Principles of Digital Communication
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Lecture – 06
Signal Spaces: Bandwidth & Degree of Freedom

In this lecture we will be talking about degrees of freedom and bandwidth.

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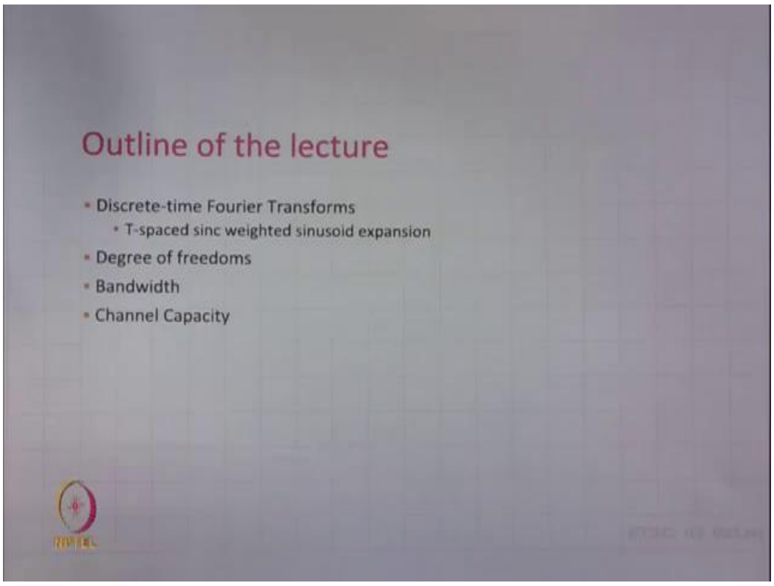

So, let us first look at the outline of the lecture. In today's lecture, we will first complete what is left from the orthogonal expansion. We will be talking about discrete time Fourier transforms and how they can be used to get what is known as T-spaced sinc weighted sinusoid expansion. We will look into what is this degree of freedom, then we will look at the bandwidth of signals and systems and then we will learn about channel capacity. So, let us first start by finishing this idea of orthogonal expansion.

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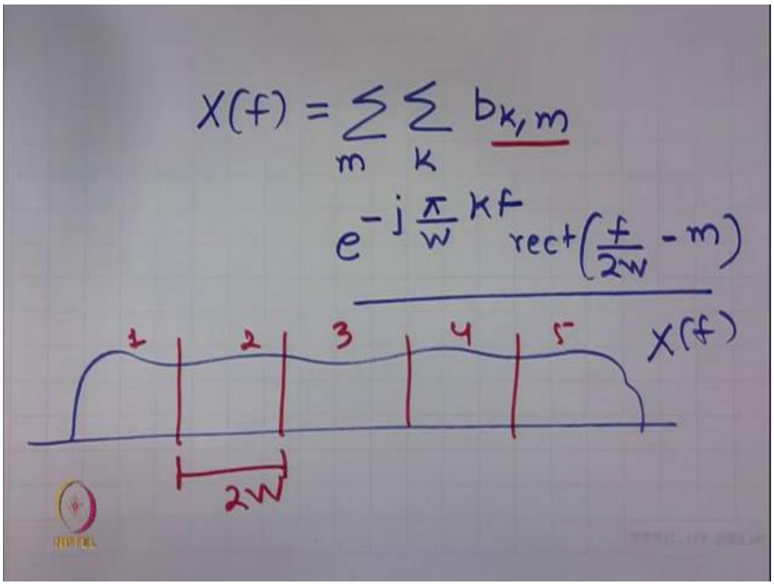

So, first of all let us write the double sum expansion built using DTFT and this expression follows from what we derived in the case of double sum expansion that was built using Fourier series which is,

$$X(f) = \sum_{m} \sum_{k} b_{k,m} e^{-j\frac{\pi k f}{W}} \mathrm{rect}\left(\frac{f}{2W} - m\right)$$

So, what we are saying in here is, again we use these as orthogonal functions to carry out a double sum expansion by using the ideas that we developed using DTFTs. So, let us assume that we have a signal and this is let us say the spectrum of the signal and if we again divide these spectrum into segments and we can have different segments and let us assume that each segment is of bandwidth 2W.

So, while considering a segment at a time, we can obtain these coefficients for these segments. So, $b_{k,m}$ is the coefficient of the m^{th} segment of the signal whose spectrum is $X(f)$ and using these $b_{k,m}$'s and these orthogonal functions we can carry out a double sum expansion. So, this is exactly same as in the case of double sum expansion of Fourier series, only we now have a different orthogonal function, everything else remains same.

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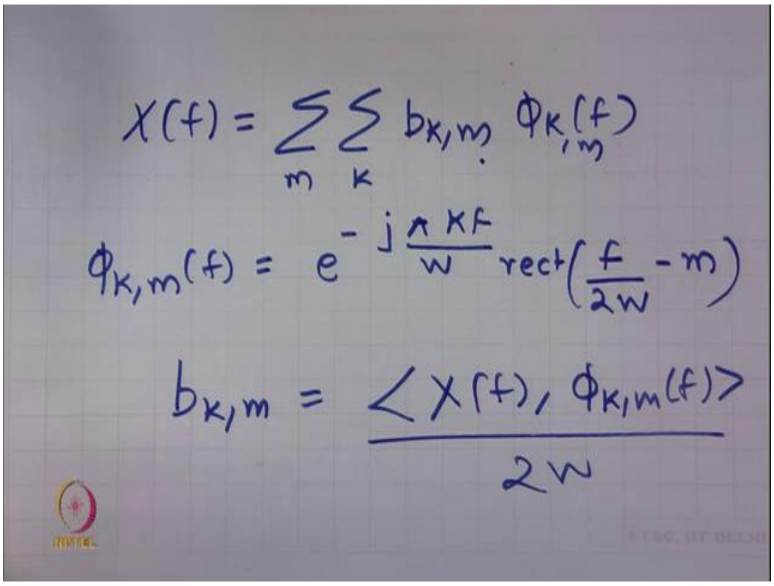

So, again you can think about this as $X(f)$ is expressed by using some coefficients multiplied by some orthogonal functions and because we are using double sum, the coefficients have two subscripts: this m subscript tells that these $b_{k,m}$'s are the coefficients corresponding to the m^{th} segment and similarly, $\phi_{k,m}$ tells that this is the orthogonal function corresponding to the m^{th} segment.

We can get these $b_{k,m}$'s by taking the inner product of $X(f)$with these orthogonal functions $\phi_{k,m}$'s, and dividing by the energy of these functions. We can prove that the energy of these functions is 2W. Energy of the functions in the case of Fourier series was T and we know that because T is getting mapped to 2W, so the energy of these orthogonal functions can be thought as 2W. Let us again now try to do what we did in the case of Fourier series. So, let us start by taking the inverse Fourier transform of this expression.

