Advanced Control Systems

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Module No. # 03

Time Domain Based Identification

Lecture No. # 02

State Space Analysis of Systems

Welcome to the lecture titled State Space Analysis for Identification of Systems. In this lecture, we shall study in detail, the state space analysis technique first; after that, how can we apply the state space analysis technique for identification of systems will be discussed. So, state space analysis means what is state space. To understand that, let us consider a few state space equations.

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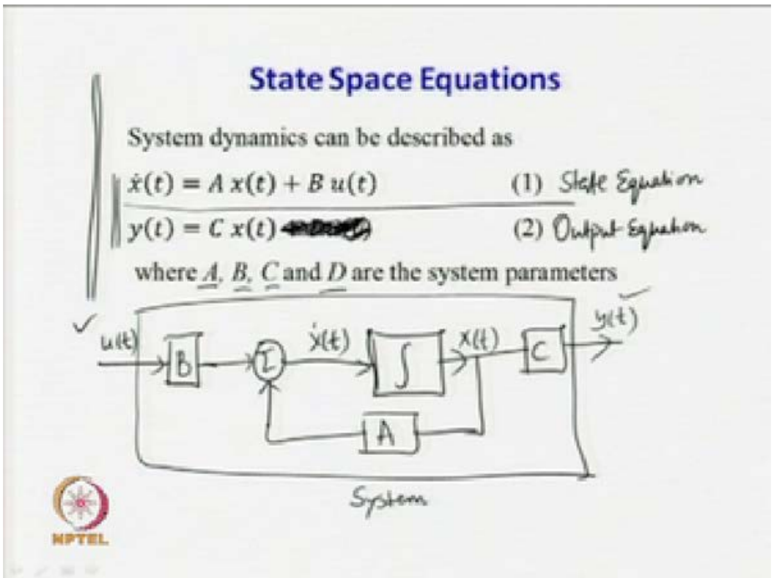

You are very much familiar to equations of this form where the first equation given by x dot t equal to a x t plus b u t is known as the state equation whereas the second one y t equal to c x t plus d u t is known as the output equation.

What we mean by the output and state of a system? Now, I have got a system whose output is y t, input is u t where from those a, b, c, d are coming. For simplicity in analysis, let us not consider d u t in the output equation and let us take output equation of the form y t equal to c x t. Now, a, b, c, d are the system parameters. Now, I have got a system here. The dynamics of this system can be given in terms of the state equation and output equation. To understand that, let us say, suppose I have got the output x t; when I have got integrator, its input to the integrator will be x dot t. Now, x dot t is equal to b u t, b u t plus a x t. So, I have got a x t. Further, y t is equal to c x t.

So, whatever we have within the bounding box is belonging to a system and the system is now subjected to the input u t and output y t and the corresponding output y t. When the system is subjected to some input u t, it results in system responses given by y t. So, the dynamic equation, the state and output equation can be obtained from this block diagram - representation of system. Now, a, b, c are the system parameters because those are coming inside the state of the system. So, that is why we call a b c are the system parameters. D we will not consider because we have set removed the d from the output equation.

Now, how practically we get, what information basically we get from this type of arrangement, from this type of block diagram? Suppose I have got a first order system without any delay, then what I get from there?

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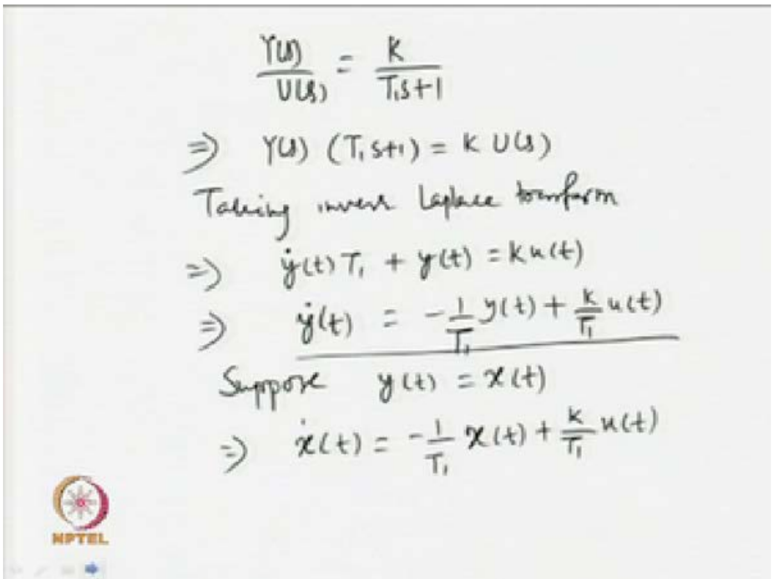

Suppose I have got some y s upon u s is equal to k upon T 1 s plus 1, I have not included the delay term in this for each end analysis. Now, when you write this again in the form of y s times T 1 s plus 1 is equal to k u s. Taking inverse Laplace transform, we get y t y dot t T 1 plus y t is equal to k u t. Now, this further can be written in the form of y dot t is equal to minus 1 upon T 1 y t plus k upon T 1 u t.

Suppose y t equal to some state variable x t, in that case the dynamic equation can be written in the form of x dot t is equal to minus 1 upon T 1 x t plus k upon T 1 u t. Then this is now given in the block diagram form.

