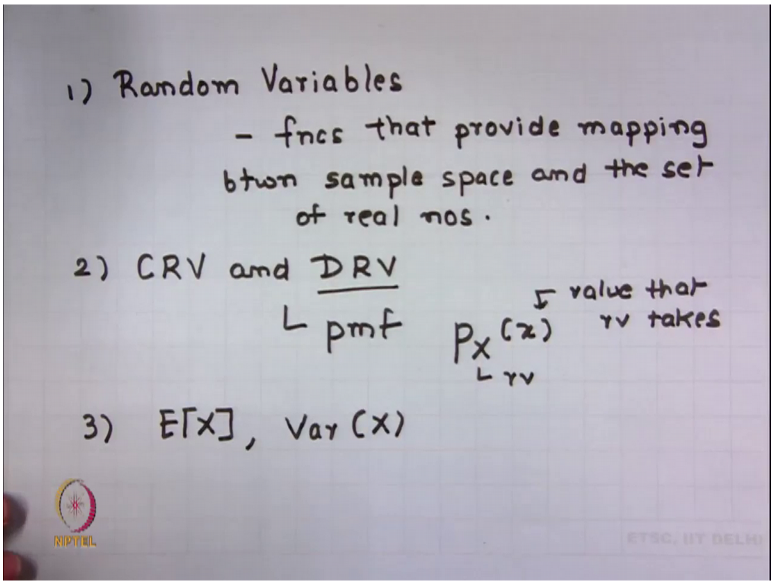
Principles of Digital Communication
Prof. Abhishek Dixit
Department of Electrical Engineering
Indian Institute of Technology, Delhi

Lecture – 08
Random Variables & Random Processes: Continuous Random Variable

So, welcome to lecture 2 on Random Processes and before starting this lecture 2, let us revisit what we have done yesterday.

(Refer Slide Time: 00:36)



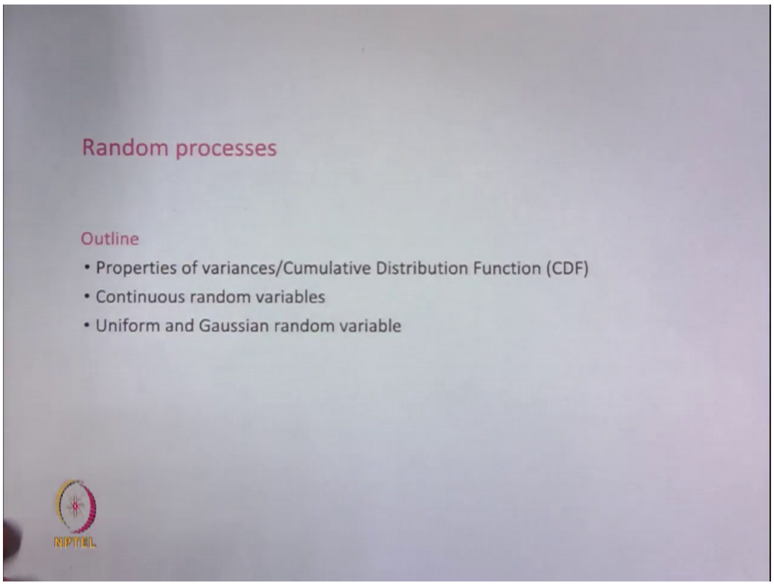
So, in the last lecture, what we have basically talked about is random variables and we have said random variables are functions that provide mapping between sample space and the set of real numbers that is the one important thing that we discussed yesterday. We said that random variables are functions, they are neither random nor variables.

Secondly, what we did is we introduced continuous random variables and discrete random variables and we basically spend most of the lecture in studying about discrete random variable. And for discrete random variable, we introduced probability mass function and we said probability mass function is represented like this. We use this notation to represent probability mass function where X, the capital X is the random variable and the small x is the numerical value of that random variable. So, it is the value that random variable takes, alright.

So, this is one big concept that we introduced yesterday, the concept of probability mass function. The third thing that we did is we introduced expectation operator. We said expectation operator could be understood as an average value of the random variable and we also introduced the concept of variance of a random variable.

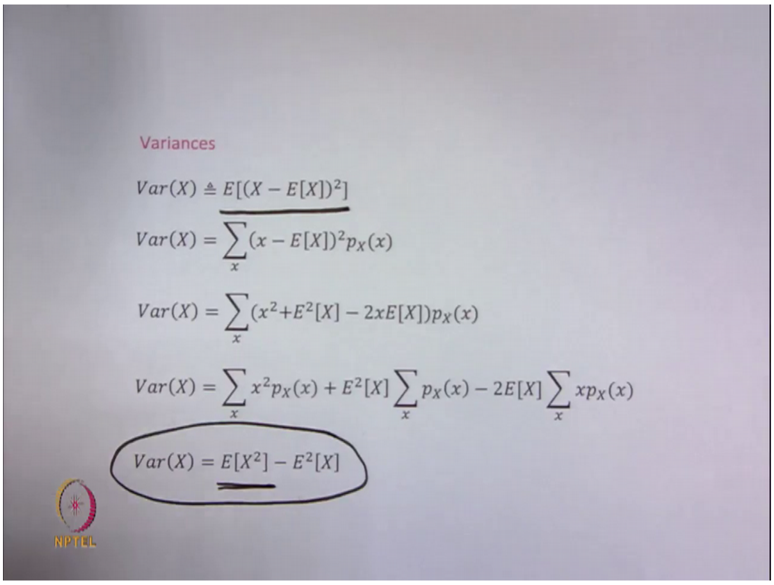
So, basically these were the three important concepts that we covered in the last lecture. So, let us see what I have got for you today.

(Refer Slide Time: 03:08)



So, in today's lecture we will talk about the properties of variances, then we will talk about the cumulative distribution function and we will then go to continuous random variables. We yesterday we focused basically on discrete random variables and today we will talk about continuous random variables and example of continuous random variables we will look at two important random variables; two important continuous random variable that is uniform and Gaussian random variables. So, this is more or less the outline for today's lecture.

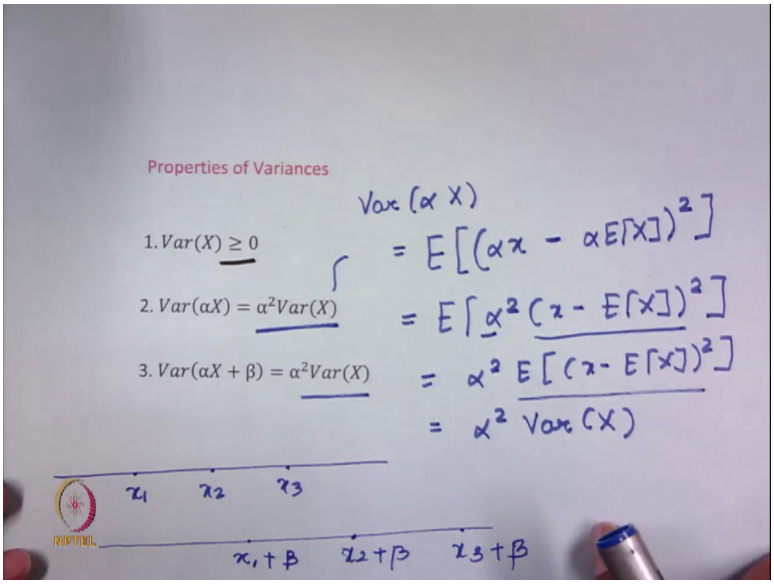
(Refer Slide Time: 03:48)



So, the first thing is that I will like to do today is to look at this slide again where we defined this variance of a random variable variance of X. So, what we said is variance is nothing, but it is the distance of the random variable from the mean. So, first thing that we have to do is we have to calculate the distance of various numerical values of random variable from its mean and then we have to do the square of it and then we have to compute the expectation.