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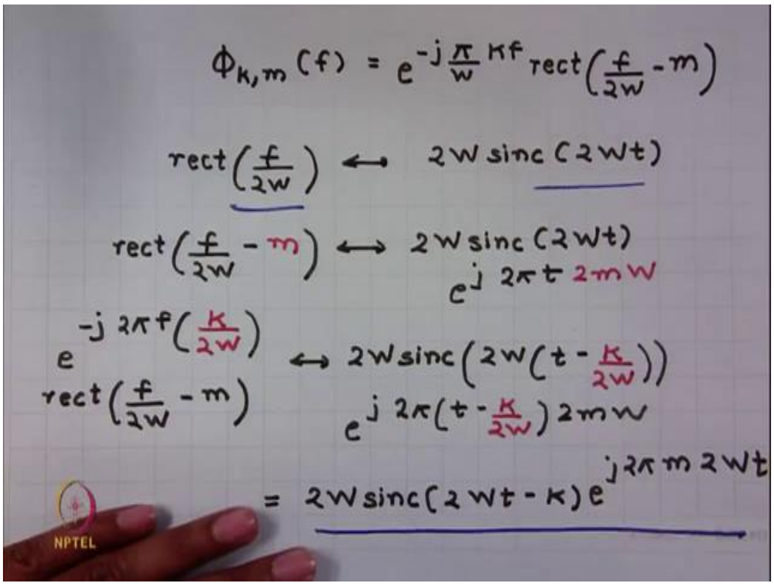

Using the ideas from Fourier world, we know that the inverse Fourier transform of a rectangular function is a sinc function. Fourier transform of shifted rectangular function can be obtained by multiplying a sinc function with a rotating complex exponential and this complex exponential will contain this factor of 2mW corresponding to this frequency shift of m2W. And the inverse Fourier transform of this shifted rectangular function multiplied by this rotating complex exponential, can simply be obtained by changing this t to $t - \frac{k}{2W}$. So, wherever you have t, there you just have to shift it by this factor of $\frac{k}{2W}$.

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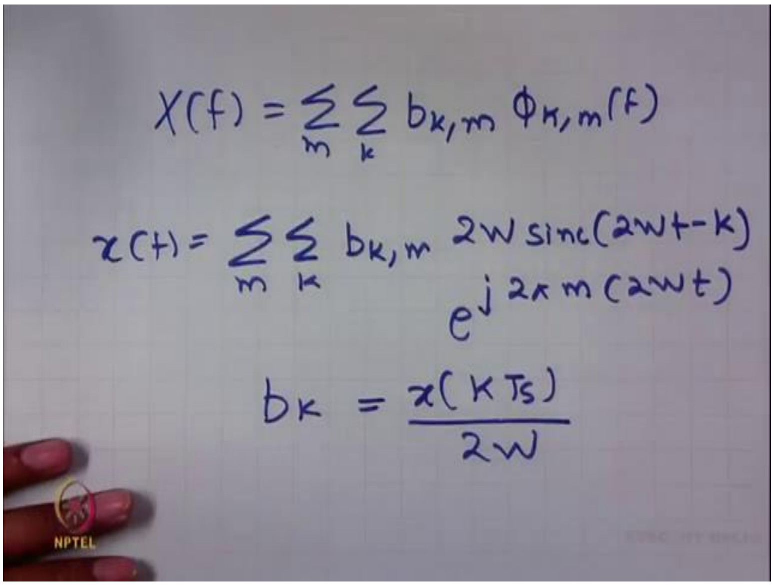

So, we are starting with these functions and we have obtained the inverse Fourier transform of these orthogonal functions as $2W\,\text{sinc}(2Wt-k)e^{j2\pi m(2Wt)}$. And by just applying mind to what these $b_{k,m}$'s could be, we can easily obtain them. For example, we have seen previously that b_k's were the samples $\frac{x(kTs)}{2W}$. So, what these $b_{k,m}$'s should be?

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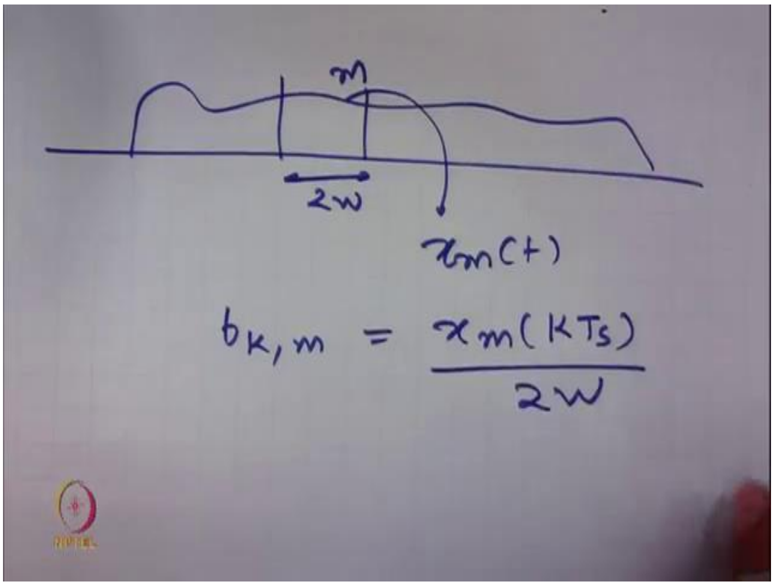

So, let us go back to that picture again. So, we are taking a segment of this signal's spectrum, the segment exist for a duration of 2W. This is the m^th^ segment. If we take the inverse Fourier transform of this m^th^ segment, we would get a time domain signal which we call as $x_m(t)$. So, these $b_{k,m}$'s are the samples corresponding to this $x_m(t)$. So, we take the samples of this $x_m(t)$ and divide it by 2W. So, this is just obtained by using our mind based on what these b_k's were.