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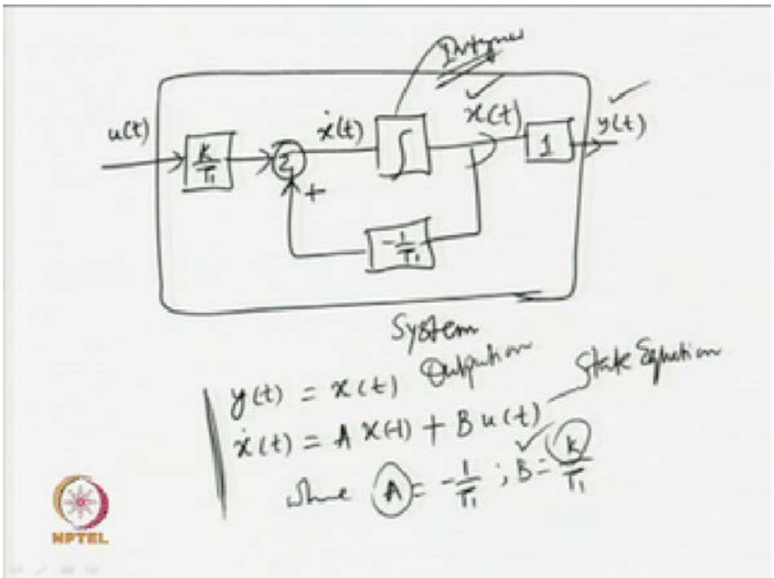

What do we get when I arranged the same in the block diagram form? I have got x t minus 1 sorry integrator x dot t, then here you have got minus 1 upon T 1 added over here and here k upon T 1 you have gone you have got unity gain here output y t and input u t (Refer Slide Time: 07:27). So, the first order system is now represented in this form. So, whatever dynamics this system model gives, that can be explained with the help of the state variables x t and the output variable y t. Now, if I look at the block diagram y t equal to x t, what about x dot t? x dot t is equal to now a x t plus b u t where a is equal to minus 1 upon T 1 and b is equal to k upon T 1.

So, these are the system parameters. What is this T 1? T 1 is the time constant of the system. What is b? b involves the steady state gain, ratio between the steady state gain and the time constant of the system. So, b again gives some another parameter of the system. So, these first order system dynamics can be given in this block diagram form. What is the beauty of this sort of representation in time domain? That one can include the initial condition in the systems. Where the initial conditions will come? This integrator is integrator can be with initial conditions. So, whenever you have got capacitors and integrators in any system, they can always contribute to initial conditions of the system because they store energy, they store information and that prior information or past information can be made use of to find the system output at any instant of time. So, with this type of representation which is known as the time domain representation of the systems in the form of state equation and the output equation, it is possible to include the effects of initial conditions in a system.

Why we are worried about the initial conditions of a system? Because the system can be started from any point or particularly in our case, the relay test can be affected at any instant of time as and when identification is necessary. When the system dynamics changes over time, it is necessary to identify the model parameters of a system or whenever retuning of controllers are necessary, at that time identification of systems are also necessary. So, the identification can be effected at any instant of time. Because of that reason, because of the initial conditions of the system, when the relay test is performed, are to be taken care of.

If one does not take into account the effect of initial conditions in a system, then the type of output wave form one will obtain analytically may not be the one, what one should have obtained when initial conditions are considered. So, to get accurate output wave form from analysis, it is necessary to include the effects of initial conditions of a system at the time of performing relay test. So, coming to the system representation, the system has been represented in this typical form, such that we can take it into account the effects of initial conditions. Now, any order systems can be represented in block diagram form. In that case, the number of integrators will be more.

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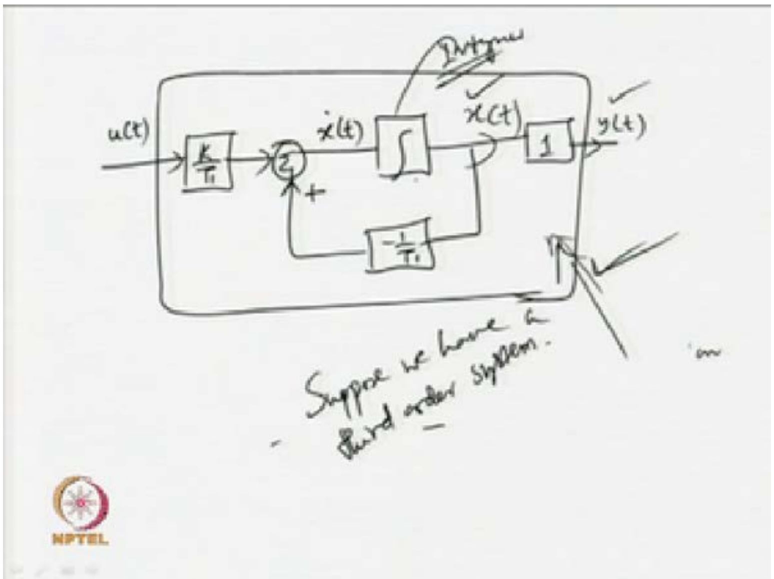

Suppose we have got, we have a third order system, then how many integrators will be there? One has to make use of three integrators to draw the simulation diagram of this form. Then the three integrators will introduce three initial conditions. Those initial conditions in a system are very important when relay based analysis of closed loop systems are done.

