Advanced Control Systems

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Module No. # 03

Time Domain Based Identification

Lecture No. # 04

State Space Based Identification of Systems

Welcome to the lecture titled State Space Based Identification of Systems. We shall continue with our discussion on system identification; that means identification of simple systems. The system could be first order plus dead time form or could be of any higher order. But, initially, we shall take off very simple system. Now, the analysis will be done for the first order plus dead time system initially, where we shall derive a set of analytical expressions, which can be used for identifying simple transfer function models of systems.

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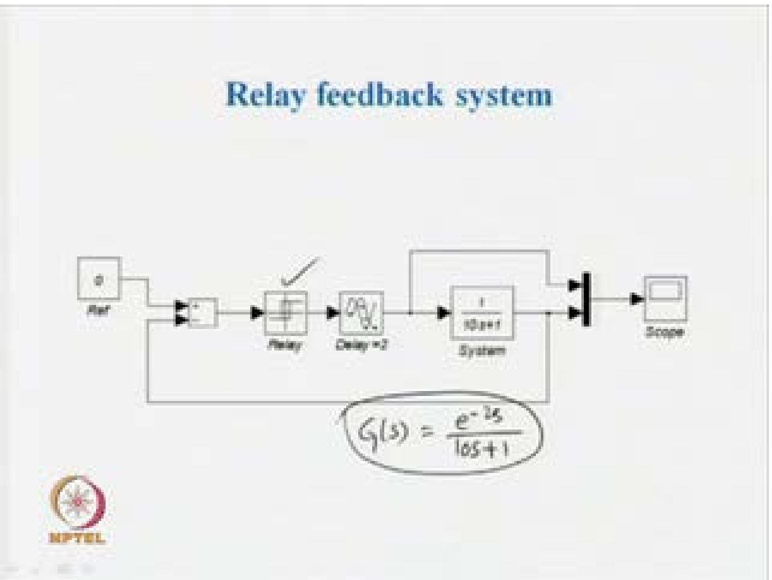

Now, let us consider the relay feedback system, where the relay induces sustained oscillatory output in the system. So, the system considered is G s is equal to e to the power minus 2 s upon 10 s plus 1. This system we have also earlier considered and we have generated oscillatory output for this system.

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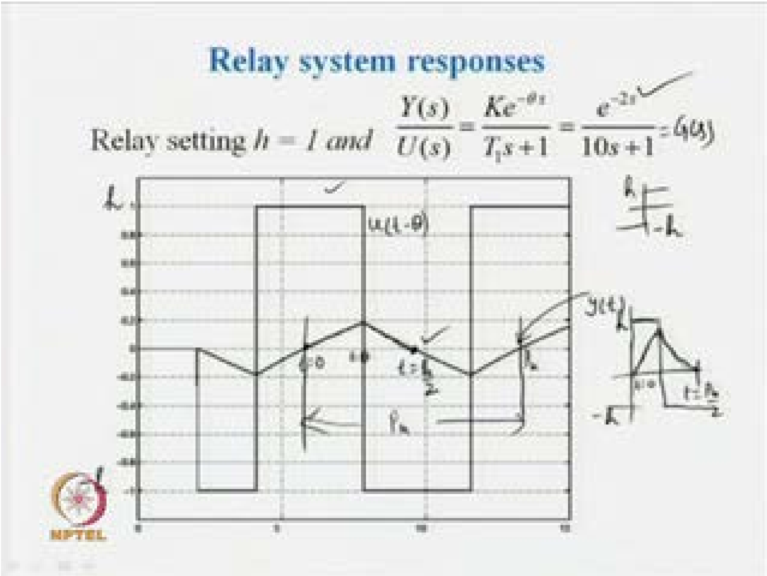

The output can be of this form, where the rectangular pluses are the input to the system; rather here we actually get the delayed input to the system given by the expression u t minus theta. Whereas, the triangular output we get here – exactly this is not triangular. But, the output we get here, which is a bit exponential, but soon here as in some triangular form, actually is the output y t, which is nothing but the sustained oscillatory output we get from the system. Now, the system dynamics is given here – G s is equal to e to the power minus 2 s upon 10 s plus 1.

Now, what do we see from here? We shall concentrate on one period rather half period of output signal, which starts from time t is equal to 0 to sometime t is equal to p u by 2, where p u is the period of the output – oscillatory output. So, the period can go from here to here. So, p u spans from time t is equal to 0 to time t equal to p u. But, the analysis will be limited to half period output of the system. Therefore, we shall concentrate on this segment of the output and this segment of the output. Now, why I am mentioning two segments? Because the input during one positive half output of the system is different. We have got positive input for time t equal to 0 to time t equal to theta; whereas, the input to the system from time t equal to theta to time t equal to p u by 2 is minus 1.

If the relay heights are set at plus h and minus h, in that case, the type of output you will be get will be plus h here and minus h here (Refer Slide Time: 03:48). So, what we basically obtain? What information we get from these outputs are the waveforms that the output y t assumes a shape of this form, which starts from time t equal to 0 and goes up to time t is equal p u by2. But, during this positive half output of the system, the input has got the form like this. So, I have got a positive h for certain duration and negative h for certain duration of positive output of the system. So, this point must be kept in mind. This will be used in analyzing the system.