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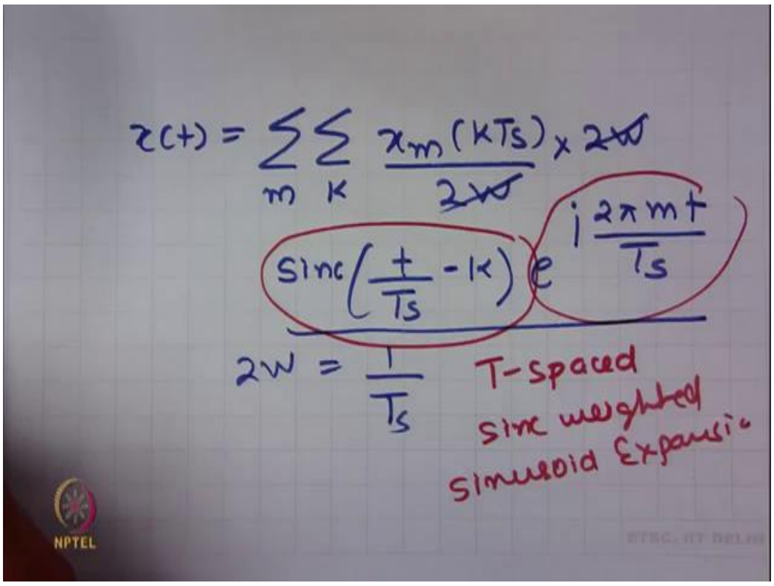

So, substituting this obtained value of $b_{k,m}$ we get

$$X(f) = \sum_{m} \sum_{k} \frac{x_m(kT_s)}{2W} \times 2W \,\text{sinc}\left(\frac{t}{T_s} - k\right) e^{j2\pi m \frac{t}{T_s}}$$

So, we have actually done two things, first we have substituted the values of these coefficients and then we have also substituted 2W as $\frac{1}{T_s}$.

So, now we get a double sum orthogonal expansion of the signal $x(t)$ using the ideas of DTFT. This goes, in the literature, by the name of T-spaced sinc weighted sinusoid expansion. So, these sinusoids are weighted by these sincs and these sincs are T spaced sincs and hence the name T-spaced sinc weighted sinusoid expansion. So, let us revise what we have done so far.

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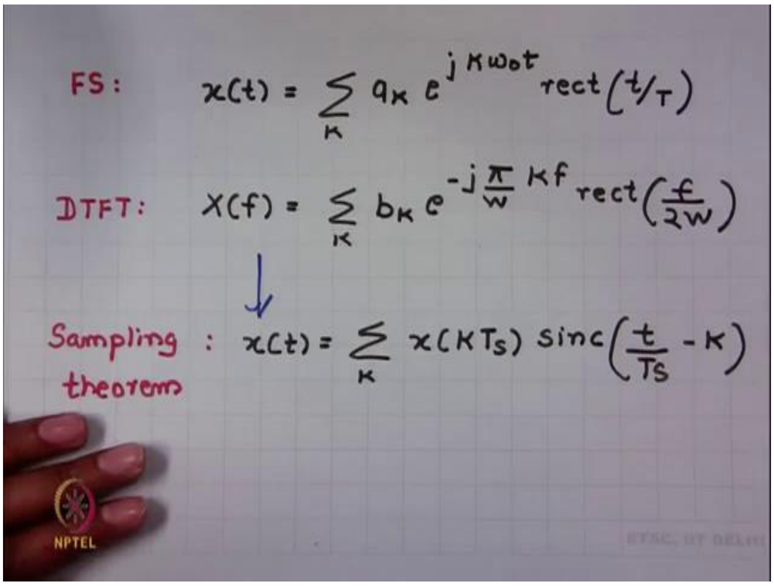

So, first we developed single sum orthogonal expansion. For example, we have looked into Fourier series. In Fourier series, we expand the signal in terms of these truncated sinusoids, then we have this DTFT which is time frequency dual of this Fourier series. If we take the inverse Fourier transform of this DTFT, we get to this sampling theorem.

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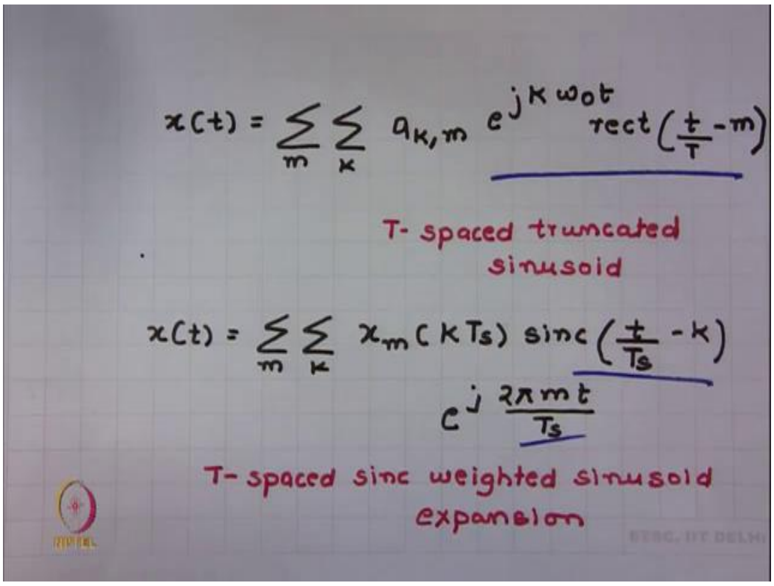

And then we also looked into double sum orthogonal expansion. So, from the idea of Fourier series we ended up with this T-spaced truncated sinusoids and using the ideas of DTFT we obtain these T-spaced sinc weighted sinusoid expansions.

So, these are the kind of orthogonal expansions that we use in practice. Let us look into the next concept. It is a very interesting concept, particularly for engineers because many rules of thumb are derived using this idea of degrees of freedom.

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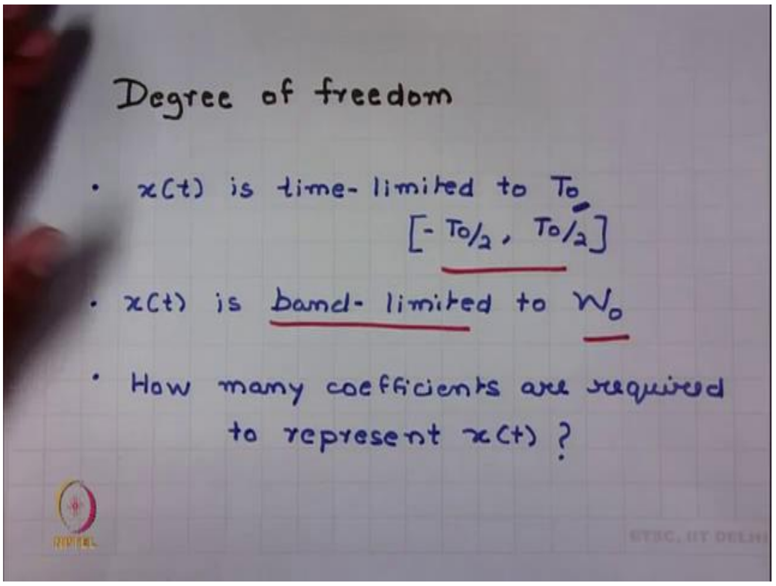

So, we assume that $x(t)$ is time limited to duration T_0 and bandlimited to bandwidth of W_0. For example, we can assume that it spans from $-T_0/2$ to $+T_0/2$, then the number of coefficients that would be required to represent the signal $x(t)$ is known as the degree of freedom of the signal $x(t)$. So, let us look at the number of coefficients that would be required.

