}{T_1} \right)$. Then, it states the substitution $\frac{1}{T_i} + \frac{1}{T_1} = \omega_g^2 \theta$. This is substituted into the first equation, resulting in $\phi_m = \frac{3\pi}{2} - \omega_g \theta - \frac{\pi}{4\omega_g} \times \omega_g^2 \theta$. The next steps show the simplification: $= \frac{3\pi}{2} - \omega_g \theta - \omega_g \frac{\pi \theta}{4}$, then $= \frac{3\pi}{2} - \omega_g \theta \left(1 + \frac{\pi}{4} \right)$. Finally, it rearranges the equation to $\Rightarrow \omega_g \theta \left(1 + \frac{\pi}{4} \right) = \frac{3\pi}{2} - \phi_m$. An NPTEL logo is visible in the bottom left corner of the slide.

So, let ϕ_m the phase margin is equal to $\frac{3\pi}{2} - \omega_g \theta$ minus $\frac{\pi}{4\omega_g} \left(\frac{1}{T_i} + \frac{1}{T_1} \right)$ I will take common here $\frac{\pi}{4\omega_g}$ if I take common $\frac{\pi}{4\omega_g}$, we are left with 1 upon T_i plus 1 upon T_1 .

Now, we will substitute 1 upon T_i plus 1 upon T_1 which is nothing but $\omega_g^2 \theta$ please keep in mind already we have found that 1 upon T_i plus 1 upon T_1 is equal to $\omega_g^2 \theta$ so substitution of that will lead to ϕ_m is equal to $\frac{3\pi}{2} - \omega_g \theta$ minus $\frac{\pi}{4\omega_g} \times \omega_g^2 \theta$, simplify little bit cancel ω_g over here giving us $\frac{3\pi}{2} - \omega_g \theta$ minus $\omega_g \times \frac{\pi \theta}{4}$ so further simplification is possible $\frac{3\pi}{2} - \omega_g \theta$ times 1 plus $\frac{\pi}{4}$ thus $\omega_g \theta$ will be $\omega_g \theta$ times 1 plus $\frac{\pi}{4}$ is equal to $\frac{3\pi}{2} - \phi_m$.

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$$\Rightarrow \omega_g = \frac{(\frac{3\pi}{2} - \phi_m)}{\theta (1 + \frac{\pi}{4})}$$

$$\Rightarrow \frac{KK_p}{T_1} = \frac{3\pi/2 - \phi_m}{\theta (1 + \frac{\pi}{4})}$$

$$\Rightarrow K_p = \frac{(3\pi/2 - \phi_m)}{\theta (1 + \frac{\pi}{4})} \times \frac{T_1}{K}$$

$$= \frac{(3\pi/2 - \phi_m)}{(1 + \pi/4)} \times \frac{T_1}{K\theta}$$

Implies, ω_g is equal to ω_g equal to $3\pi/2 - \phi_m$ divided by $\theta (1 + \pi/4)$.

Now, why I have brought the analytical expression to this form if you look at the gain condition from the gain condition we have found that, ω_g is equal to KK_p/T_1 so substitute ω_g is equal to KK_p/T_1 over there. So, left hand side will be KK_p/T_1 is equal to $3\pi/2 - \phi_m$ divided by $\theta (1 + \pi/4)$, implies the proportional gain of the PID controller K_p as K_p is equal to $3\pi/2 - \phi_m$ divided by $\theta (1 + \pi/4)$ into T_1/K so finally, which can be simplified and written as $3\pi/2 - \phi_m$ by $1 + \pi/4$ times $T_1/K\theta$ it does not matter. So, finally, what we have got? we have found expression for K_p this is how the proportional gain of the PID controller will be estimated using the formula derived over here. So, what are known things in right half of this expression T_1 is known θ is known K is known these are the parameters of the transfer function model for the second order unstable transfer function model.

So, for the unstable second order plus dead time transfer function model T_1 K θ are there, now assuming some user defined phase margin suppose I wish to have phase margin for the closed loop system as 45 degree 60 degree and so on assume that one so that way K_p can be estimated using this formula so we have found.