So, variance of a random variable it is nothing, but it is the expectation of a square distances of the random variable where the distances we measure from mean and we also said that variance of a random variable it is nothing, but it is the second moment. So, this is the second moment of a random variable minus a square of the first moment of a random variable. So, this is the interpretation of variance of a random variable that we developed.

(Refer Slide Time: 04:53)



So, today we will like to finish this variance by talking about various properties that the variance have. For example, the first property that variance of a random variable has is that is strictly non-negative. The variance can be 0 or it should always be either 0 or it should be greater than 0, but it can never be negative. And why is that? So, to understand that you just have to look at this expression where the variance is calculated as the expected value of a square of a quantity.

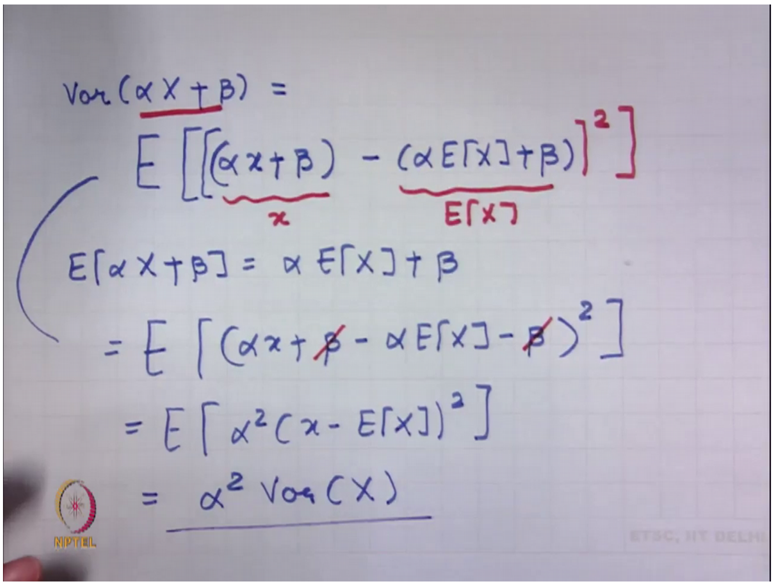
Now, the square of a quantity can never be negative and thus the average value of a positive numbers is always positive it can be 0, but it can never be negative. So, that is the first property of variance of a random variable. Let us look at this property. What is the variance of a constant times a random variable? So, to think about this let us plug this into the expression.

So, variance of a constant times a random variable is nothing, but it is the expected value of alpha times x minus; so, why we have alpha times x because if you multiply a random variable with alpha with the constant alpha, then its numeric value will also be multiplied by alpha and the mean of the random variable will also be multiplied by alpha. Because we are multiplying a random variable with a constant, then the mean of the random variable gets multiplied the numerical value of that random variable is also getting multiplied with alpha and then we have to take it is square. This is from the definition of variance of a random variable.

Now, if I do this more because alpha is a constant I can take this out, I get this and now we have seen in the last lecture that expectation of a constant times a random variable is nothing, but constant times the expected value of that random available. So, this is constant and this is the random variable. So, expectation of a constant times a random variable we have seen is nothing, but it is the constant times expected value of that random variable and as you can see this is nothing, but this is alpha square and this quantity is variance of a random variable.

So, we have seen an important reason that is the variance of a constant time here random variable is nothing, but it is the square of the constant times variance of X variance of that random variable. Let us see now this property, the third property what is variance of alpha X plus beta. So, we can plug again this into.

(Refer Slide Time: 08:00)

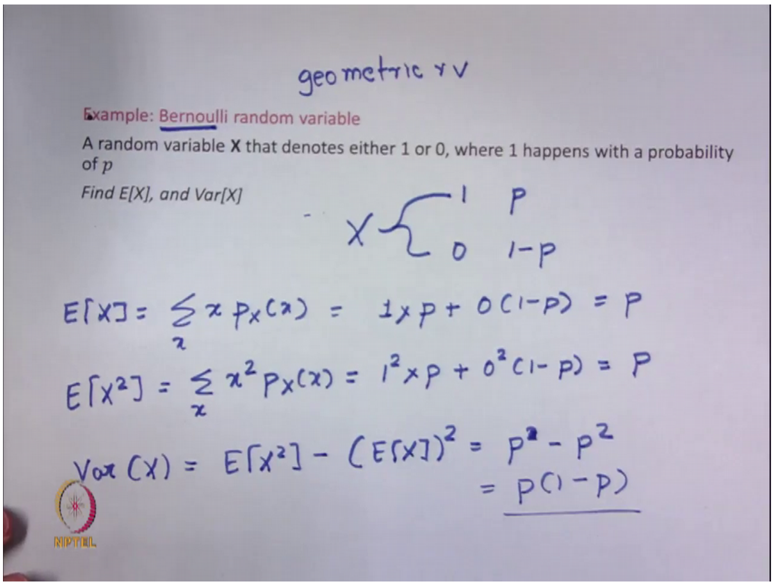


So, if we have to solve, what is variance of alpha X plus beta, we can work this out. So, now, the numerical value changes to alpha X plus beta, right. So, instead of X you would have this thing. And how does the expected value of this quantity be? So, we have seen in the last lecture that the expected value of alpha X plus beta is nothing, but alpha times expected value of X plus beta. So, the expected value of this quantity is this. So, we have taken a numerical value. So, this is of the form X and this is of the form E to X. So, you take a numerical value which is in this case alpha plus beta and then you take expected value of this quantity which is alpha times expected value of X plus beta you subtract the numerical value from the expected value as we have done and then you square this and calculate the expectation.

So, this I can write as and as you can see from this expression that I can cancel beta with beta and then I have alpha X minus alpha E X square I can again take out alpha and this is nothing, but it is alpha square times variance of X,. So, what we have done is we can derive this expression that a variance of a constant times random variable plus a constant is nothing, but alpha square times variance of that random variable.

What is interesting to see is that if you shift a random variable if you add a constant to a random variable, then the variance is not changed. So, you have got the same variance as in this case. And what is the difference? You have some constant added to a random variable alpha X. So, if you add a constant to a random variable, then its variance remains unchanged. And why is this? Because when we are adding a constant what is happening to the random variable. So, you had some numerical values. When you add a constant, these numerical values shift by beta. So, the numerical value shift by beta, but the distance of these values from the mean will not change, right. These values numerical values have changed the mean will also have shifted by the same amount beta and thus the distance of these numerical values from the mean would not change and thus the square of the distance will also not change and thus the variance remains unchanged, ok.