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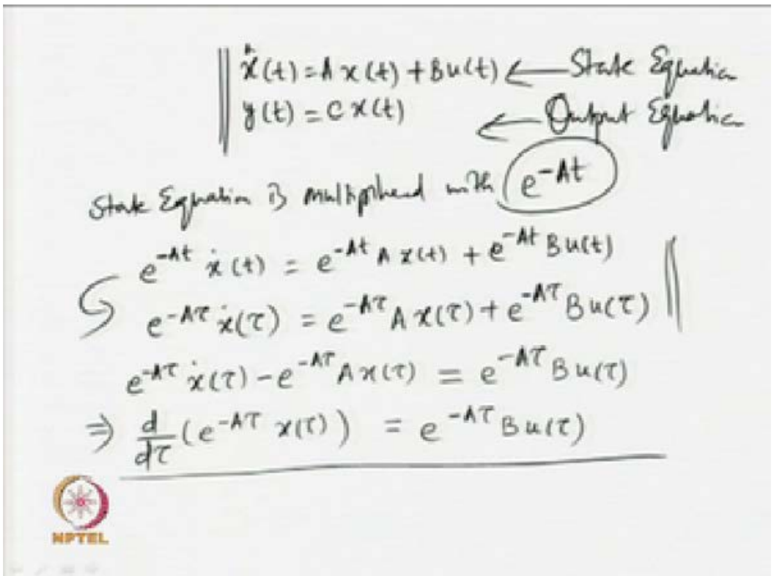

Now, we get a dynamic equation of the form x dot t equal to a x t plus b u t for a system and the output equation of the system is given as c x t. Now, to find the output of the system at any time instant, what you need to know? You need to have information about the states of the system at that instant of time. So, x t gives us us the set of states involved with the system. How can we find expression for x t? If we can find expression for x t, then simply y t is equal to c x t can be made use to find the output of the system analytically. Now, either to get the wave form, output wave form or to get the analytical expressions, it is necessary to first establish the relationship between the states and system parameters. So, now, effort will be made to find expression for the state variables x t.

Consider the state equation first. So, when the state equation state equation is multiplied with multiplied with e to the power minus a t, we get e to the power minus a t x dot t is equal to e to the power minus a t a x t plus e to the power minus a t b u t. This e to the power minus a t is often known as eternal function. eternal function Why it is called eternal function because it plays an important role in mathematics as well as in solving many difference differential equations; that is why it is called an eternal function.

So, with the help of the eternal function, I can write this expression in the form of e to the power minus a tau x dot t. Now, I will use some other variable x dot tau is equal to e to the power minus a tau a x tau plus e to the power minus a tau b u tau. What has been done here? We have changed the variable only; the t has been substituted by tau. There are no changes to the system dynamics or to the state equation, as far as both equations are concerned.

Now, when I collect the terms in this fashion, e to the power minus a tau x dot tau minus e to the power minus a tau a x tau is equal to e to the power minus a tau b u tau, this enables us to write the left hand side of the upper equation in the form of d over d tau e to the power minus a tau x tau is equal to e to the power minus a tau b u tau. Now, upto this point, I believe you should have no doubt. I will modify the state equation little bit because we will be dealing with delayed state equations in analysis of relay control systems or for identifying systems using time domain analysis, we shall deal with delayed state equations or differential equations.

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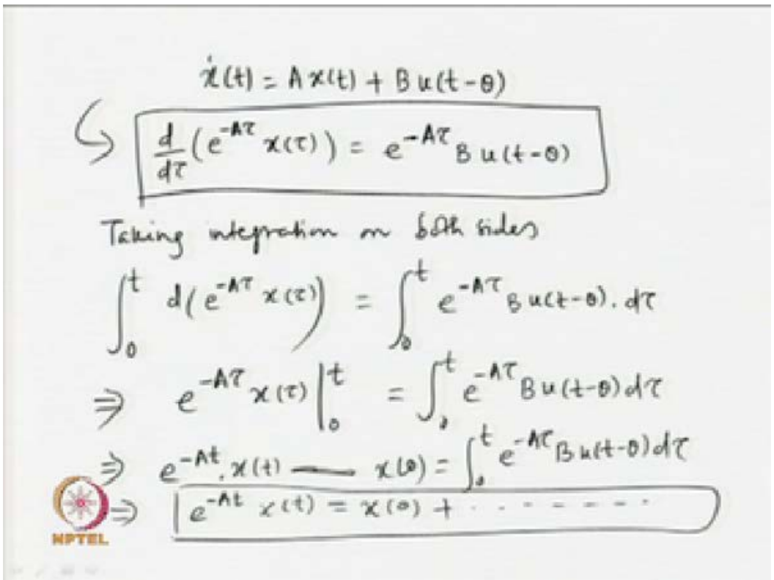

Therefore, when the state equation is written in the form of x dot t x dot t is equal to a x t plus b u t minus theta because of time delay in the system, then the same can be written, the one we have got earlier can be written, this expression can be written in the form of d upon d tau e to the power minus a tau x tau is equal to e to the power minus a tau b u t minus theta. So, this equation can further be solved taking integration on both sides. Then, we get integration from 0 to t, d over d of e to the power minus a tau x tau is equal to integration from 0 to t e to the power minus a tau b u t minus theta d tau. Why there is no d tau here? Because d tau this d tau and this d tau cancelled out (Refer Slide Time: 20:18). When you take the integration, limits of integration will be there and the limiting with respect to the variable is to be there. So, I have removed these two terms, thus giving us the expression of this form.

So, when the integral is taken, then what do we get? In the left hand side we get e to the power minus a tau x tau with limits 0 to t is equal to integration from 0 to t, e to the power minus a tau b u t minus theta d tau. Now, the left hand side often simplification gives us e to the power minus a t x t minus e to the power minus a times 0 and x 0 equal to integration from 0 to t e to the power minus a tau b u t minus theta d tau. Further, this e to the power minus a times 0 will be equal to 1 or identity matrix depending on the order of a. If a is a scalar, if a is scalar, then you get this as e to the power 0 becomes 1 and if a is a matrix, in that case e to the power minus a 0 will give you some identity matrix of the order of a. So, I removed this one (Refer Slide Time: 22:24) and I get the terms e to the power minus a t times x t minus x 0 is equal to integration from 0 to t e to the power minus a tau b u t minus theta. Again, this can be written as e to the power minus a t x t is equal to x 0 plus the remaining term.