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Handwritten derivation of PID controller parameters:

- ① $K_p = \frac{\left(\frac{3\pi}{2} - \phi\right)}{\left(1 + \frac{\pi}{4}\right)} \cdot \frac{T_i}{K\theta}$
- ② $T_d = T_2$
- ③ $T_i = \frac{T_1^2}{K^2 K_p^2 \theta - T_1}$

The derivation starts with the equation $\frac{1}{T_i} + \frac{1}{T_1} = \omega_g^2 \theta = \frac{K^2 K_p^2}{T_1^2} \theta$. It then rearranges to $\frac{1}{T_i} = \frac{K^2 K_p^2 \theta}{T_1^2} - \frac{1}{T_1} = \frac{K^2 K_p^2 \theta - T_1}{T_1^2}$, and finally takes the reciprocal to get $T_i = \frac{T_1^2}{K^2 K_p^2 \theta - T_1}$.

So far, explicit expression for K_p and T_d so K_p is found to be $\frac{3\pi}{2} - \phi$ by $1 + \frac{\pi}{4}$ times T_1 by $K\theta$ and T_d is equal to T_2 one parameter of the transfer function model. So, we have already found explicit expression for two parameters of the PID controller, how to find expression for the remaining parameter of the PID controller? it is not difficult to go for the expression we have got earlier please keep in mind that $\frac{1}{T_i} + \frac{1}{T_1}$ is equal to $\omega_g^2 \theta$, this is what already we have got so I shall make use of that one, now $\frac{1}{T_i} + \frac{1}{T_1}$ is equal to $\omega_g^2 \theta$ and ω_g is how much? $K K_p$ you know ω_g is also $K K_p$ by T_1 . So, this is $K^2 K_p^2$ by $T_1^2 \theta$ implies $\frac{1}{T_i}$ is equal to $\frac{K^2 K_p^2 \theta}{T_1^2} - \frac{1}{T_1}$. So, in the numerator you will have $K^2 K_p^2 \theta - T_1$ divided by T_1^2 finally, giving T_i as T_1^2 square divided by $K^2 K_p^2 \theta - T_1$.

So, we have got T_i as T_i is equal to T_1^2 ? Yes. T_1^2 square divided by $K^2 K_p^2 \theta - T_1$ so this is the third formula so using 1, 2 and 3 it is possible to estimate all the parameters of the PID controller in terms of the unstable second order plus dead time transfer function model.

Now, I will go for some other technique a simple controller, how a simple controller can be designed for a first order plus dead time transfer function model using the phase margin criteria.

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Optimum gain controller for a FOPDT unstable process

$$G_m(s) = \frac{K e^{-\theta s}}{T_1 s - 1}$$

$$G_c(s) = K_p$$

Loop gain $G_c G_m(s) = \frac{K K_p e^{-\theta s}}{T_1 s - 1}$

At the gain crossover freq, ω_g

$$|G_c G_m(j\omega_g)| = 1$$


So, how to design optimum gain controller? for a first order plus dead time unstable process. So, let $G_m(s)$ is equal to $K e^{-\theta s} / (T_1 s - 1)$ and $G_c(s)$ is equal to simply a proportional controller we will find the optimum value for the K_p using the phase margin condition. Now, the loop gain is $G_c G_m(s)$ is equal to $K K_p e^{-\theta s} / (T_1 s - 1)$.


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$$\frac{K K_p}{\sqrt{1 + (\omega_g T_1)^2}} = 1$$

$$\Rightarrow 1 + (\omega_g T_1)^2 = (K K_p)^2 \quad \text{--- (1)}$$

$$\phi_m = \pi + \angle G_m G_c(j\omega_g)$$

$$= \pi - \theta \omega_g + \tan^{-1}(\omega_g T_1)$$

$$\frac{d\phi_m}{d\omega_g} = -\theta + \frac{T_1}{1 + (\omega_g T_1)^2} = 0$$


Now, we know that at the gain crossover frequency ω_g $|G_c G_m(j\omega_g)|$ magnitude equal to 1, thus giving us $K K_p$ by $\sqrt{1 + \omega_g^2 T_1^2}$ is equal to

$1 + \omega_g T_1^2$ is equal to $K K_p^2$, I will use this expression later on coming to the phase margin. So, the by definition phase margin equal to π plus angle of $G_m G_c(j\omega_g)$ which is equal to π you look at this expression the phase angle you will get from here minus θ minus θ plus $\tan^{-1}(\omega_g T_1)$, we will find the optimum you will use the optimum phase margin condition to find the proportional gain of the controller or of the proportional, we design the proportional gain associated with the controller using the optimum phase margin condition. So, I will differentiate ϕ_m with respect to the gain crossover frequency leaving me minus θ plus $\tan^{-1}(\omega_g T_1)$ in the denominator so this is to be equated to zero for finding the gain crossover frequency that will ensure optimum phase margin of the loop.