(Refer Slide Time: 11:58)

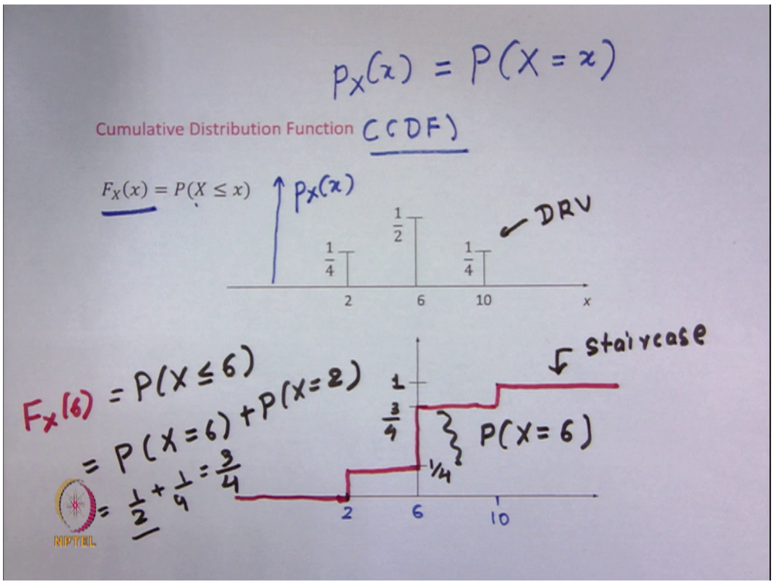


Let us now do an example. Yesterday's lecture we introduced the notion of geometric random variable and today, we are visiting another important random variable which is Bernoulli random variable, ok. So, Bernoulli random variable is a random variable that either takes a value 1 or 0 and where 1 happens with a probability of p, ok. So, it takes either 1 or 0 and it takes 1 value 1, where it 1 happens with a probability of p and so, you see that this Bernoulli random variable models data transmission because when we transmits data, we transmit mostly in the form of 1 and 0's and the 1 happens with a certain probability and the 0 happens with a certain probability.

So, let us not worry about that at this moment. Let us focus on this random variable and let us say that is random variable takes two values 1 or 0. So, it takes either a value 1 or 0 and it takes a value 1 with a probability p. So, it will take a 0 with a probability 1 minus p and this question is what is the expected value of this random variable. So, expected value of this random variable we can use this. So, here it is simple X takes only two numerical values and the one is 0, it takes one with a probability of p and it takes 0 with the probability of 1 minus p. So, the expected value of this random variable is nothing, but p, easy. Is it not?

So, now let us calculate the second moment of this random variable which is this and when you are calculating the second moment of this random variable, you have to put x square. So, you have to put 1 square, it happens in the probability p then you take 0 square which happens with a probability 1 minus p and the second moment also turns out to be p, good. Then let us see what is the variance. Variance is the second moment minus square of the first moment. So, we have p square sorry, we have p minus p square and this is nothing by p times 1 minus p. So, this is the variance of a Bernoulli random variable,.

(Refer Slide Time: 14:50)



Let us now introduce a new concept of cumulative distribution function and cumulative distribution function is a simple distribution function. So, till now we were talking about probability mass function. What was probability mass function? Probability mass function was, what is the probability that a random variable X takes in a value small x, ok. So, we were talking about probability mass function. Now, we are talking about a second function a different function; this is known as cumulative distribution function in short this is also known as CDF – Cumulative Distribution Function.

Now, cumulative distribution function tells me or worries about basically what is the probability that a random variable X takes in a value which is less than or equals to x ok. So, this is the definition of the cumulative distribution function. So, we worry not about what is the probability that X takes a value x, but we worry about what is the probability that X takes a value less than or equals to x, alright. So, it is simple and this is a notation that we will use to denote CDF or Cumulative Distribution Function.

Let us work out for this case. So, on this axis y axis let us assume that I have PMF and on x axis I have a numerical values that use random variable takes. So, if I have to draw the CDF for this case so, let us just first put the point 2 here, 6 here and 10 here. So, what is the probability that this random variable takes a value less than 2? Let us say at this point very nearer to 2, what is the probability that this random available takes a value let us say 1.9? The probability is 0, because this random variable takes only these three values 2, 6 or 10. It does not take any other numerical value.

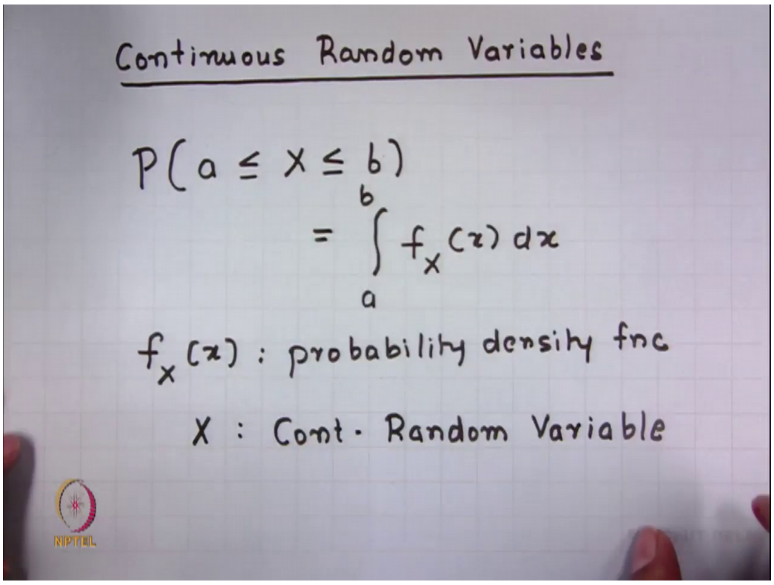
So, probability of a random variable taking a value less than 2 is 0 like. So, I can draw a 0. What is the probability that this random variable takes a value 2 or less than 2? It cannot take any value less than 2, but at 2 the probability is 1 by 4. So, I goes up to 1 by 4, let us say this is 1 by 4, ok. What is the probability that this random variable takes a value less than 2.1 is also 1 by 4 because it can then only have a value 2; it does not have any numerical value between 2 and 6.

So, the probability between 2 and 6 remains constant and at 6 what we are saying is F x of 6 is what is the probability that this random variable takes a value less than or equals to 6. And so, for here we say that what is the probability that this random variable takes a value 6 or it takes a value 2 because they are it cannot take any other numerical values. So, these are the only two numerical values that it can take. It can take either 6 or it can take 2. So, probability of a random variable taking a value less than or equals to 6 is this and this is half plus 1 by 4 which is 3 by 4.

So, the probability at 6 at point 6 you have to add half to this. So, they shoots up at 6, then it again becomes constant till 10 and at 10 it shoots up again and it remains constant. What is the probability? So, this point is 3 by 4 and this point is 1. So, after 10; this point, after 10 it has a probability 1. So, from this example what we have seen is that the cumulative distribution function of a discrete random variable; so, we have been thinking about a discrete random variable. So, this is a discrete random variable and what we see is cumulative distribution function of a discrete random variable is a staircase. It looks like a staircase, ok.

So, if you see a CDF function a cumulative distribution function and cumulative distribution function looks like a staircase, then you better guess that it is a discrete random variable. Because discrete random variable takes only a countable number of elements and at each numerical value it has a finite probability and at that point it this probability at. So, the amount of discontinuity that you see here is the probability that the random variable takes a numerical value 6. So, amount of discontinuity would be same as this PMF at 6. So, we have almost finished everything about discrete random variable and then we will like to talk about continuous random variables, ok.