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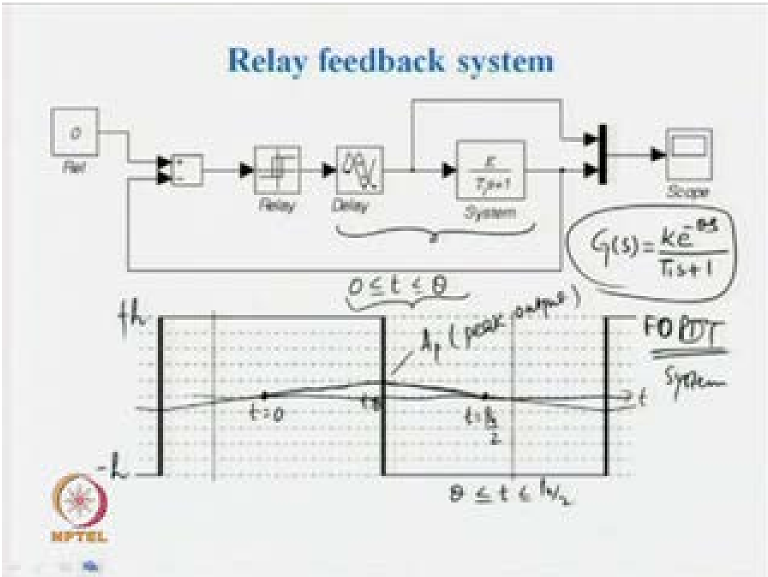

Now, when the output waveform is zoomed here, we see that the output has a zero crossing over here. So, let us starts with time t is equal to 0. And again, we have a zero crossing at time t is equal to p u by 2, where p u is the time period of the output signal. Now, at time t equal to theta, interestingly, we get the peak output, which is designated by A p. A p stands for peak output of the system. So, whenever a first order plus delay system is subjected to relay test, then the typical type of output we obtain from the test is of this form; and, this is the x-axis. Whereas, during the positive output of the signal, we find the input to the system to be having two parts, two piecewise constant inputs. One is plus h for certain duration from time t goes from 0 to theta. So, that I write in the form of t in between 0 to theta. Whereas, also, the input to the system is going from theta to p u by 2 during when the input is minus h. So, it is evident that we have to consider two different inputs, two piecewise constant inputs to the system during positive half output of a system. The output is sustained oscillatory output. Therefore, we need not consider whole period of the output signal, rather concentration will be there for one half – positive half period of the output signal that carries enough information as for as dynamics of the system under relay test is concerned.

Now, I shall start the analysis, which the assumption that the system now, is represented in the general form having a starting gain of (Refer Slide Time: 07:09) k, a delay of theta seconds and with time constants T 1. So, this is the general form of a first order plus dead time system. So, in short form, in acronyms, we put it as FOPDT system. Now, how can we generate this type of output from analysis? That we shall discuss now.

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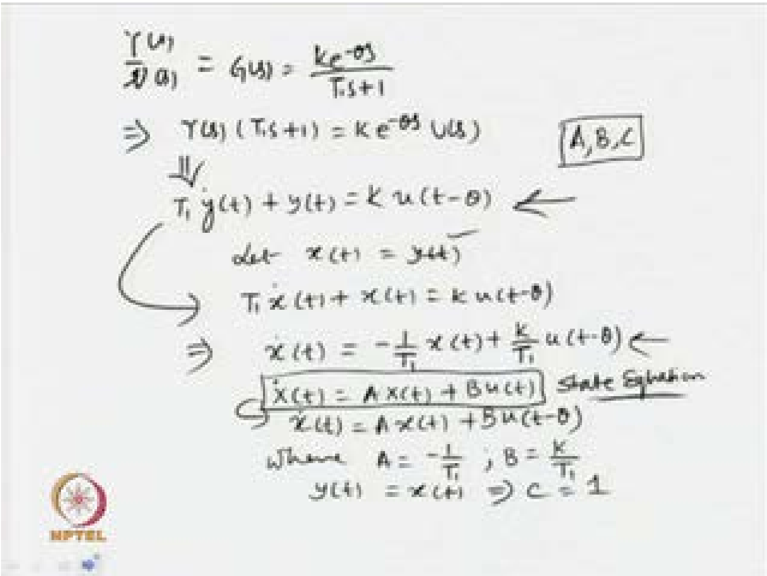

Now, the system dynamics is given by y upon U s is equal to G s, is nothing but k e to power minus theta s upon T 1 s plus 1, which can ultimately we written in the form of Y s time T 1 s plus 1 is equal to k e to the power minus theta s U s. So, taking inverse laplace transform, it is possible to write the same expression in the form of T 1 y dot t plus y t is equal to k u t minus theta. So, the type of input to the system is a delayed input.