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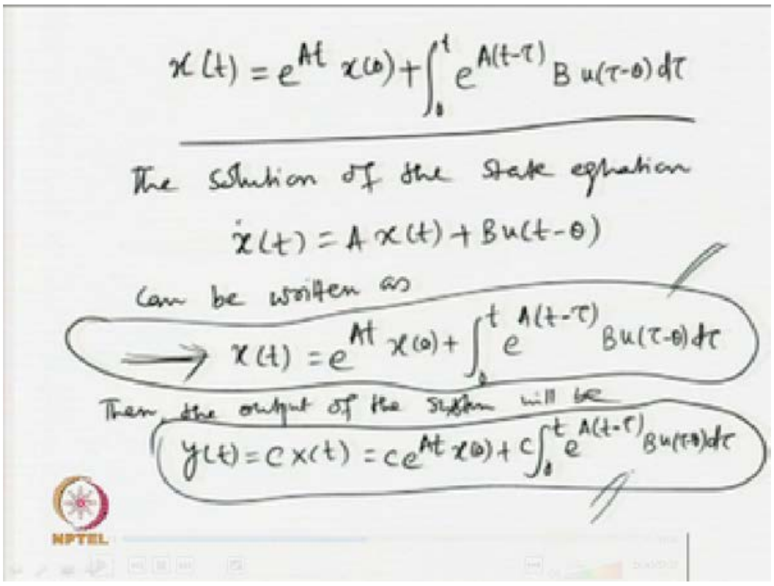

Then, when both sides of this equation are multiplied with e to the power a t, then we will get in the left hand side x t, but in the right hand side, it will be e to the power a t x 0; yes, e to the power a t x 0 plus integration from 0 to t e to the power a t minus tau b u tau minus theta d tau. e to the power a t will get multiplied here. Therefore, you get e to the power a t minus tau b u t min minus tau minus theta; I have made some mistake somewhere. This will be b tau minus theta. All those t will be tau; we have made some mistake; somewhere it will be. This becomes u tau. Then you have got u tau. You have got u tau over here. Then this will be u tau minus 0; u tau minus no t will be there; tau with the u.

Thus, finally, what we will get? We will get an expression of this form. So, what we have got? The solution, the solution of of the state equation x dot t equal to a x t plus b u t minus theta can be written as x t equal to e to the power a t x 0 plus integration from 0 to t e to the power a t minus tau b u tau minus theta d tau. Then the output of the system will be y t is equal to c x t is equal to c e to the power a t x 0 plus c integration from 0 to t e to the power a t minus tau b u tau minus theta d tau.

Thus, we develop the expressions for the state of a system, the state variables of a system. Solutions of the state equation basically give the expression for state variables of a system in this form and consequently the output of a system can be expressed in this form. These two expressions are very important in state space analysis of systems. Now, we shall make use of these expressions for analysis of relay control system. How these two powerful equations are quite helpful for analyzing closed loop relay control systems? That we shall see next.

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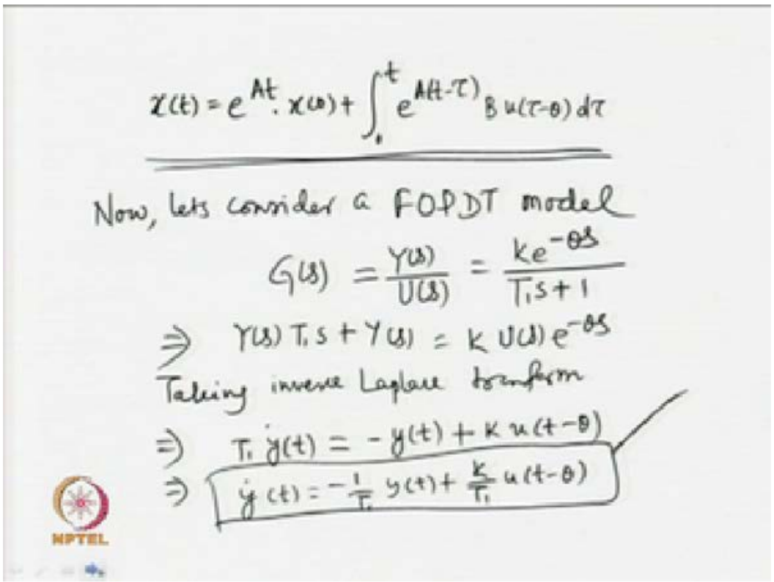

So, let me write down the two basic equations once again. x t is equal to e to the power x t is equal to e to the power a t x 0 plus integration from 0 to t e to the power a t minus tau b u tau minus theta d tau. Now, let us consider a first order plus dead time model. The model in transfer function form is given as y s upon u s is equal to k e to the power minus theta s upon T 1 s plus 1. So, the transfer function model of a system has got three unknowns now and those are k, theta and T 1. When attempt will be made to estimate these parameters k, T 1 and theta, definitely we have to have some relationship with the system parameters with the waveform, output waveform of a system; then only it will be possible to estimate the model parameters of the system.

Now, when I cross multiply the terms, then I get the same expression y s T 1 s plus y s is equal to k u s e to the power minus theta s. Taking inverse Laplace transform, what we get? We get T 1 y dot t is equal to minus y t plus k u t minus theta. Further, giving us y dot t is equal to minus 1 upon T 1 y t plus k upon T 1 u t minus theta. This we have derived in our previous lecture also, but we have not made use of state space analysis there.

Now, we shall extend state space analysis technique for the analysis of this expression.