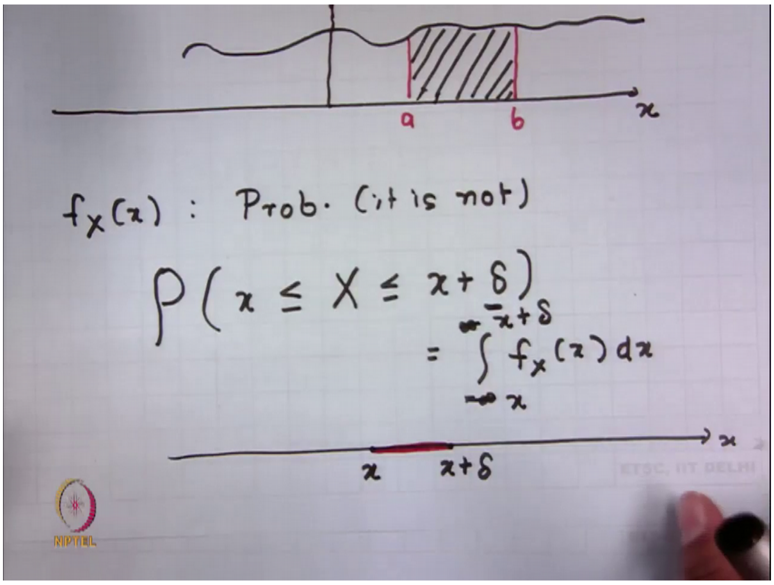
(Refer Slide Time: 20:51)



So, now, let us start looking at continuous random variables and let us see what they are. So, for example, if I have a random variable which is a continuous random variable and I ask a question what is the probability that this random variable takes a value between the limits a and b, ok. So, we are thinking about what is the probability of this random variable taking a value between a and b and if this probability is given by this expression. So, what is first let me introduce what is this f X of x. So, again just see that we have used a capital letter X to denote the random variable and the small x again represents the numerical value that that random variable takes and f X of x is defined as probability density function.

So, this f X of x is a probability density function and we will see what it means in a short while. But, let us see if I can calculate the probability of a random variable taking a value between limits a and b as integration of this probability density function then this random variable X is known as or is a continuous random variable, alright. So, let us see what this all means. So, let us take an example of a probability density function.

(Refer Slide Time: 23:17)

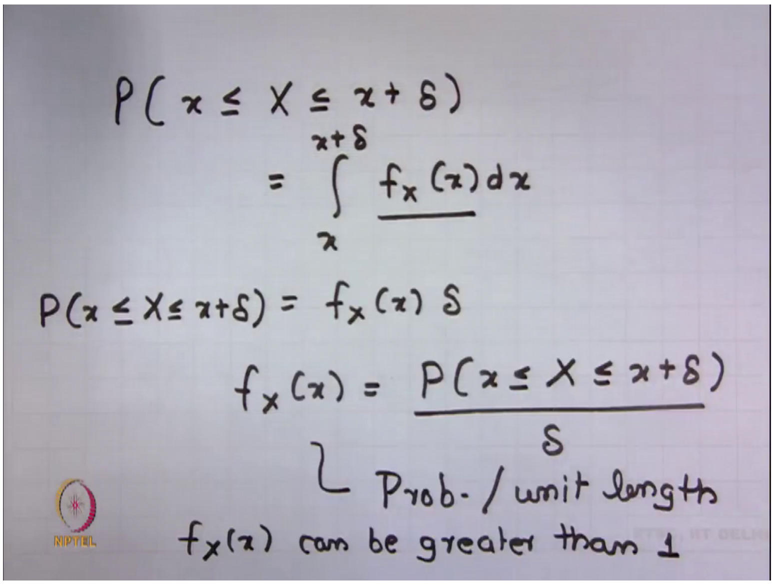


So, on x-axis I have the numerical values that random variable can take and I have on y-axis some probability density function and let us mark the limits from a and b. So, if I look at this integration formulation, to calculate the probability that this random variable takes a value between a and b just what I have to do is I have to integrate this and get this area. So, this area tells me the probability that a random variable takes a value between a and b, right.

So, let us define this probability density a bit more. The first question is this property density function is not probability itself; it is not probability, ok. It is not probability it is a probability density function and how can we interpret this. So, let us go back to that integration and try to calculate the probability that a random variable takes a value between X and x plus delta, ok. This is a very small quantity and now, I am interested in finding what is the probability that the random variable lies between x and x plus delta.

So, what I mean is, let us say my x point is here and let us take a small interval delta. So, x plus x plus delta is this line. So, what I am asking is what is the probability that a random variable takes a value on this line and to calculate that if I use that expression here I have to change the limit going from x to x plus delta instead of minus infinity to plus infinity or instead of a and b because I am interested in finding the probability of a random variable taking values between x and x plus delta. So, the limit has to go from x to x plus delta.

(Refer Slide Time: 25:56)

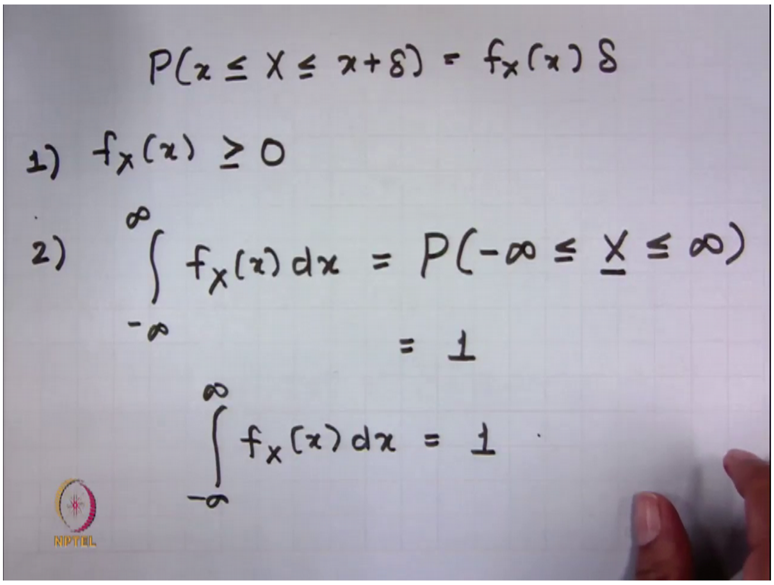


And, if I work this out, what let me write this again and because delta is a very small quantity let me assume that this probability density function is constant in this limit and if it is constant I can pull this out and what I will have is probability density function evaluated at x times delta, alright.

So, I can think of this probability density function in this way. So, what I am saying is probability density function is the probability of a random variable taking a value between x and x plus delta divided by delta. So, if I am interested in the units of this quantity the unit of this quantity is probability per unit length. So, this is how we can interpret this probability density function. Probability density function is not probability, but it is probability per unit length. Now, because it is probability per unit length and not probability itself the probability density function can be greater than 1, ok.

So, let us see what other properties this probability density function would satisfy.

(Refer Slide Time: 28:15)

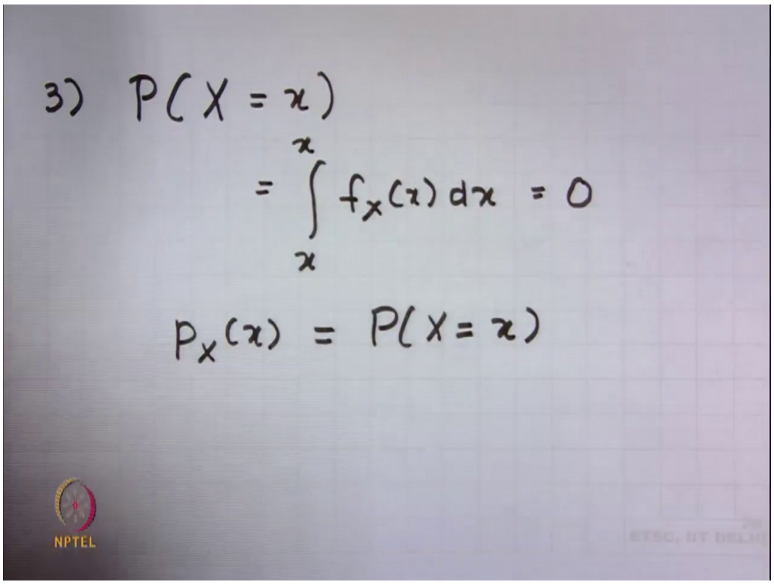


So, let me write that expression again, so that is extremely clear what we are talking about. So, if you look at this expression and if you reason out because the probabilities can never be negative themselves. So, this probability density function is also always positive, right because probabilities can never be negative the probability density function can also never be negative. It is always a non negative; it is always greater or equals to 0.