Now, we shall introduce some state variable. Let the state variable x t is equal to y t. Then, the dynamic equation can be written as T 1 x dot t plus x t is equal to k u t minus theta. Now, I will put it in some specified form; x dot t is equal to minus 1 upon T 1 x t plus k upon T 1 u t minus theta. Now, the dynamic equation can be written in the standard form, our state equation form, which is given by x t is equal to A x t plus B u t. And, since we are dealing with the only one state variable, the same state equation can be now written as x dot t is equal to A x t plus B u t. Now, that becomes x dot t is equal to A x t plus B u t minus theta in our case. So, this is the standard form for the state equations of systems – state equation. Whereas, in our case, the state equation we get for our first order plus dead time system is that x dot t is equal to A x t plus B u t, where A is equal to minus 1 upon T 1 and B is equal to k upon T 1. Further, since the output equation y t is equal to x t, therefore, C is equal to 1. So, A B C are the constant matrices or vectors or scalars for the state and output equation of a system. So, we have got the A, B, C for this.

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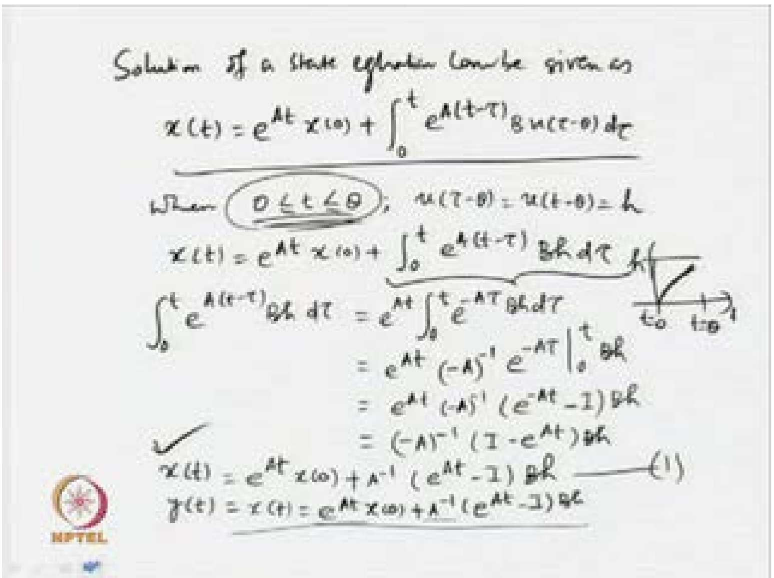

Why we are finding in this form? Because we know the solution of a state equation, which is given in the form of… Solution of a state equation can be given as x t is equal to e to the power A t x 0 plus integration from 0 to t e to the power A t minus tau B u tau minus theta d theta. So, this is how we have got the solution for state equation when we have got a variable theta. But, we have to avoid this variable, because we are reserving theta for the delay associated with the system. Therefore, I will write down the output equation in some other convenient form; so, t e A t minus tau B u tau minus theta d tau. So, this is the solution of a state equation. How can we use the state solution for the state equation for our case? When we considered the time range t between 0 to theta, during that, the input u tau minus theta is equal to u t minus theta, is h. Then, the state equation can be written as x t is equal to e to the power A t x 0 plus integration from 0 to t e to the power A t minus tau B h d tau.

Now, let us solve the integral part. This can be simplified (Refer Slide Time: 12:46). We know that 0 to t e to the power A t minus tau B h d tau can be simplified further and written in the form of e to the power A t integration from 0 to t e to the power minus A tau B h d tau, which again can be written as e to the power A t – now, directly I will integrate this one – minus A inverse e to the power minus A tau with the limits 0 to t with B h at the end. Then, it can be simplified as e to the power A t minus A inverse. Then, we will have two terms now. With the limit t, we get e to the power minus A t. And, with the limit 0, we get this as I B h, which ultimately gives us upon simplification, in the form of minus a inverse I minus e to the power A t B h. So finally, I can write the solution to the state equation for the time range t between 0 to theta as e to the power A t x 0; then, plus A inverse e to the power A t minus I B h. So, this is the solution of the state equation for the time range t between 0 to theta.

Now, the output of the system for the same time range can be obtained as (Refer Slide Time: 15:06) y t is equal to x t is equal to e to the power A t x 0 plus A inverse e to the power A t minus I B h. When this is plotted, then definitely we are expected to get the part of the output waveform. Which part of the output waveform? That is, spanning from time t equal to 0 to time t equal to theta, because you cannot go beyond theta. This output expression is derived up to time t equal to theta. Therefore, the output can be obtained in this form. When this is plotted with the substitution of A, B values, then definitely we will expect the output of this system in this form, when the input to the system remains plus h. Similarly, let us try to find the analytical expression that can be used to obtain output for the other part of the positive output of the sustained oscillatory output of a system.

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For the time range, theta is less than equal to t, is less than equal to p u by 2. We know that the input u tau minus theta, which is nothing but u t minus theta, will be minus h. In that case, the expression for the solution to the state equation becomes x t is equal to e to the power A t minus theta x theta; then, plus integration from theta to t e to the power A t minus tau B u tau minus theta d tau. Now, which can ultimately be written in the form of B to the power A t minus theta x theta plus integration from theta to t e to the power A t minus tau B minus h d tau.