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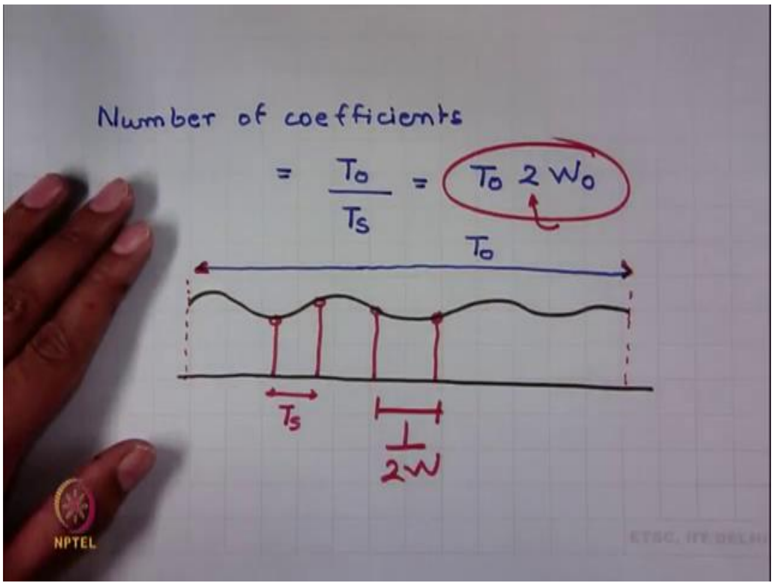

It follows very simply from the ideas of sampling theorem that we have just seen. So, if a signal is of duration T_0 and if W is the bandwidth of the signal then we have already seen that we need to collect the samples of the signal at a duration of $\frac{1}{2W}$. Then the number of coefficients that would be sufficient for the representation of the signal, can be simply obtained by dividing the total duration by the period at which samples needed to be collected and this number would be $2W_0T_0$.

So, this is typically the degree of freedom of a signal and this factor 2 is popping up here because of the notations and the way in which we define the duration of a signal in time domain and in bandwidth domain. When we define the duration of a signal in bandwidth domain, we only define for the positive side of the spectrum, whereas when we define the duration of a signal in time domain, we take into consideration the total duration of the signal. We can also interpret this degree of freedom in another way.

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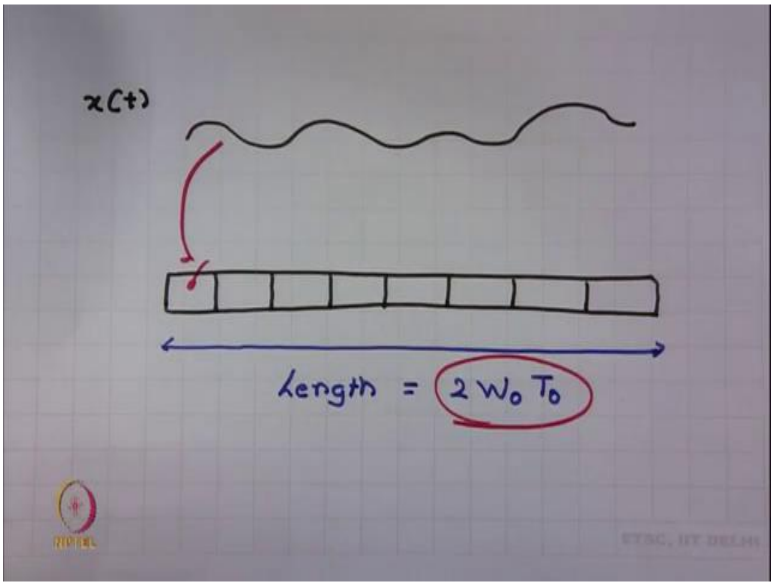

So, if we have a signal $x(t)$ and we want to store this signal into an array and we have already said that this is important for signal processing ideas. We cannot store a continuous time signal. We need to convert this signal into sequences and so, the question is: what is the length of the array required to store this signal? And again the length will correspond to $2W_0T_0$.

The kind of numbers fill in here depends upon what kind of signal is $x(t)$. So, if $x(t)$ is a real valued signal these numbers are real numbers. So, for a real valued signal $x(t)$, which is time limited to a duration of T_0 and bandwidth limited to a bandwidth of W_0, the number of real numbers that we require is $2W_0T_0$. If $x(t)$ is a complex valued signal, then the number of complex numbers that we require to represent the signal $x(t)$ is again $2W_0T_0$.

But there is a mathematical intricacy that we have avoided so far in this discussion and this mathematical intricacy is that the time limited signals can never be frequency limited signal or bandwidth limited signal. We have seen this in signals and systems that if you squeeze a signal in time domain, then the signal is spread out in frequency domain. If you spread a signal in time domain, then the signal gets squished in frequency domain. This is because time and frequency are reciprocal spaces. So, making a signal time limited would make it bandwidth unlimited. Thus, this assumption that the signal is time limited and bandwidth limited at the same time does not hold.

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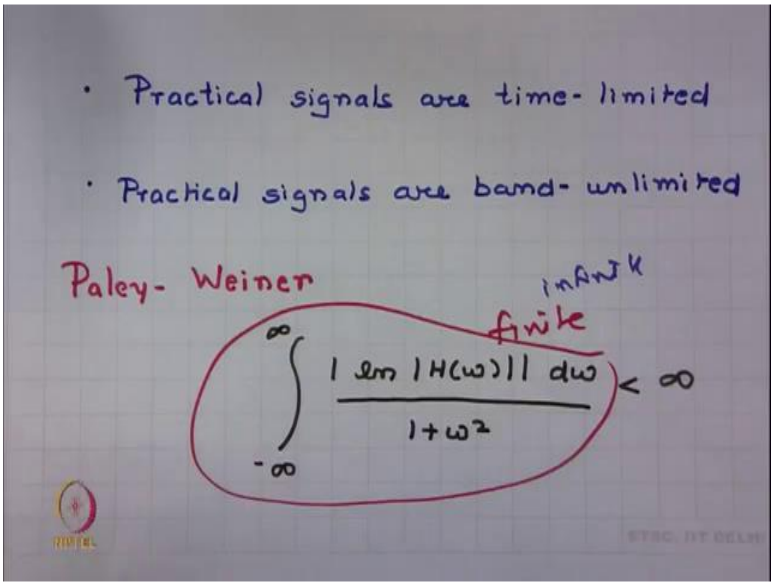

Moreover, we can see that practical signals will always be time limited. If we want to transmit a signal, then duration of the signal has to be finite. We cannot transmit a signal spanning from - ∞ to + ∞ and also we can see that these practical signals will be band unlimited and this follows from this Paley-Weiner condition which says that, for the practical realizability of the signals or systems,

$$\int_{-\infty}^{+\infty} \frac{\left|ln|H(\omega)|\right| d\omega}{1+\omega^2} < \infty$$

So, Paley-Weiner theorem says that this complicated integration should be finite for practically realizable signals or systems.