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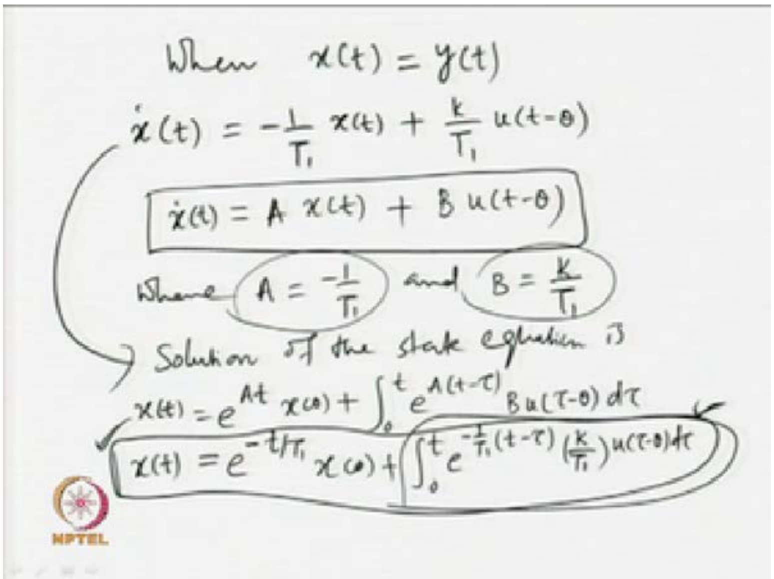

Now, y dot t using state space analysis can be expressed in the form when the state variable x t is chosen to be y t. In that case, the equation become x dot t is equal to minus 1 upon T 1 x t plus k upon T 1 u t minus theta, which in general form can be written as a x t plus b u t minus theta. So, it is giving us the standard dynamic equation of the form x dot t equal to a x t plus b u t minus theta, where a equal to minus 1 upon T 1 and b equal to k upon T 1. What is the solution of this state equation? We know that the solution of solution of the state equation is x t is equal to e to the power a t, x 0 plus integration from 0 to t e to the power a t minus tau b u tau minus theta t tau. Then, the solution of this equation using the standard solution for the state equation can be obtained as e to the power minus t upon T 1 x 0 plus integration from 0 to t, e to the power minus 1 upon T 1 t minus tau and k upon T 1 u tau minus theta d tau.

So, when I substitute the values a and b, then the solution of the state equation gives us an expression of this form - x t equal to this one (Refer Slide Time: 33:00). Thus, the first order system will be in state variable, a scalar variable x t of this form. The value of the state at any instant of time can be found using this expression. Now, x t can further be written after simplification of the second term. What is that second term? When I write the second term, then I get (Refer Slide Time: 33:38). So, let us simplify the second term integration.

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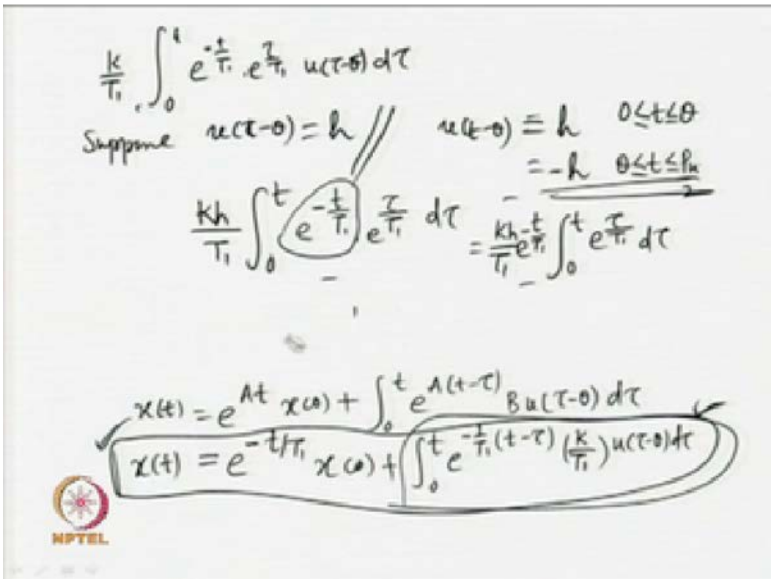

From 0 to t e to the power minus t e to the power let us write minus t upon T 1, then e to the power tau by T 1 and let us take out the constant k upon T 1 and we are left with the term u tau minus theta d tau over here. Now, suppose u tau minus theta equal to h, how we find these values? As we have discussed earlier, u t due to the delay, u t minus theta will be equal to h for the time range 0 is less than equal to t is less than equal to theta and will be equal to minus h from theta is less than equal to t is less than equal to and a period till is half period of the signal. So, this part we will not discuss.

So, when u the input signal u t minus theta becomes h. At that time, the second part can be written as k h upon T 1 integration from 0 to t, e to the power minus t upon T 1 times e to the power tau upon T 1 e to the power tau upon T 1 d tau. Again, this term can be taken out of the integral giving us k h upon T 1 e to the power minus t upon T 1 with integration from 0 to t e to the power tau upon T 1 d tau, which can further be simplified and found in the form of this is equal to equal to k h upon T 1 e to the power minus t upon T 1; then it will be into one upon T 1. So, T 1 will come and we will have e to the power tau upon T 1 with limits of integral from 0 to t.

So, thus this gives us k h e to the power minus t upon T 1 with a term e to the power t upon T 1 minus 1. So, which ultimately enables us to write x t x t in the form of...