Now, the next property: so, this is one property of the probability density function that is always non-negative quantity let us try to see what is this expression. So, what this integration tells me that I am finding the probability of random variable X to take a value between minus infinity and plus infinity, alright. So, I am interested in finding the probability, what is the probability that this random variable takes a value between minus infinity to plus infinity and as you can see that this probability will be 1, right.

So, we can conclude from this that the integration of probability density function from minus infinity to plus infinity is always 1. So, that is the second property of a probability density function.

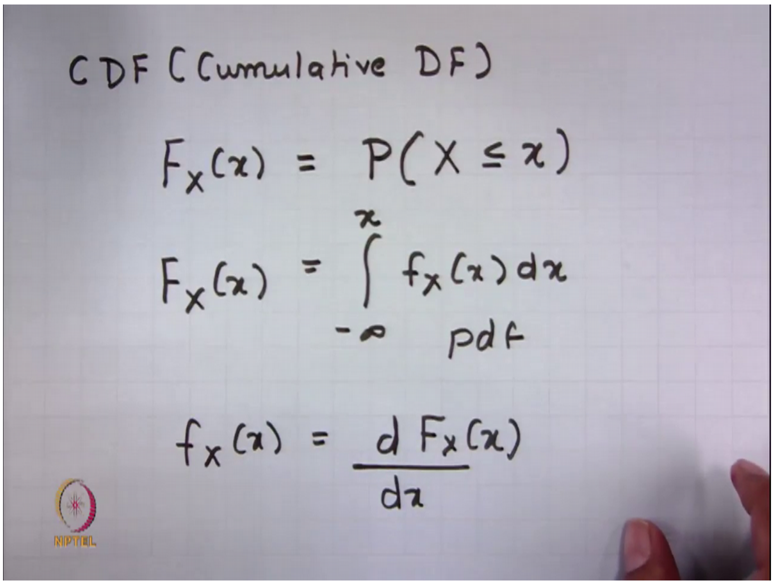
(Refer Slide Time: 30:01)



Let me ask a third question. What is the probability that a continuous random variables. So, we are just focusing on continuous random variables at this moment what is the probability that this continuous random variable takes a particular or a specific numerical value, right. Now, we are not asking the question what is the probability that a random variable lies between X and x plus delta, but we are trying to investigate what is the probability that these random variable takes a specific value x and to calculate that the limit will now go from X to x and as you can see that this is a integration at one point there is no width in that area. So, this integration will turn out to be 0.

And, this might surprise you that in the case of a continuous random variable we do not talk generally about the probability of a specific numerical value because that is probability is always 0. In discrete random variable on the other hand, the probability mass function that we introduced in the last lecture was talking about probability of a random variable taking a specific numerical value x that is the probability mass function. But, this expression will have no meaning here because the probability of a random variable taking a specific numerical value will always be 0, until and unless their singularities and other things sitting in the probability density function and which we normally do not have.

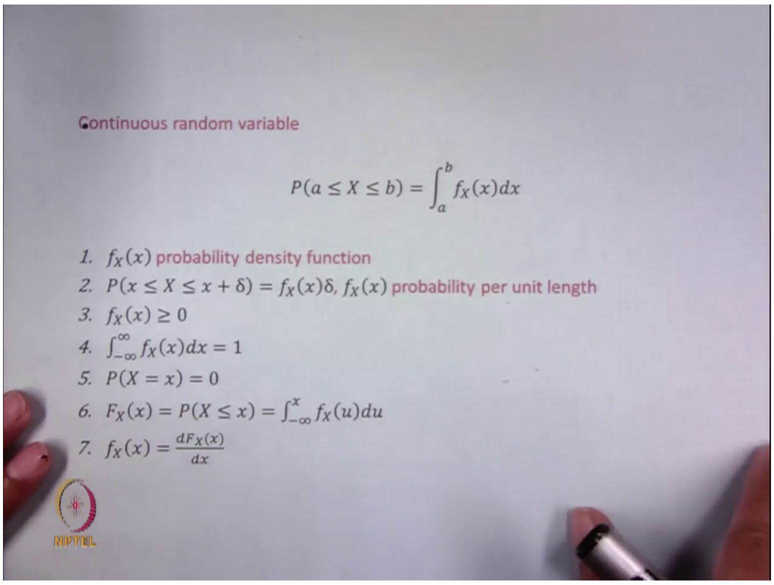
(Refer Slide Time: 31:45)



So, like in the discrete case we can also define CDF in continuous case which is cumulative distribution function and the cumulative distribution function is defined same as before. So, cumulative distribution functions tells me what is the probability that a random variable takes a value less than or equals to x and if you want to evaluate this you just have to put it into the integration that we have seen before. So, this is nothing, but this, right. So, this gives us an expression of CDF in terms of probability density function probability density function is also known as PDF in short Probability Density Function.

So, this expression relates cumulative distribution function with respect to probability density function and you can go back and forth. So, if you have to find probability density function in terms of cumulative distribution function you just have to differentiate it with respect to x, ok.

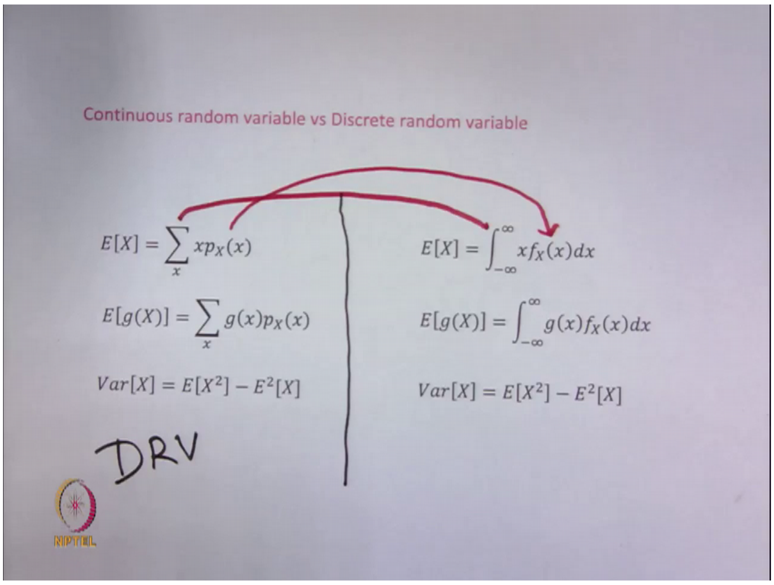
(Refer Slide Time: 33:05)



So, let us revise all properties that we have seen. So, we have introduced continuous random variable in which we are talking about the probability of a random variable taking the values between a and b and this probability can be obtained in terms of probability density function. We have said that the probability density function is not probability, but in this case it is probability per unit length. We have set probability density function is always non-negative because probabilities are non-negative we have seen that area and a probability density function from minus infinity to plus infinity is 1. We have said we do not talk about a specific probability in the case of continuous random variable because that probability is 0.

We have also said that you can think about the cumulative distribution functions in terms of probability density function and you can also obtain the probability density function from cumulative distribution function.