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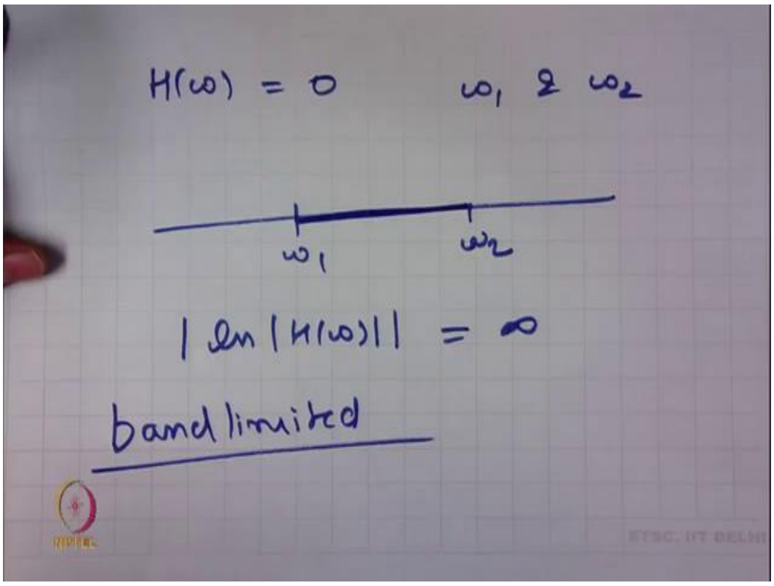

In another word, it says that if $H(\omega) = 0$ for $1 < \omega < 2$ then $\left|ln|H(\omega)|\right| \rightarrow \infty$ in this frequency range and hence this integration would become infinite. Hence if $H(\omega) = 0$ for a continuous range of frequencies then this integration would become ∞ and Paley-Weiner condition would not hold. Thus, for Paley-Wiener condition to hold, $H(\omega)$ cannot be 0 for a continuous range of frequencies and thus a signal can never be band limited. So, what we are saying is all practical signals need to be band unlimited and time limited.

So, then how can we talk about this degree of freedom which was a nice idea that we can represent the signal by using these number of coefficients? And the idea is simple. When we are saying that this can be represented using these many coefficients, we assume that whatever we are losing out does not matter for us. So, from practical sense you would lose out energy when you do this conversion, but the loss in the energy is of no practical consideration and from engineering sense this number makes sense. So, what are the examples of the signals that can be approximated as band limited signal?

No signals or no systems will be band limited ideally, but we will investigate what are the good examples of signals or systems which can be approximated to be band limited signals or systems.

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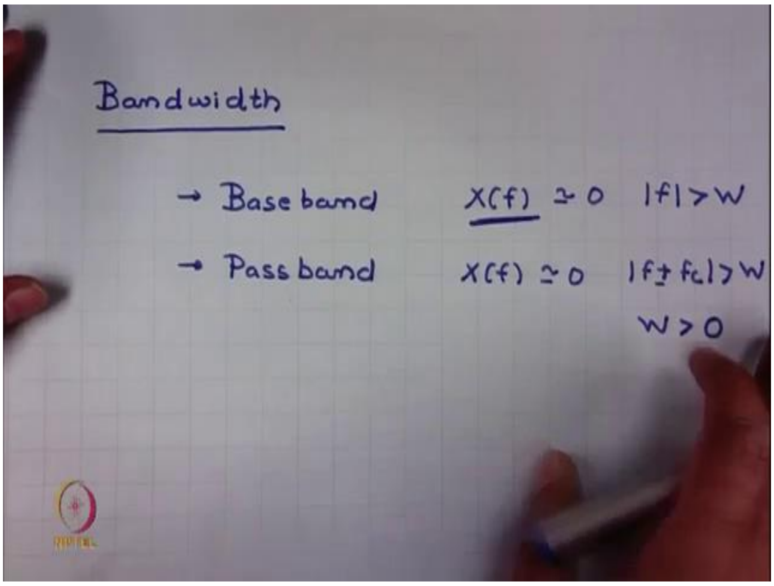

But before all of this, let us talk about these two kinds of signals/systems: the baseband signals/systems and the passband signals/systems. So, mathematically we can say a basement signal/system is a system whose spectrum is mostly concentrated at around dc and a passband signal/system is a system whose spectrum is mostly concentrated at around certain frequency f_c. This is easy to understand pictorially.

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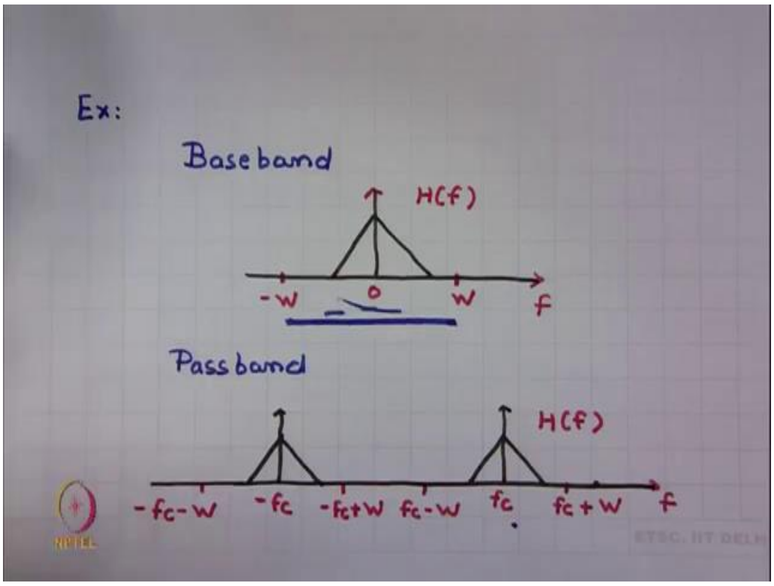

So, this is the spectrum of the basement signal or a system where most of the energy is concentrated at around dc or is confined in this range. For passband, most of the energy will be concentrated at around a higher frequency f_c and these two kinds of signals and systems are important.