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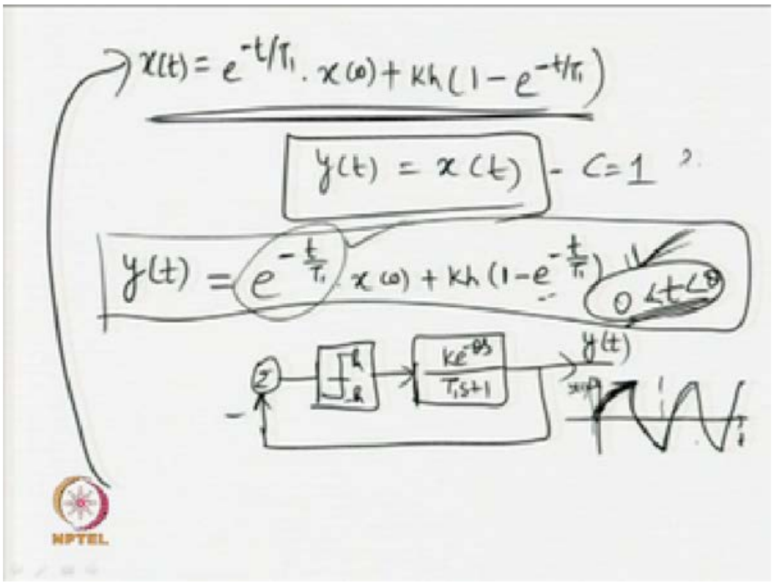

So, finally, after simplification these can be written in the form of x t is equal to e to the power minus t upon T 1 times x 0 plus; so, this in place of this I will write when I multiply this then I am getting this in the form of k h; then 1 minus e to the power minus t upon T 1.

So, this is what we get; an expression for the state variable of the system. Then, the output of the system can be found using the output equation. What is c in that case? For the first order plus delay system, we have assumed and we have found this expression with the assumption that y t is equal to x t. So, if you do not if you have not forgotten, then in that case c is equal to 1. Then the output of the system y t becomes e to the power minus t upon T 1 times x 0 plus k h 1 minus e to the power minus t upon T 1. This is the expression for output of the first order plus delay system under relay control.

So, what I mean by that? When a symmetrical relay with heights or amplitudes h and minus h experiences one first order plus delay system of the form k e to the power minus theta s upon T 1 s plus 1, then the type of output you will get from this arrangement is governed by the equation y t and that is the y t we have here. So, suppose this results in some t for particular values of k theta and T 1 and h, we get some typical wave form of this form from here. Then, this wave form is basically the result of plot of this function at different instant of time. This is the time axis this is our y t (Refer Slide Time: 40:07).

So, if I take a plot of y t, function y t for time range between 0 to theta, keep in mind, we have the assumption with time between 0 to theta only we have been able to find this expression. And when t is beyond theta, we do not get this expression. So, till time t equal to theta, I get an output of this form (Refer Slide Time: 40:43). Suppose this is your theta, then the first order system will definitely result in some exponential output. Why I call this exponential output? Because if I carefully observe, there are exponential terms here; so, in spite of the constants we have in the expressions due to the presence of exponential here, the output will take the shape of exponential curve.

Now, what are the benefits of getting the output expressed in this form? So, when we conduct or perform a relay test, we get periodic output of some typical form from first order system. Let the output be given by this form this is your y t (Refer Slide Time: 41:36). So, if this this out this is the output wave form and if we can establish that the peak amplitude has some relationship with some parameter or many parameters of the transfer function model, then it will be possible to estimate the transfer function model parameter based on measurement of the peak amplitude. That is the purpose of doing all these analysis.

So, is it possible to get the out form from analysis? Yes, it is possible because the analysis, state space analysis shows that, for a first order system when t is between 0 to theta, the output assumes this form. And when this is plotted for various values of t for time 0 to theta, definitely we get an output of that form what practically we obtain from conducting a relay test. So, a relay test will also definitely yield a waveform of this shape. So, shape of waveforms will definitely enable one to find the model parameters of a system. Now, how to establish those relationships? That we shall try now.

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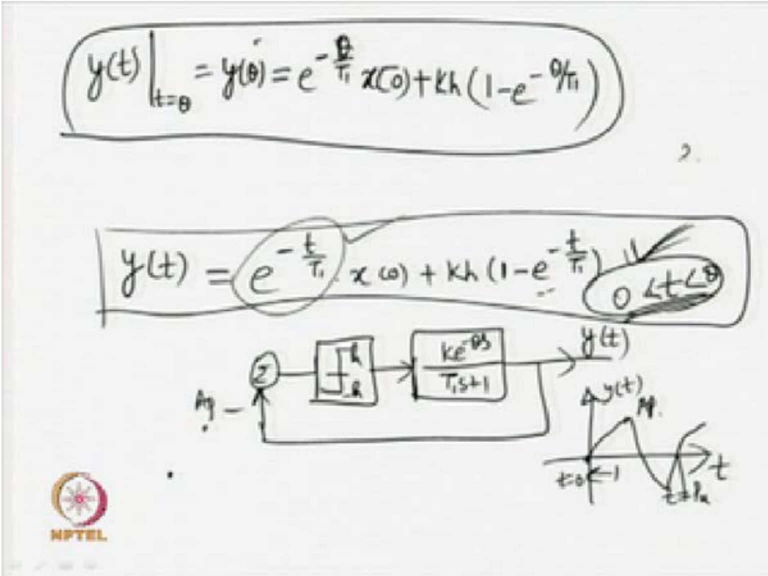
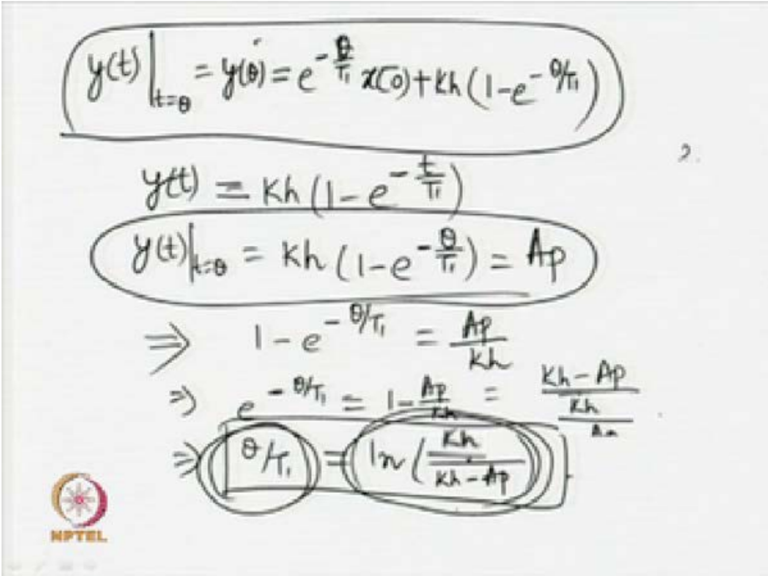