Now, like the earlier case, let us simplify the second term. So, simplification of second term can be done in this fashion. Integration from theta to t e to the power A t minus tau minus B h d tau is equal to e to the power A t. Again, you takeout this term – e to the power A t, because the integral variable is tau. So, it gives us theta to t e to the power minus A tau minus B h with d tau. Let us takeout this (Refer Slide Time: 18:39) minus B h term to the end, so that we can conveniently find the integral – e to the power minus A tau d tau minus B h at the end. Then, it can be written in the form of e to the power A t. Again, minus A inverse e to the power minus A tau with the limits theta to t and minus B h at the end. So, this gives us e to the power A t minus A inverse e to the power minus A t minus e to the power minus A theta minus B h. So, I will collect the term and simplify this one. So, I will keep this initially minus A inverse; then, e to the power A t and times e to the power minus A t will give you the identity matrix – I minus e to the power A t minus theta; then, minus B h.

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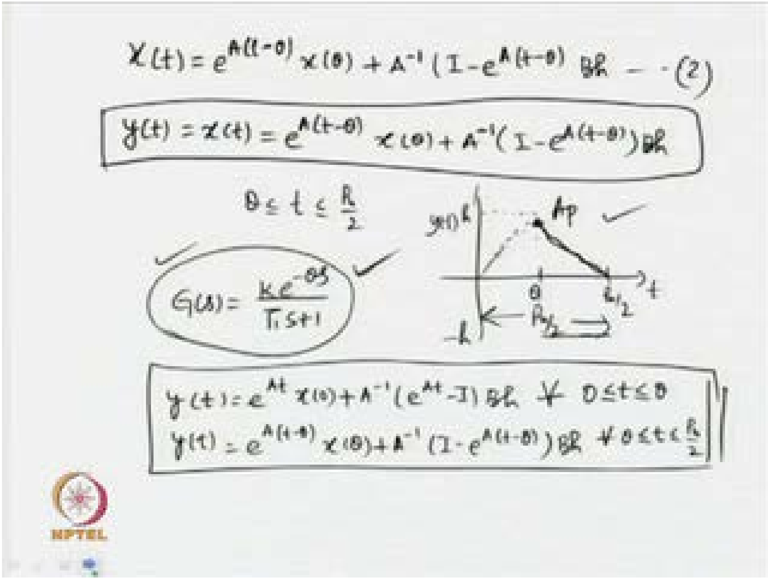

Finally, we get the output expressed in the form of x t is equal to e to the power A t minus theta x theta plus A inverse I minus e to the power A t minus theta B h. This is the second equation. Then, corresponding output expression y t, which is nothing but, is equal to x t – becomes e to the power A t minus theta x theta plus A inverse I minus e to the power A t minus theta B h. So, when again this expression is plotted for the time range – time between theta to p u by 2, what type of output we expect? The plot will be from time t is equal to theta. This is the time axis. Let this be p u by 2. Then, in that case, the plot will start from the earlier peak value and it will go down in this form, some exponential form. So, the earlier one, we have obtained is like this. So, finally, thus, it is possible to obtain the positive half output of the sustained oscillatory output of a system under relay control in this form. This the plot for y t.

And again, I would like to mention that the dotted part of the output is obtained with the help of positive input h; whereas, the solid part of the output is obtained with the input to the system remaining, minus h. So, input to the system is minus h. So, if it is possible to obtain explicit expressions for different parts of the output signal, then also it is possible to correlate the parameters of the first order plus dead time model with that of the output signal. So, the measurements made on the output signal can be translated into the model transfer function form, because there is direct correspondence between the two. What I mean by that? Since the system is now G s is k e to the power minus theta s upon T 1 plus 1, when the system is subjected to a relay test, then the system produces a typical output of this form (Refer Slide Time: 23:11).

And, we have been able to derive analytical expression for the output of the system. So, the measurements made on the output means what measurements we can make? We can measure the peak amplitude of this output. Also, we can measure the half period of the output using zero-crossing detectors. So, when the measurements are made and when the expressions are found to be involving those measured quantities, then it is not difficult to solve the analytical expressions and find the unknowns associated with the transfer function model.

Finally, let me summarize both the equations I have obtained for obtaining the typical oscillatory output. So, y t is equal to e to the power A t x 0 plus A inverse e to the power A t minus I B h, for the time range 0 is less than equal to t, is less than equal to theta. And, y t is equal to e to the power A t minus theta x theta plus A inverse I minus e to the power A t minus theta B h, for the time range theta to p u by 2. So, these are the two powerful equations (Refer Slide Time: 25:08) that can be used for finding the transport function model parameters. Now, we shall analyze these two equations further and try to find simpler expressions that can be exploited to find or estimate the parameters of a transform function model.

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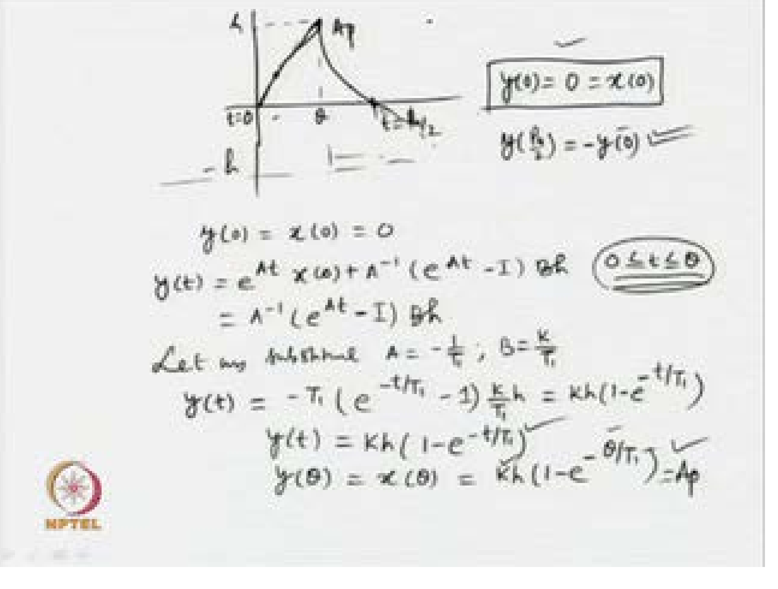