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We have already seen this picture of attenuation versus frequency profile of various kinds of channels and here you see that optical fiber has a low attenuation for very high frequencies. So, optical fiber is a passband channel. Coaxial cable is also a passband channel. Twisted pair can be used both as passband or baseband channel. So, you can transmit digital data directly over twisted pair and that is what is used in Ethernet LANs or you can use discrete multi-tone modulation where you transmit the signal in form of passband signals.

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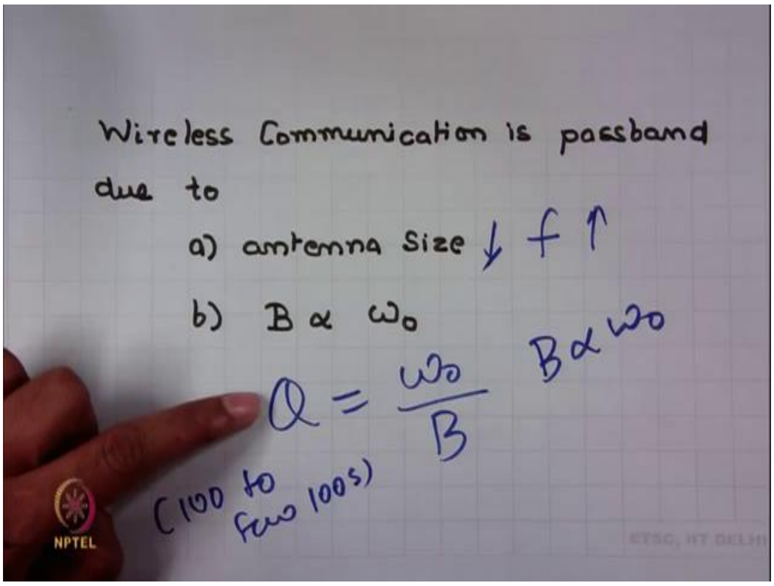

Wireless communication is mostly passband because we know that if you can operate communication over higher frequencies, antenna size reduces. And so, there is an incentive to have communication systems at higher frequencies, so you can reduce the antenna size. And because we now want to make our devices smaller and smaller, passband communication is preferred in wireless systems.

Also we know from electrical world, that the quality factor of electrical circuits is given by $Q = \omega_0/B$ and because this quality factor is rather a constant quantity, it lives with between 100s to few 100s. So, we can approximately treat this as constant and hence bandwidth is proportional to ω_0, where ω_0 is the frequency of operation. So, as the frequency of operation increases, bandwidth of the system also increases and that is the second reason why we prefer this passband communication because it has the possibility to offer higher bandwidth.

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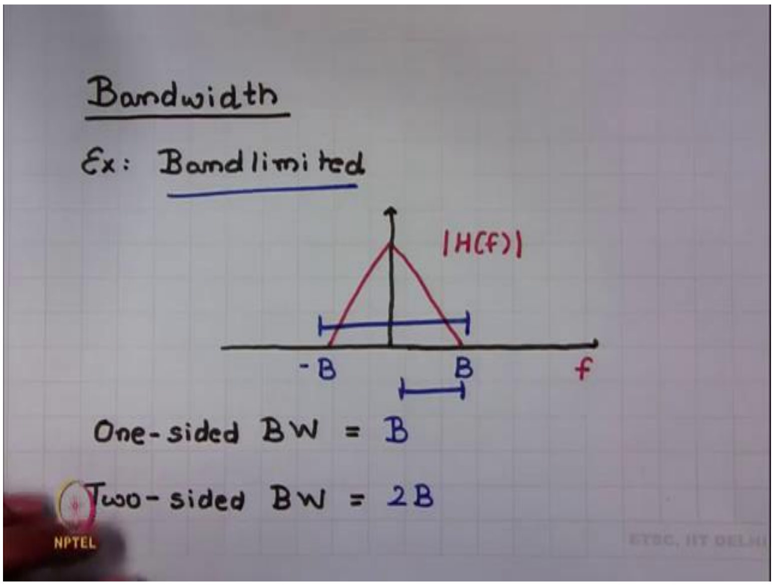

We have not yet talked about this bandwidth concisely, so let us look into what is this bandwidth and the definition of bandwidth is quite easy if the signal in consideration is band limited signal. For example, if the spectrum of a signal or mod of a spectrum of a signal is like this we know that it is a band limited signal or it is a good approximation towards band limited signal. So, how do we define bandwidth? One idea is just to look at the positive side of the spectrum and then the bandwidth will be B. If you look at both positive and negative side of the spectrum, then the bandwidth will be 2B. We call the former as one sided bandwidth and the latter as two sided bandwidth.

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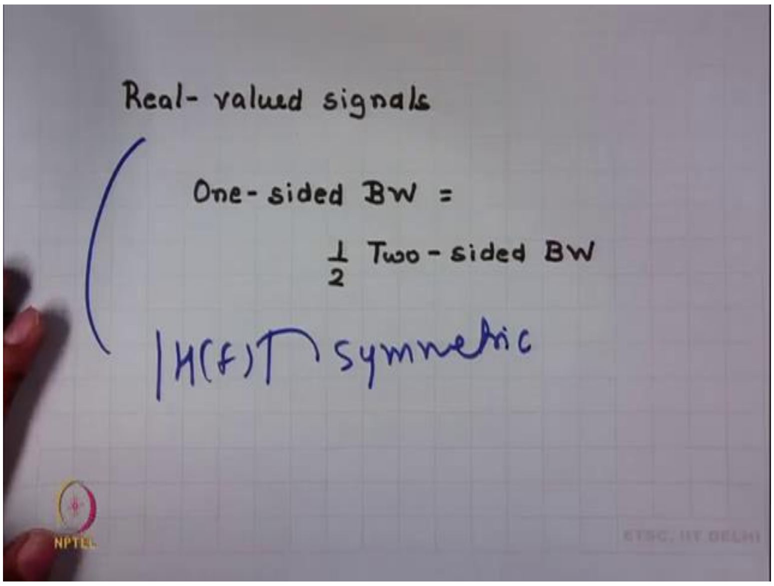

And we know for real valued signals, because $|H(f)|$ is symmetric, one sided bandwidth is always going to be half of two sided bandwidth.