(Refer Slide Time: 34:06)

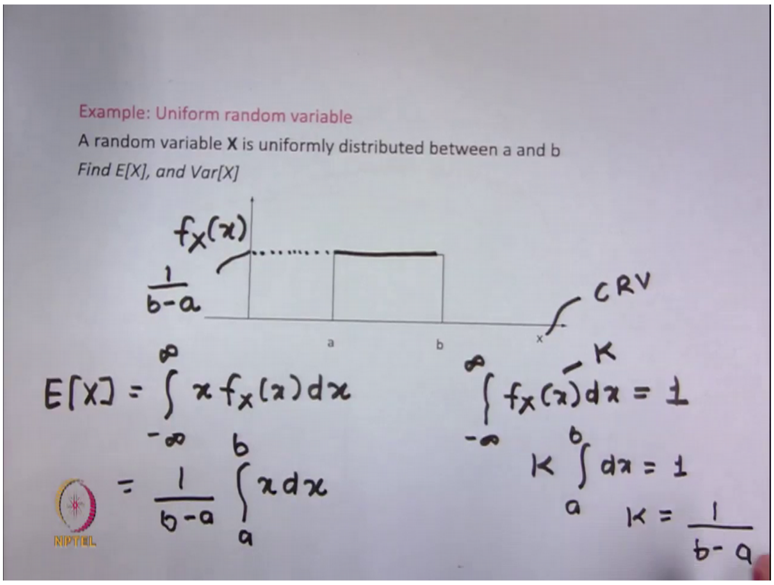


Let us again talk about the concept of expectation and variance in the case of continuous random variables we already discussed this in the case of discrete random variables. So, this is what we have covered in the last lecture, we have already talked about these things. So, we have already talked about the expectation of a discrete random variable which we obtain like this. In the continuous case, it is exactly similar what changes is that this probability mass function changes to probability density function that is the one change and the summation is replaced by integration, ok. So, it is exactly same with these changes.

So, you can think about the expectation of a continuous random variable in terms of probability density function and you need to replace the summation with integration everything else remain same. So, expected value of a function of a random variable we obtained in discrete case like this. It remains exactly same, but the changes are that this probability mass function is replaced by probability density function, this summation is replaced by integration the expression of the variance also remains unchanged that is exactly same, ok.

Now, let us do one example.

(Refer Slide Time: 35:33)

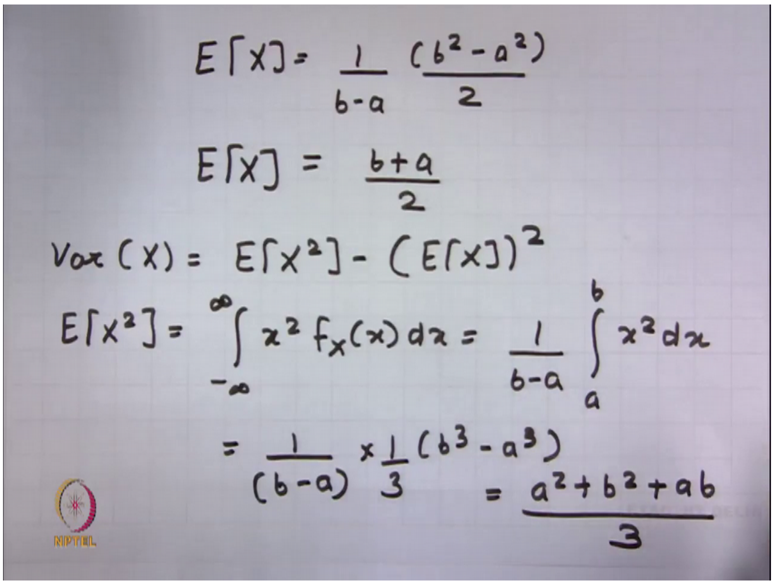


And so, what we have here is probability density function of a continuous random variable and what we are interested in is finding the expectation of this random variable and a variance of this random variable. So, first let us see why is this a probability density function of a continuous random variable because the x takes all numerical values between a and b. So, the range of x is unaccountably infinite right. So, because it takes all values between a and b and these values are real numbers, right. So, x is a continuous random variable because its range is uncountable infinite and let us now see what is the expected value of this x.

So, expected value of x is nothing, but we can use the expression x into probability density function. Now, what you may worry is that this probability density function though it looks uniform, but we do not yet have a value what value does it take at this point, right. But, you can easily obtain this if we use the property that integration of probability density function from minus infinity to plus infinity is 1, alright. So, let us assume that this is some constant K. So, K times minus infinity, but you see that it goes only from a to b and from this you can obtain that K is 1 upon b minus a. So, what we have got is even though this value was not given, but we can compute it to be like this.

Now, finding the expected value of this is easy. So, probability density function is constant is constant only in the limits a and b, after that it is all 0. So, we get this and you can solve this up.

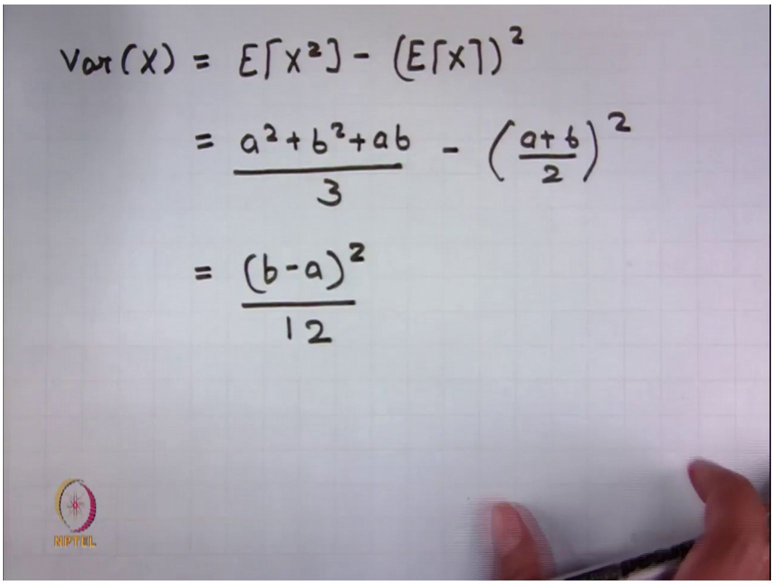
(Refer Slide Time: 38:08)



So, we had this and this gets out to of the form b square minus a square by 2 and from this we get b plus a by 2. So, expected value of uniformly distributed random variable we have obtained it as a plus b by 2 and it is of no surprise because we have said that expected value of a random variable can also be thought as a centre of gravity. And, if you look at this structure the centre of gravity of this structure will fall at this point a plus b by 2 think more about this. Let us now calculate the variance of this random variable.