As we have seen, what could be the maximum output from this first order plus delay system? The maximum output we get is at time t equal to theta; time is gradually increasing; so, at time t equal to theta y t at time t equal to theta will be equal to y theta, which will be given by e to the power minus t upon T 1 x 0 plus... So, this will be theta because when I substitute t by theta, I get theta over here plus k h 1 minus e to the power minus theta upon T 1. So, the output of the system is given by this expression.

One important point we need to consider is that, whenever we make measurements on the system output, assuming that the system output is of the form as shown over here (Refer Slide Time: 44:19), we target measurements from time t equal to 0 till measurements up to time t equal to p u; at least one complete period. So, to get the peak amplitude and ultimate period, what we have to concentrate on? We have to concentrate on the output wave form, stable output wave form for one period at least and by convention what we take? We we we assume that the first 0 crossing occurs at time t equal to 0. Therefore, y 0 will be equal to 0. (Refer Slide Time: 45:16)



With this, when y 0 is assumed to be 0 the output expression at time t equal to 0 gives us y 0 is equal to x 0. If you substitute t by 0, it becomes 1 and then rest of the things you have got k h 1 minus 1. So, y 0 becomes x 0 and since we start considering the output wave form from time t equal to 0 which occurs at the first 0 crossing, then it is assumed that for a first order system x 0 is equal to 0. Then, in that case, ultimately the output of the wave form can be given by the expression y t is equal to e to the power minus t upon T 1 times 0 plus k h 1 minus e to the power minus t upon T 1.

So, this will be the expression for output wave form of first order plus time delay system for the time range 0 to theta. Now, thus I have got finally, the expression for the output in the form of y t is equal to k h 1 minus e to the power minus t upon T 1 only for first order plus dead time system. I cannot guarantee this expression for any other order of transfer function model. It can vary from system model to system model, of course, with the limiting values for T 1 k and so, we can extend this expression for any other type of transfer function model. Now, when y t is this much, then the peak amplitude of the output signal y t at time t equal to theta is equal to k h 1 minus e to the power minus theta upon T 1 which becomes the peak amplitude now. This is how I find one relationship between parameters of the first order plus delay model with that of the peak amplitude of the output signal.

So, what we get from this expression? Simplifying this expression, one get 1 minus e to the power minus theta upon T 1 is equal to a p by k h or e to the power minus theta upon T 1 is equal to 1 minus a p by k h, which is nothing but k h minus a p by k h. So, theta by

T 1 is equal to natural logarithm of k h upon k h minus a p. This shows that there exists relationship among the parameters of the transfer function model, first order plus dead time model with that of peak amplitude of the wave form output signal we get from relay test.

So, finally, let me write down the expression once more to get inside about the expression or what benefit we get from the expressions.

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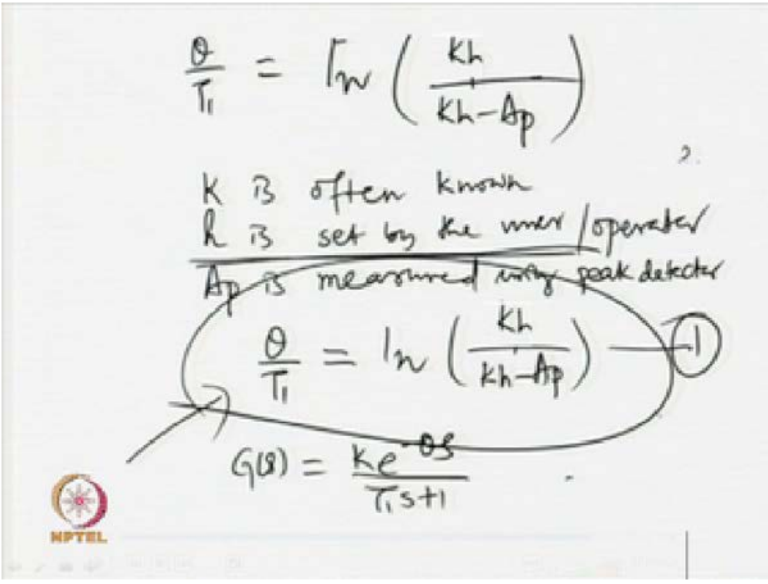

So, theta upon T 1 can be written as natural logarithm of k h upon k h minus a p. What are known and what are unknown in this expression? The steady state gain of the system can be obtained by various methods. So, k is often known; h is set by the user. So, the relay amplitude is set by the user or operator; a p is measured using some peak detector. So, thus the right hand side of the expression is known. Using the right hand side and left hand side, it is not difficult to get the transfer function model parameters theta, ratio between the transfer function parameter theta upon T 1 lan k h upon k h minus a p. So, thus, the model parameter that we had in the transfer function model k e to the power minus theta s upon T 1 s plus 1 - ratio between the two parameters is obtained using this expression.