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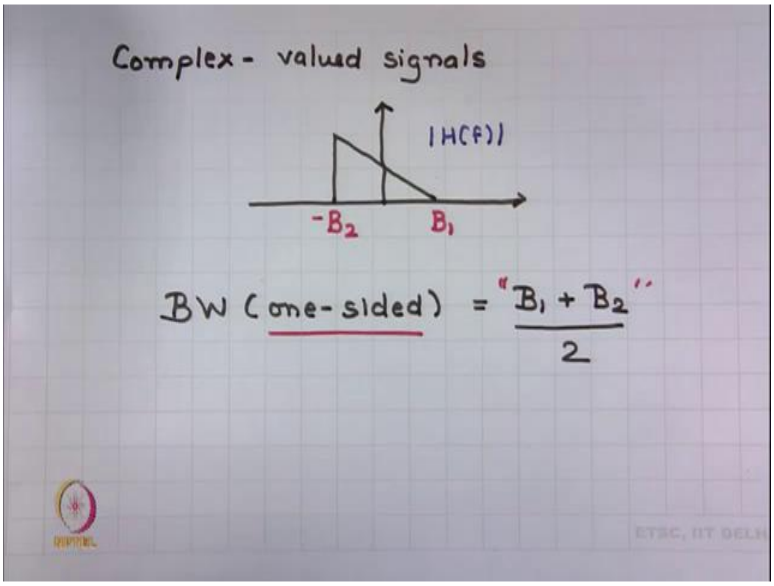

However, for complex valued signals, $|H(f)|$is not symmetric. So, $|H(f)|$ might look something like this. Now in this case our good old rule that one sided bandwidth is half the two sided bandwidth does not hold. So, in literature, the definition of one sided bandwidth, in the case of complex valued signals is obtained by looking at the total support; that means, we consider both positive side of the spectrum and negative side of the spectrum. So, we get the total support as $B_1 + B_2$ and divide this by 2. So, this is the definition of one sided bandwidth in case of complex valued signals. But if nothing more has been specified about the bandwidth, whether the bandwidth is one sided or two sided bandwidth, we consider the bandwidth as one sided bandwidth.

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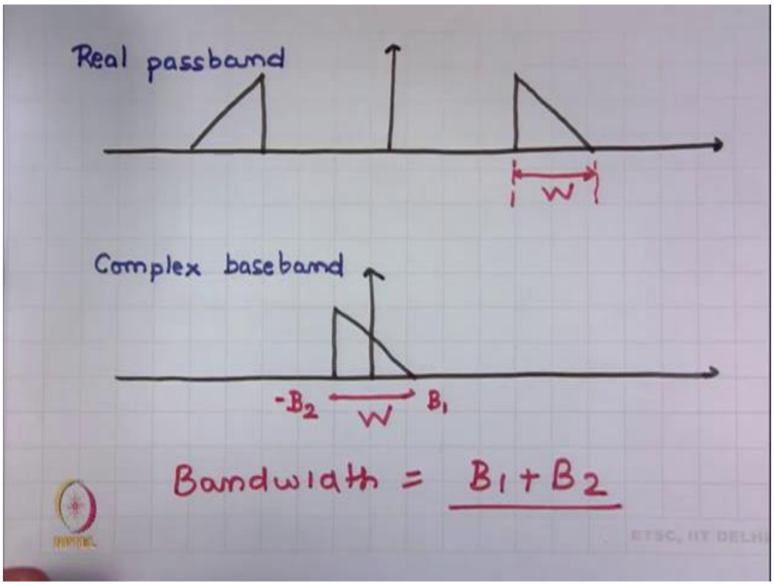

Another definition of bandwidth for complex valued signals can flow from the idea of equivalence between the real passband signal and complex baseband signal. We will see later on that a real passband signal has an equivalent representation in baseband domain and this representation of a real passband signal in baseband domain is normally complex. The complex valued signals arise because of this equivalence between passband and baseband signals because no signal can exist in reality as a complex signal.

So, whenever we are saying complex valued signals, it is an analytical signal which models the behavior of a real passband signal. So, for the bandwidth of this real passband signal, we can look at only the positive side of frequencies if we are talking about the one sided bandwidth. For the complex baseband signal, the bandwidth should be considered by looking at both side of the spectrum. Thus from the equivalence point of view between the real passband signals and complex baseband signal, the bandwidth of a complex baseband signal can be considered as the total bandwidth. Let us now try to define bandwidth for band unlimited signals.

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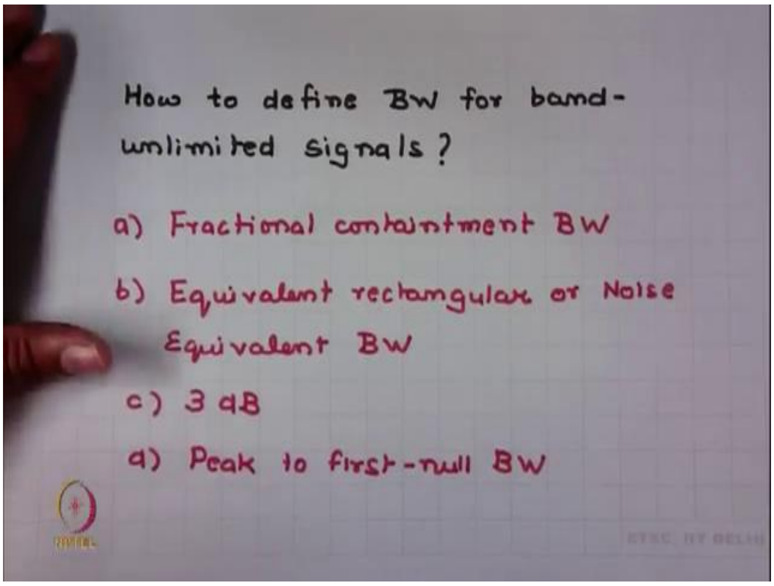

So, when we are thinking about band unlimited signals, there are various definitions of bandwidth that we can think about. We will take all these four definitions one by one. So, we have fractional containment bandwidth, equivalent rectangular bandwidth, 3 dB bandwidth and peak to first null bandwidth.

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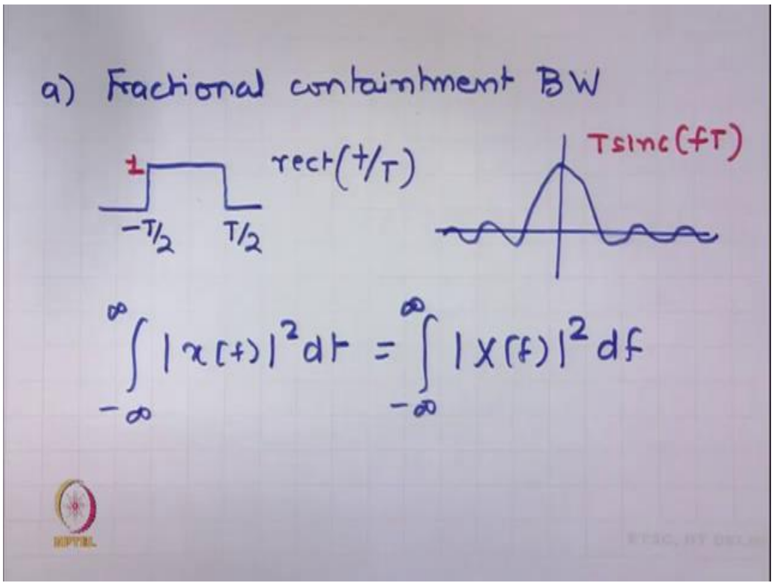

So, let us start with the idea of fractional containment bandwidth which says that we are interested in the range of frequencies for which a fraction of energy is contained. Suppose we have a rectangular pulse so, we denote this as rect (t/T) and and it takes an amplitude 1 between – T/2 to + T/2.