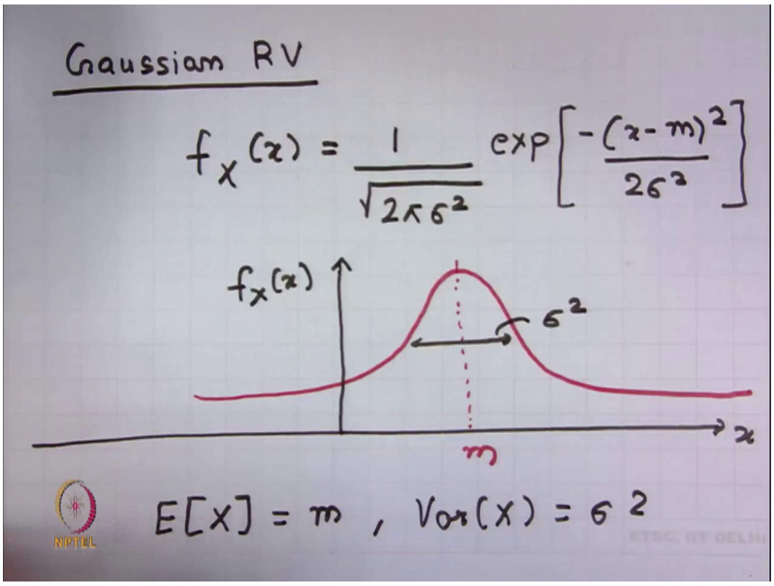
So, variance of this random variable is the second moment of the random variable minus the square of the first moment of the random variable. So, what is E X square again we can use the definition in which; so, we have already calculate the value of probability density function it is only constant in the limit a and b and this will become, ok. So, this will be. So, from this we can calculate the variance.

(Refer Slide Time: 40:28)



We already calculated the second moment. We have also calculated the first moment and simplifying this expression what we will get is this, ok. So, we have calculated the variance of a uniformly distributed random variable.

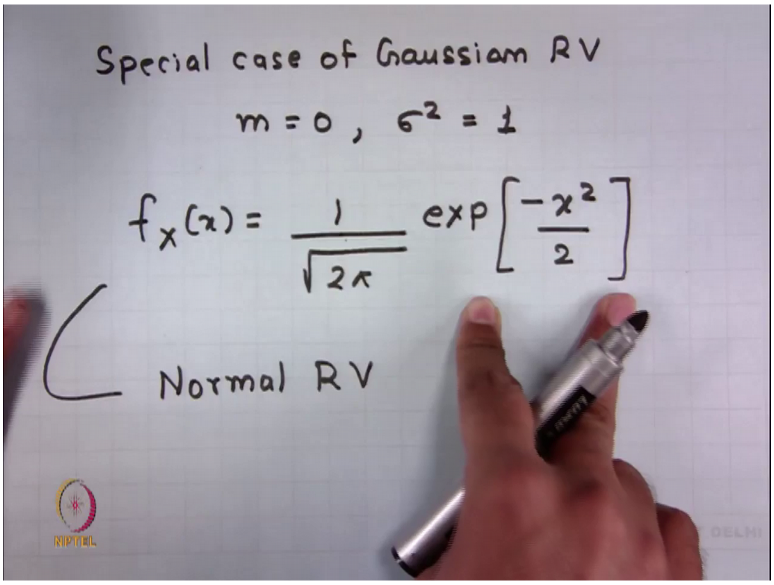
(Refer Slide Time: 41:20)



Let us now talk about another important random variable which is Gaussian random variable. So, Gaussian random variable is a random variable which have a probability density function of this form, and if you plot this probability density function this looks like a bell curve. This is a bell curve and the peak of this density function is at m and this fluctuation is proportional to sigma square and you can see and prove that expected value of this random variable is m which is the mean and the variance of this random variable is sigma square, ok.

So, this probability density function is completely specified in terms of the mean and its variance. So, if you tell me the mean and a variance of a Gaussian random variable I can write the expression of the probability density function it is completely specified the only there are only two unknowns; this m and sigma square. So, once you have m and sigma square this probability density function is completely specified and this is a very important random variable which is the Gaussian random variable.

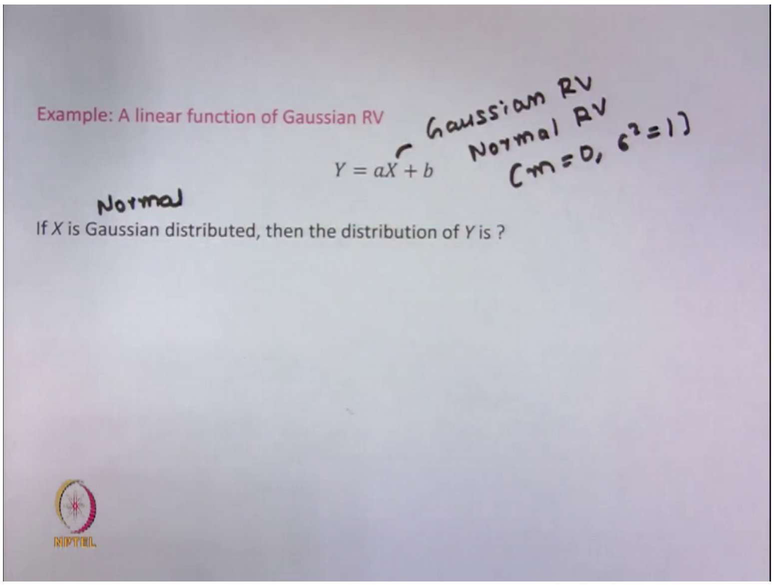
(Refer Slide Time: 43:41)



Let us now look at the special case of Gaussian random variable the special case is when you plug in this m as 0. So, the mean we are assuming to be 0 and the variance if we assume it to be 1, then what we get is the probability density function is. So, what we have done is just put m as 0 and sigma square as one and we have got a simple probability density function and this random variable is known as normal random variable, ok.

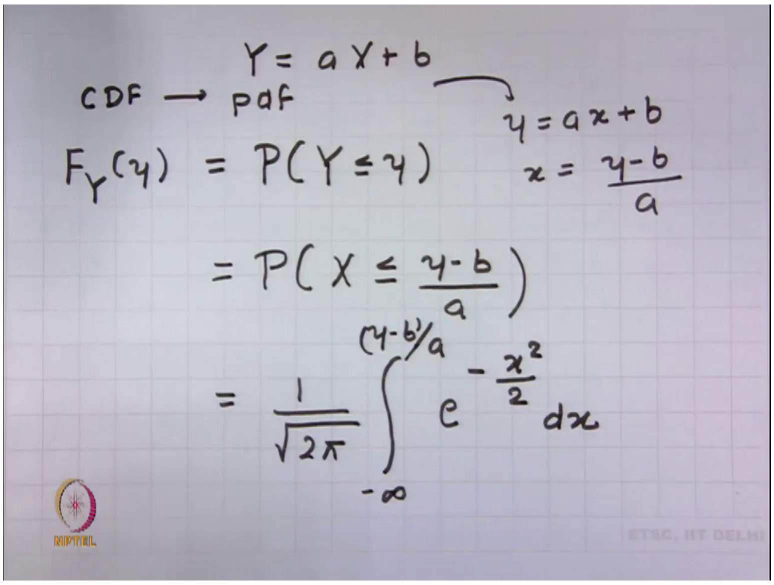
So, normal random variable is just nothing, but it is a special case of Gaussian random variable when you assume mean to be 0 and sigma square to be 1. So, there are various ways in which we can analyze this expression in better ways means we can simplify this expression further and we will use those all simplifications when we talk about detection and estimation of deteriorate, but I will like to talk about this at that point in time.

(Refer Slide Time: 45:03)



Let us come to the last concept of this lecture in which what we are given is we are given a linear function of a Gaussian random variable. So, X is a Gaussian random variable and let us take this special case of Gaussian random variable. Let us assume for simplicity X to be normal random variable; that means, it is Gaussian with the mean of 0 and a variance of 1. So, let us assume a normal random variable X and let us assume Y and Y is a linear function of this normal random variable X. So, the question that we are investigating is if X is Gaussian distributed or let us say normal distributed, what is the distribution of Y? So, this is what we have to investigate.