We know that the form of the output signal is like this, where we have got the inputs: plus h and input minus h. And, we are starting from time t is equal to 0 till time t equal to p u by 2. So, looking at this positive half output signal of the system under relay test, we can set some conditions. Conditions like y 0 is equal to 0, but y 0 is nothing but x 0, since the state variable x has been introduced and we have assumed that x t is equal y t. So, for this first order plus dead time system, definitely this is true that x 0 is equal to y 0. And, that is also equal to 0. Further for the limit cycle condition, we know that the positive half and the negative half you get have got symmetricity; they are symmetrical. That means we have got half-wave symmetricity and therefore, y p u by 2 will be equal to minus y 0. What I mean to say by that is that whatever output you get at any instant of time suppose t equal to beta seconds, definitely you will get minus of that output at time t equal to p u by 2 plus beta. So, what I mean by that is that output y at time beta is equal to minus of output at time p u by 2 plus beta. Thus, we get some typical wave form, which has definitely got half-way symmetricity.

I shall make use of these conditions of limit cycle to further explore the way we can use the earlier expressions and get simpler analytical expressions. So, when y 0 is equal to x 0 is equal to 0, that time the expression y t will be equal to e to the power A t x 0 plus A inverse e to the power A t minus I B h, for the time range theta, can be written in the form of A inverse e to the power A t minus I B h. Let us substitute the A, B, C we have got. So, substituting A is equal to minus 1 upon T 1, B is equal to k upon T 1, we get the output expressed in the form of minus T 1 e to the power minus t upon T 1 minus 1; B is k upon T 1 – h; which becomes k h 1 minus e to the power minus t upon T 1. So, this can be plotted now. Finally, what we have obtained? An expression for the output for the time range t between 0 to theta as y t is equal to k h 1 minus e to the power minus t upon T 1.

When time t equal to 0, how much we get? Whether we are satisfying that condition or not let us see. So, when t equal to 0, y 0 becomes k h 1 minus – T 0 is 1. So, that way y 0 becomes 0 (Refer Slide Time: 30:32). So, this expression is correct as far as the positive output of the system under relay control is concerned. Then, y theta is nothing but x theta for us, can be given as k h 1minus e to the power minus theta upon T 1. So, this is one important expression for us. Why that is so? Because we know that we have a plot for the time range starting from time t equal to 0 to time t equal to theta, during which the form of the output is guided by this analytical expression y t is equal to k h 1 minus e to the power minus t by T 1.

Now, when time t is equal to theta, at that time, the output we get is (Refer Slide Time: 31:42) k h 1 minus e to the power minus theta by T 1. So, that will be the maximum output we get, which we denote by A p. So, this is equal to A p. And, we know that we can easily measure the peak output signal. And therefore, that information can be made used to estimate the unknowns associated with the transfer function model. What are the unknowns? The unknowns are k, theta and T 1.

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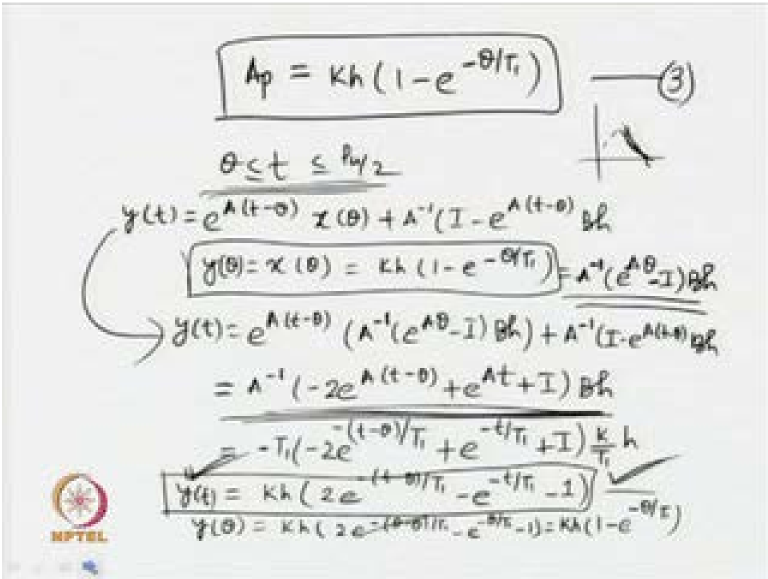

We have developed one analytical expression, which is given as A p is equal to k h 1 minus e to the power minus theta upon T 1. So, this is an important expression for us, which will be used further for estimating transfer function model parameters. Let us carry on with the analysis for the other part of the output for the time range t between theta to p u by 2. We know that the output for that is given as y t is equal to e to the power A t minus theta x theta plus A inverse I minus e to the power A t minus theta B h. But, we know that x theta is equal to k h 1 minus e to the power minus theta upon T 1. Or, in a general form, x theta is given as A inverse e to the power A theta minus I B h. Substitution of this in the expression y t, we get y t is equal to e A t minus theta. I will substitute over here the expression for x theta. x theta has got two parts. Now, it is having A inverse e to the power A theta minus I B h, neglecting the part containing x 0.