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$$\frac{T_1}{1 + (\omega_g T_1)^2} = \theta$$

but $1 + (\omega_g T_1)^2 = (K K_p)^2$

$$\Rightarrow \frac{T_1}{(K K_p)^2} = \theta$$

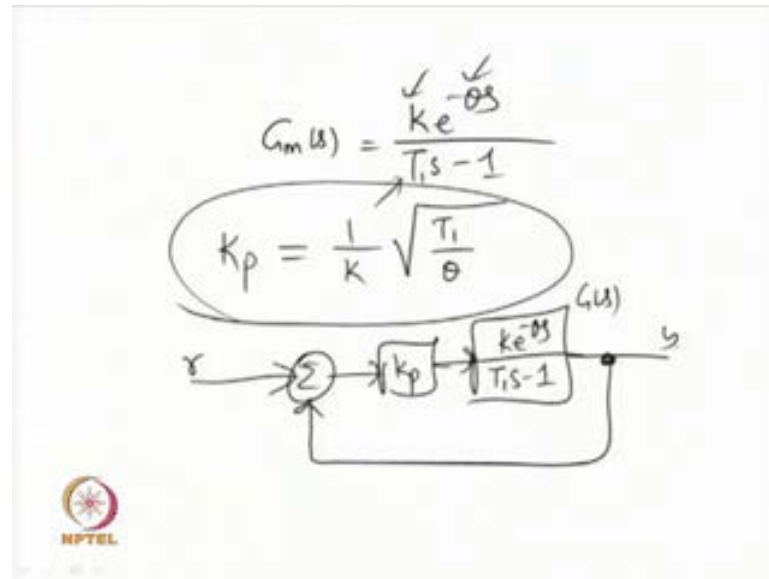
$$\Rightarrow K_p^2 = \frac{T_1}{K^2 \theta}$$

$$\Rightarrow K_p = \frac{1}{K} \sqrt{\frac{T_1}{\theta}}$$

K, θ, T_1 are the dynamic model parameters of the unstable plant

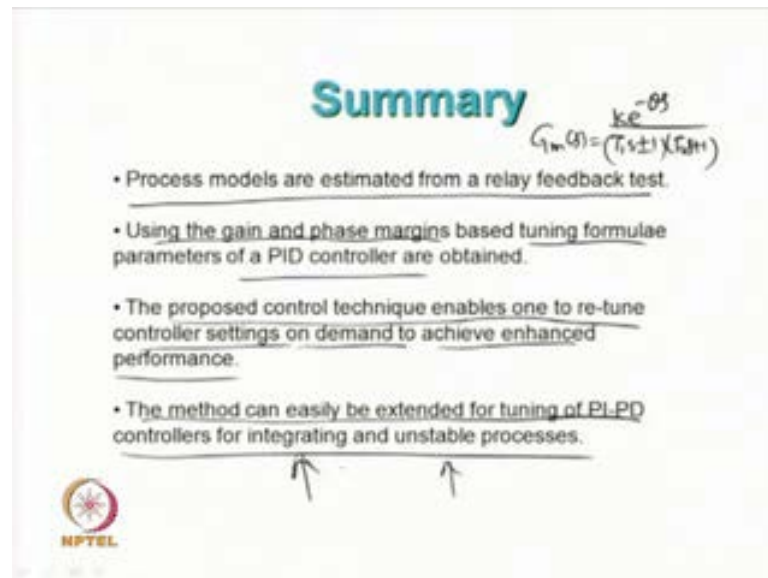
So, from this equation θ is now equal to T_1 by T_1 by $1 + \omega_g T_1^2$ is equal to θ , but earlier we have found $1 + \omega_g T_1^2$ is $K K_p^2$ so substitution of that will give you T_1 upon $1 + \omega_g T_1^2$ upon $K K_p^2$ is equal to θ implies K_p^2 is equal to T_1 by $K^2 \theta$, implies K_p is equal to 1 upon K root of T_1 by θ thus a proportional controller that ensures optimum phase margin of the loop is designed for an unstable first order plant or process. So, the final expression for K_p is equal to 1 upon K root of T_1 and θ divided by θ , now K θ and T_1 are the dynamic model parameter model parameters of the unstable plant.