Similarly, further expressions can be developed using state space between T 1 and theta and other parameters like are ultimate period that we measure from the output wave form. Then, using two expressions it is always possible to estimate two unknowns or if we can develop three equations using the state space analysis, then the three unknowns of the transfer function model can easily be estimated. That is how accurately we obtain the estimate for the transfer function model. The beauty of using state space analysis is that it is possible to estimate the parameters of the transfer function model accurately.

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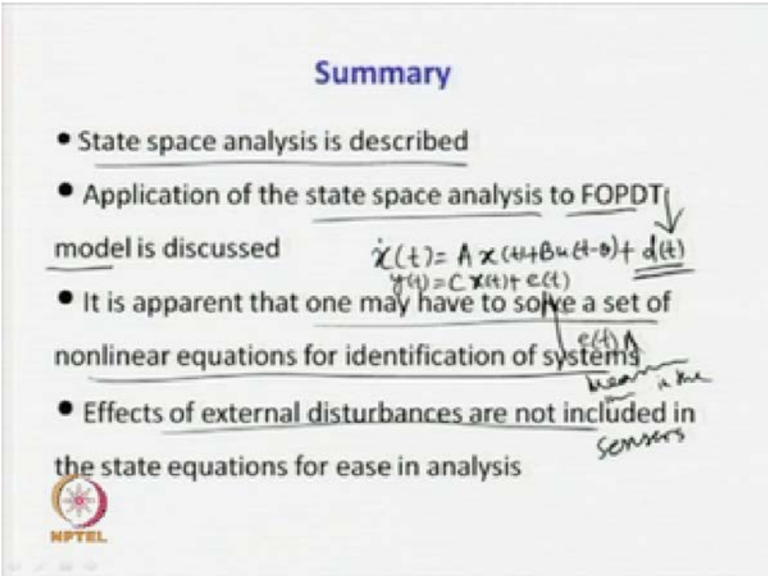

Now, what we have learnt from this lecture? That what a state space is, state space equation and output equation is; then what benefit we get from state space analysis of system. When the state space analysis of system has been extended to a first order plus dead time model, we have seen that we can establish relationship between the parameters of the model with that of the output we get from the system under relay control. Also it is apparent that one may have to solve a set of non-linear equations for identification of systems. As I have told that for higher order systems, we may get a set of non-linear equations from the state space analysis. The state space analysis will get give you exact equations, but the equations might be highly non-linear in nature. Then how to handle such non-linear equations? Care must be taken to choose judiciously the initial conditions or initial solutions when solving such set of non-linear equations.

Again, like the previous lecture, the external effects of external disturbances have not been considered in this lecture. Because we begin with simple expression to avoid to make the analysis easier, intentionally we have not considered the effects of external disturbances in our analysis. Otherwise, the state equation could have been given in the form of x dot t is equal to a x t plus b u t minus theta plus d t where d t are the real time disturbances in the system states. Similarly, the output would have been also disturbances given by y t equal to c x t plus e t, where this e t will be nothing but our measurement errors; e t stands for all sorts of measurement error due to inaccuracy in the sensors.

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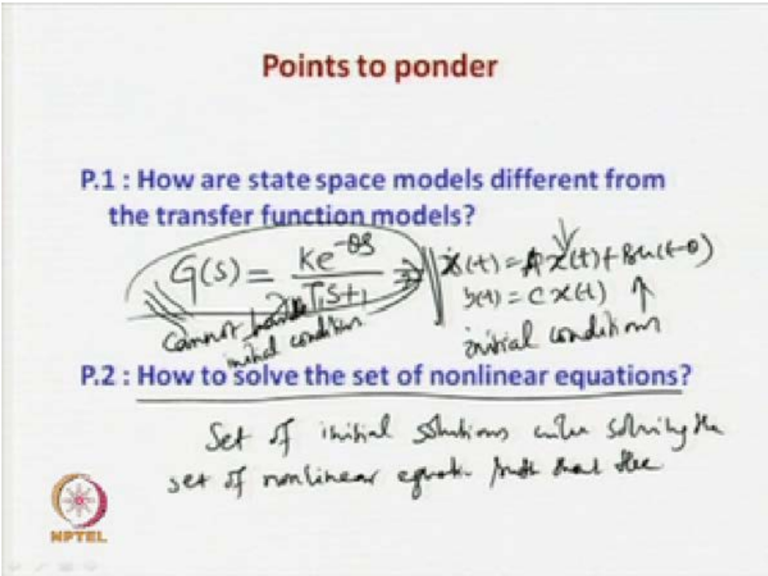

Some points to ponder. How are state space models different from the transfer function models? Let me give you one simple example. A transfer function model is given by g s k e to the power minus theta s upon T 1 s plus 1. Whereas, the state space model has got two expressions one state equation and the other one is the output equation and you get these expressed in the form of x dot t equal to a x t plus b u t minus theta, y t equal to c x t. So, how they are different? Using the state space model it is possible to take into account the effects of initial conditions initial conditions whereas, transfer functions cannot handle handle initial conditions. So, there is no scope for considering the initial conditions when you have transfer function model whereas that is not so with the state space model. That is the main difference between the two type of models we have for many real time systems.

And when the question comes like this - how to solve the set of non-linear equations? As I have already mentioned, it is all about choosing judiciously the set of initial solutions initial solutions while solving the set of non-linear equations, equations such that such that the solutions do not converge to erroneous values. So, it is all about writing proper codes, mat lab codes for solving the set of non-linear equations. There are some functions under mat lab library like f min, f solve to help solve non-linear equations effectively. That is all.