And then, we have got the second part given as A inverse I minus e to the power A t minus theta B h.

Let us collect the term and simplify this expression y t. So, y t ultimately can be given in the form of A inverse minus 2 e to the power A t minus theta plus e to the power A t plus I times B h. So, simplification of the expression y t gives us this second half of the equality. Then, y t further upon substitution of e and B can be written as y t is equal to minus T 1 minus 2 e to the power t minus theta by minus T 1 will be there – so, minus t minus theta by T 1 – plus e to the power minus t upon T 1 plus I; then, k by T 1 h. So, we will further… With the collection of terms, can be written in the form of y t is equal to k h 2 e to the power minus t minus theta by T 1 minus e to the power minus t by T 1 minus 1. So, this is the expression we have obtained for the time range t remaining between theta to p u by 2. Again, let me reiterate; when this expression is plotted for this time range, we get the second part of the output. First part we have already obtained from the plot of the earlier expression; the second part can be obtained.

Now, how can we make use of this one? Let us check validity of this expression first; whether at time t is equal to theta, we get the same expression or not, what we are obtained earlier. At time t equal to theta, y t is how much? At time t equal to theta, y theta is equal to x theta is equal to (Refer Slide Time: 37:34) k h 1 minus e to the power minus theta by T 1. Whether we are getting the same expression or not, when time t equal to theta, y t becomes y theta k h 2 e to the power minus theta minus theta upon T 1 minus e to the power minus theta by T 1 minus 1, which is equal to k h. So, this term will give you 1; e to the power 0 is 1. So, 2 minus 1 is 1 – 1 minus e to the power minus theta upon T 1. Therefore, this expression is correct. We have checked the correctness of this expression. Then finally, what we have obtained? We have obtained two analytical expressions.

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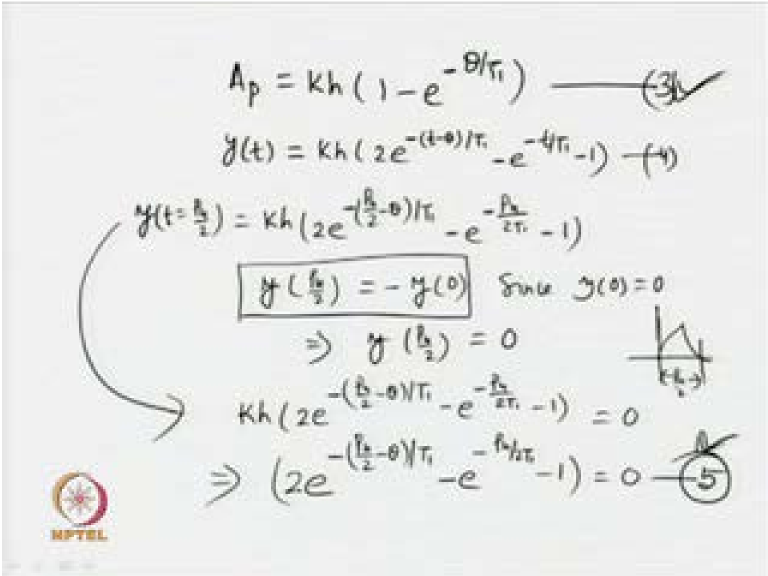

What are those two analytical expressions? A p is equal to k h 1 minus e to the power minus theta by T 1. So, this is third. And furthermore, we have obtained an expression of the form y t is equal to k h 2 e to the power minus t minus theta upon T 1; then, minus e to the power minus t upon T 1 minus 1. Let this be equation is 4.

Now, what will be the output at time t equal to p u by 2? The output of the system at time t equal to p u by 2 will be k h times 2 times e to the power minus p u by 2 minus theta by T 1 minus e to the power minus p u by 2 T 1 minus 1. But, for us, for maintaining sustained oscillatory output, the limit cycle condition gives us that output y at time t equal to p u by 2 is equal to minus y 0. This condition must be made to induce limit cycle oscillations. The output can be sustained oscillatory output provided this condition is satisfied.

Then, since (Refer Slide Time: 40:21) y 0 is equal to 0, then y p u by 2 has to be also 0. So, setting y p u by 2 to 0, this gives us an equality of the form k h times 2 e the power minus p u by 2 minus theta by T 1 minus e to the power minus p u by 2 T 1 minus 1 is equal 0, which can be obtained in the form of 2 times e to the power minus p u by 2 minus theta by T 1 minus e to the power minus p u by 2 T 1 minus 1 is equal to 0. So, this expression is very important for us. Why this is so important? If sustained oscillatory output is obtained and we make measurements on the sustained oscillatory output, measurements like the peak altitude and the half period p u by 2, then this condition must be satisfied. Now, we have developed two expressions. Expression number 3 and 5, which can be used together or can be solved simultaneously to estimate unknown parameters associated with the transfer function model.