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So, when you have got unstable plant, what you should do given an unstable plant obtain its dynamics using the transfer function model $G_m(s)$ is equal to $K e^{-\theta s}$ upon $T_1 s - 1$. So, these are the three parameters associated with the transfer function model and then design a proportional controller K_p gives as 1 upon K root of T_1 by θ , and this K_p will definitely give you stable performances for the unstable system so when you connect this now with this proportional controller K_p with an first order plus dead time unstable process whose dynamics is given by this is the real time system $G(s)$ then for any reference inputs are you will get satisfactory time response y . y , I cannot guarantee satisfactory response rather I can tell that we will get stable response because you are using the optimum phase margin criterion.

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The slide is titled "Summary" in a large, bold, blue font. To the right of the title is a handwritten transfer function: $G_m(s) = \frac{K e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)}$. Below the title, there are four bullet points, each preceded by a small blue square. The first bullet point is "Process models are estimated from a relay feedback test." The second is "Using the gain and phase margins based tuning formulae parameters of a PID controller are obtained." The third is "The proposed control technique enables one to re-tune controller settings on demand to achieve enhanced performance." The fourth is "The method can easily be extended for tuning of PI/PD controllers for integrating and unstable processes." At the bottom left of the slide is the NPTEL logo, which consists of a stylized sun or flower icon with the word "NPTEL" underneath it. Two blue arrows point upwards from the bottom of the slide towards the last two bullet points.

Summary

$$G_m(s) = \frac{K e^{-\theta s}}{(T_1 s + 1)(T_2 s + 1)}$$

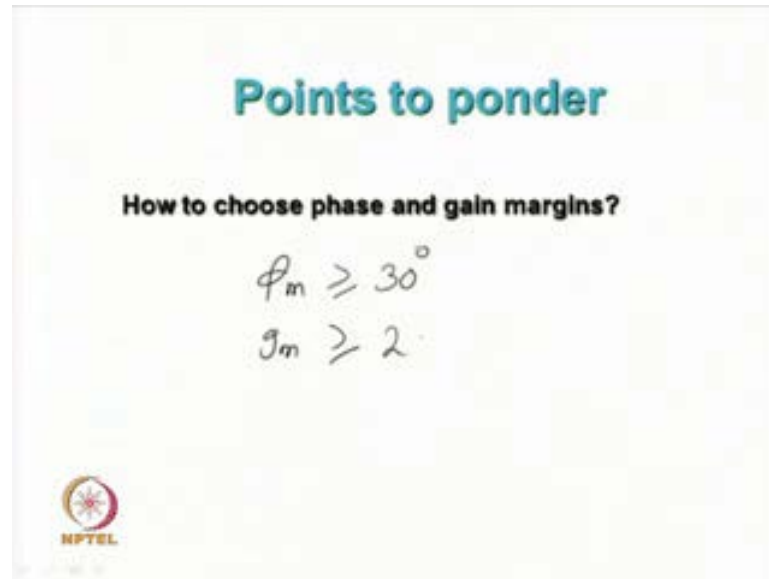
- Process models are estimated from a relay feedback test.
- Using the gain and phase margins based tuning formulae parameters of a PID controller are obtained.
- The proposed control technique enables one to re-tune controller settings on demand to achieve enhanced performance.
- The method can easily be extended for tuning of PI/PD controllers for integrating and unstable processes.

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Now, we will go to the summary so what we have discussed in this lecture process models are estimated from a relay feedback test. So, initially you identify the dynamics of a real time system by some transfer function model given in some form such as $G_m(s)$ is equal to $K e^{-\theta s} / (T_1 s + 1)(T_2 s + 1)$. Now using the gain and phase margins tuning formulae for the PID controllers are obtained.

Now, use the tuning formulae derived based on the gain and phase margins for closed loop operation of the system. The proposed control technique enables one to retune controller settings on demand to achieve enhanced performance, why enhanced performance? Because you are using phase and gain margin based PID controller, and you will get robust performances from the controlled systems. Now the method can easily be extended for tuning of other type of controllers, such as PI/PD controller and so on, even for integrating and unstable processes. Initially we have discussed the way PID controller parameters can be designed for a second order plus dead time stable transfer function model, then we extended it to second order plus dead time unstable process model; it can also be extended to second order or first order integrating processes with dead time.

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Now one point to ponder, how to choose phase and gain margins, basically there are lots of guidelines given in the literature; the way one can find or choose phase and gain margins. So, from intuition and from the experience, the operators usually set the phase and gain margins, but generally the phase margin chosen should be greater than equal to 30 degree, and the gain margin chosen should be greater than equal to 2 that is all.

Thank you.