(Refer Slide Time: 46:06)

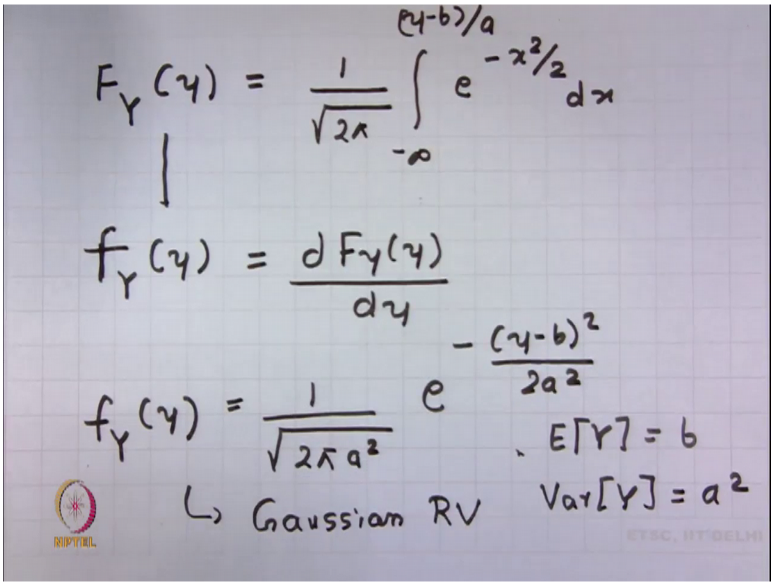


So, the question is if we have a linear function. So, Y is a linear function of a normal random variable X. So, we have to find the distribution of Y and a useful strategy to think about such situations is to try to think in terms of CDF first and then from CDF we know we can get PDF, right. So, if I have to find the CDF of y the CDF of y; that means, I am trying to find what is the probability that this random variable Y takes in a value less than small y, ok. So, that is the meaning by definition that is the meaning of the CDF of a random variable. So, we are thinking about what is the probability that this random variable Y takes in a value less than small y.

And, if you see this let us write it this equation in terms of numerical values and from this you get this; so, a simple transformation. So, if we are interested in probability that Y takes in the value smaller than this y, this is this probability is same as the probability that a random variable X takes in a value which is smaller than y minus b by a, ok. So, from this transformation and we know we can easily find this probability because we have been given the distribution of X because X is known to us it is a normal random variable.

So, let us it write this probability and the variance is 1. So, I do not have any sigma square here. It will go from minus infinity to y minus b by a and this is e to the power minus x square by 2 into dx, alright. So, this is the probability that a random variable X takes in a value less than y minus b by a. So, we have to integrate from minus infinity to y minus b by a, this is in bracket.

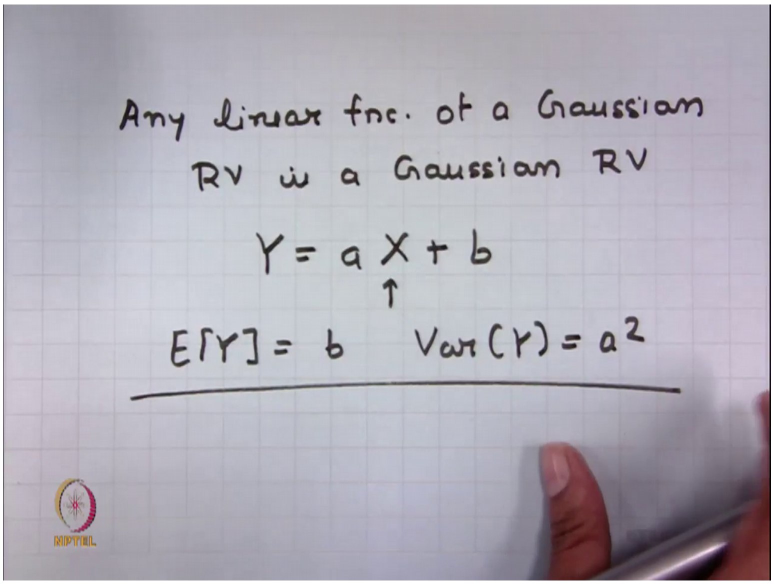
(Refer Slide Time: 48:49)



So, let me write this again what I have written is I have said that the probability that random variable Y takes in a value less than y is nothing, but is and you know how can you get from this expression how can you get PDF, we know we can we have to simply differentiate this with respect to y. So, probability density function; so, we use small f to denote that is probability density function. So, from going from here to here you just have to differentiate. So, here you can differentiate the CDF to get the probability density function of y.

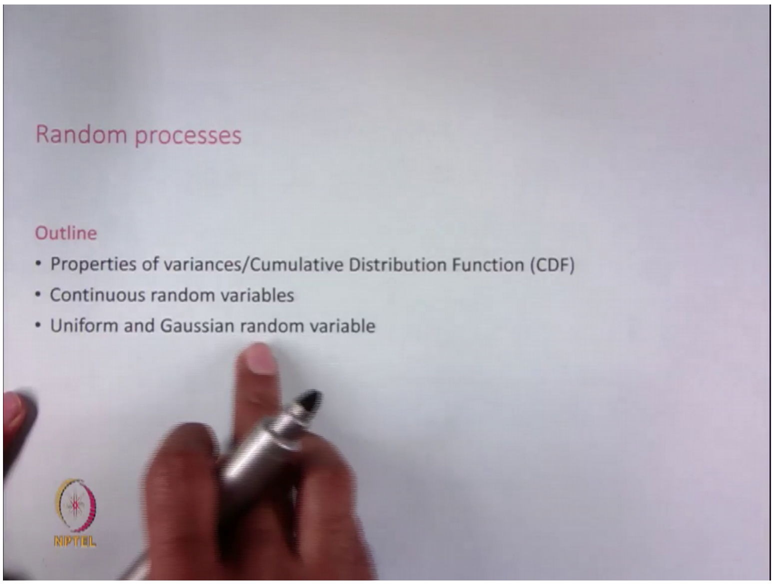
So, I just have to differentiate this expression which will give me. So, this is the probability density function of y, we have obtained this and look something interesting has happened here. The interesting thing is that the probability density function of y is a still Gaussian. So, y is a still a Gaussian random variable, ok. However, the mean of y has changed. So, mean has changed to b. So, if we write mean of Y the mean of Y is b and the variance of Y has now become a square, ok. So, the variance and mean of Y has changed, but the Y has still remained a Gaussian random variable.

(Refer Slide Time: 51:04)



And, this is an important conclusion that any linear function of a Gaussian random variable is a Gaussian random variable. So, what we have said is we have define a random variable Y which is a linear function of a random variable X: X was assumed to be normal distributed random variable, Y we found out is a still a Gaussian distributed. So, X is Gaussian distributed, Y is Gaussian distributed what has happened is the mean of Y in this case is b and the variance of Y is a square. This is very important conclusion that any linear function of a Gaussian random variable is a Gaussian random variable with a modified mean and variance. So, how does this mean and variance modifies, we will see in the next lectures.

(Refer Slide Time: 52:21)



So, this concludes the lecture for today and what we have essentially covered in this lecture is we have discussed the property of variances and we have also introduced the notion of cumulative distribution function. We have introduced what are known as continuous random variables and they are little bit tricky to handle then discrete random variables. We have also introduced two special kind of continuous random variables which is the uniform random variable and a Gaussian random variable and we have proved an important property with respect to Gaussian random variable that is a linear function of a Gaussian random variable is a Gaussian random variable. We will see more about this in the next lecture.

Thank you.