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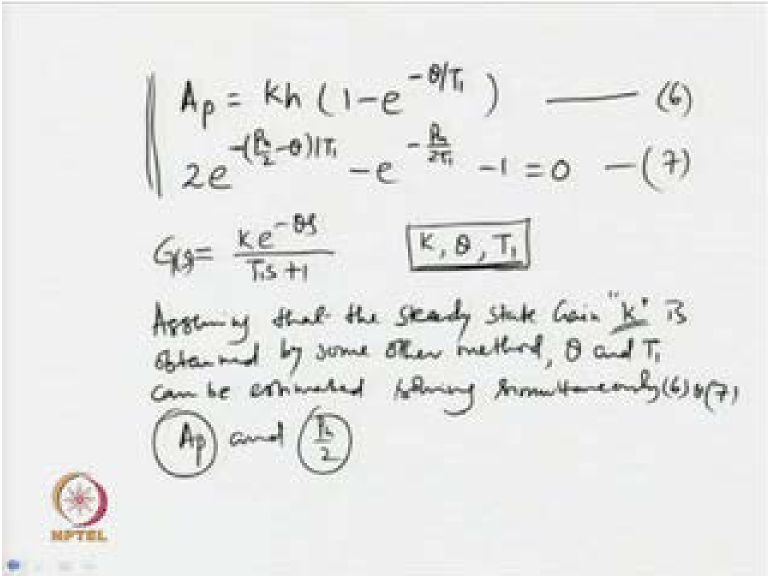

Let me write down the two equations, final expressions we have obtained from the analysis. The analysis gave us A p is equal to k h times 1 minus e to the power minus theta by T 1. And, further the limit cycle condition requires that 2 e to the power minus p u by 2 minus theta by T 1 minus e to the power minus p u by 2 T 1 minus 1 has to be 0. So, these are the… I can put now equation numbers 6 and 7.

Now, let us go back to the transfer function model, which parameters are to be estimated. This G s is equal to k e to the power minus theta s upon T 1 s plus 1. We have got three unknowns associated with this first order plus dead time model. Those unknowns are k, theta and T 1. So, once you have got information about k, theta and T 1, that means you have been able to identify the dynamics of the system accurately. Now, how to estimate k, theta and T 1? Assuming that the steady state gain k is obtained by some other method, we are left with two more unknown parameters: theta and T 1. Then, theta and T 1 can be estimated solving simultaneously 6 and 7. This is the technique, the way the parameters of a first order plus dead time model are estimated. So, we assume that k is either obtained by some other method or k is known a priori. 6 and 7 can be solved simultaneously using the measurements of A p and p u by 2. So, we measure A p and p u by 2. So, use those two in the expressions and solve for theta and t 1. This is how the transfer function model parameters are estimated.

Now, further assuming that k is known a priori, it is possible to develop explicit expressions for the unknowns also. Let us try to make use of the expression 6. The expression 6 can be made use of… Now, I will use expression 6 now to develop explicit expressions that can be used further to estimate transfer function model parameters. So, 6 can be used in the form of…

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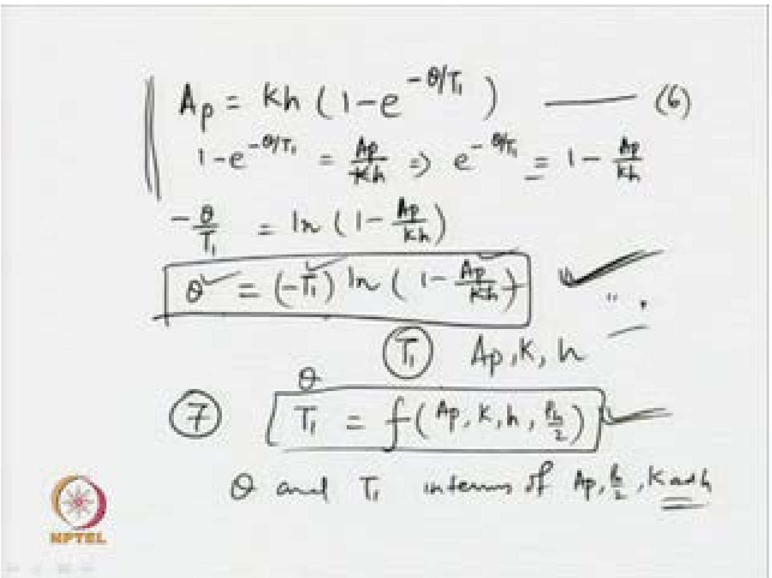

Now, write this equation 6 in the form of 1 minus e to the power minus theta upon T 1 is equal to A p by k h. Or, e to the power minus theta upon T 1 is equal to 1 minus A p by k h. Take natural logarithm of both sides; then, minus theta upon T 1 is equal to ln 1 minus A p by k h. Or, theta upon T 1 is equal to… or directly I can write now, theta is equal to – it will be minus T 1 time ln of 1 minus A p by k h. So, this simple expression can be used for estimating theta. How can you estimate? If I know T 1, then it is possible to make use of this expression using A p, k and h to estimate theta. So, this seventh expression can similarly be expressed in the simplified form, which will give you in form of T 1 as a function of A p, k, h and p u by 2 to use 7 and estimate T 1. After that, use T 1, A p, k and h to estimate theta. That is how we can derive explicit expressions for theta and T 1 in terms of the measurements A p, p u by 2 and the unknowns k and h.

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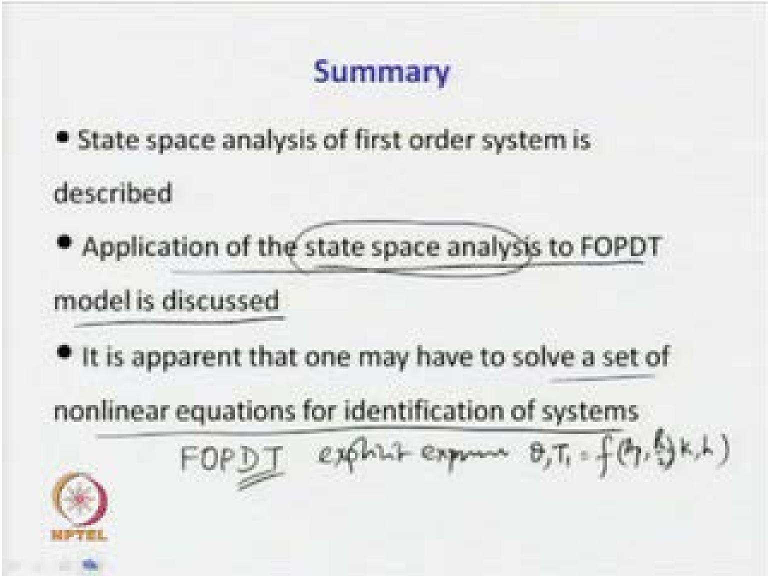

What we have learnt from this lecture? That state space analysis of first order system is described, where we have been able to derive simple expressions that can be solved simultaneously to estimate unknown parameters of a first order plus dead time model. Also, we have seen that application of the state space analysis can be extended to identify simplified models. How that can be done? That we shall discuss in the subsequent lecture.

Now, application of the state place analysis to first order plus dead time model is discussed. The state space analysis, solution of the state space equation and output equation can easily be applied to analysis of first order plus dead time model. Now, it is also apparent that one may have to solve a set of non-linear equations for identification of systems. This is not necessary for the first order plus dead time system. Fortunately, we can derive explicit expressions for the unknowns of the first order plus dead time model theta and t 1 in terms of the measurements A p, p u by 2 and the unknowns k and h.

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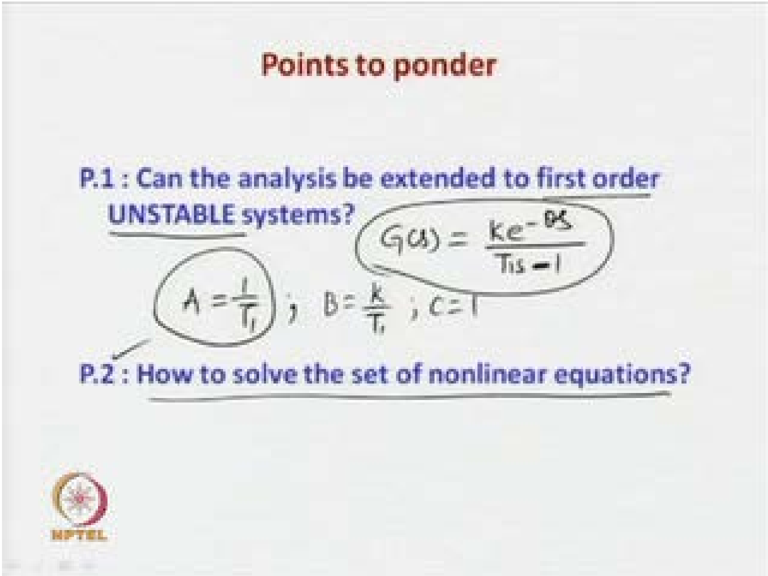

Certain points to ponder regarding the material we have discussed in the lecture are like this. Can the analysis be extended to first order unstable systems? Often we can use the transfer function model G s is equal to k e to the power minus theta s upon T 1 s minus 1 to the present of the dynamics of a first order unstable system. So, easily we can extend the analysis for first order unstable system, in which case the A matrix or scalar will be simply 1 upon T 1. If you work out, find the state equation and output equation for a first order system, definitely, you see that A has to be 1 upon T 1; whereas, B becomes k upon T 1 and C is equal to 1. So, only there will be few sign changes; otherwise, the same analytical expressions those has been derived for finding transfer function model parameters of stable first order plus dead time systems can be extended to first order unstable system.

Second question might be how to solve the set of nonlinear equations? This has been discussed earlier also. One has to take care that the solution does not converges to false solutions. For that, care must be taken or the initial values we give during solving the set of nonlinear equations matter most. Therefore, the initial values should be chosen in such a way that we do not lead to false solutions. That is all in this lecture.

Thank you.