If we take its Fourier transform, we get a sinc pulse and the expression for the sinc pulse is $T\,\mathrm{sinc}(fT)$. If we are interested in the energy of this pulses, we can use Parseval's theorem to either think about the energy in terms of a time domain signal $x(t)$ or in terms of frequency domain signal $X(f)$.

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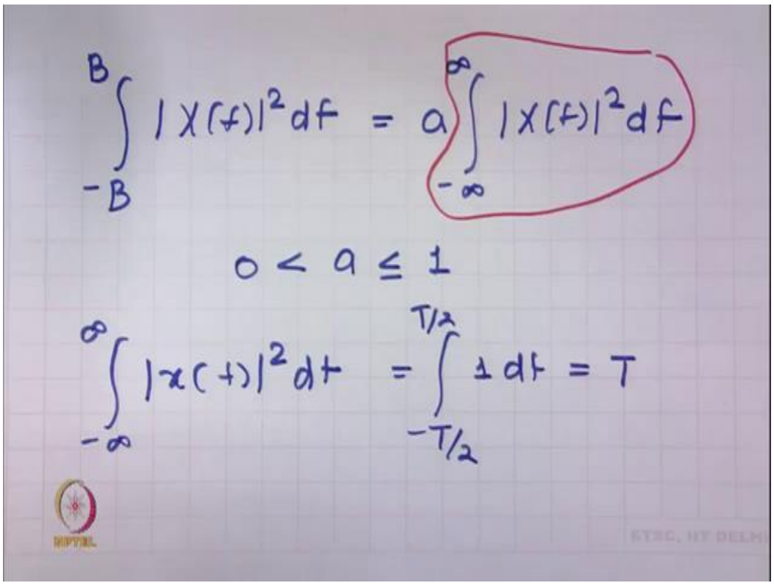

Now, we can define bandwidth as B for which this equality follows. If we carry out this integration between -B to +B, we get certain energy and this energy should be some fraction times the total energy. Here, a is a fraction whose value will go from 0 to 1 and for this equality whatever the value of B we get, that is the bandwidth.

Now, thinking about this total energy directly is little bit inconvenient, but these two are equivalent expressions from Parseval's theorem. Now this is easier integration to do because $x(t)$ is a rectangular function which is 1 only between – T/2 to + T/2. So, we can simply go from – T/2 to + T/2, have 1 in here and this would easily integrate to T.

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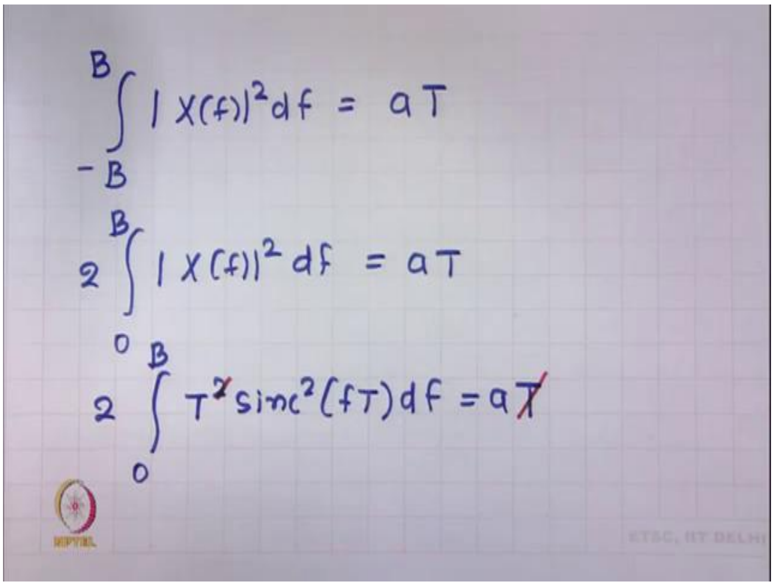

That means we can think about the bandwidth by solving for this expression and now let us try to simplify this bit more before we resort to computer to solve it, we cannot solve it by hands. It should be solved using computers but let us simplify this little bit further by thinking about that $|X(f)|$ is a symmetric function.

So, rather than carrying out this integration, it would be simpler to just carry out the integration between 0 to B and just multiply with 2. So, from this we have to solve for this expression. Substituting the value of $|X(f)|$, we see that T cancels with one T and finally, we have to solve for this expression.

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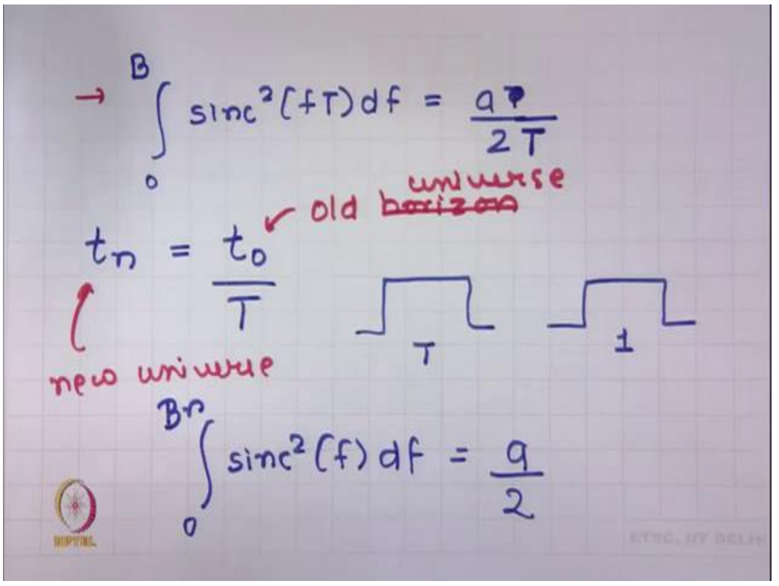

Now we can even simplify this further by changing the units of the universe. So, we can define a new universe in which we have changed the time scale. So, in the new universe, the time scale can be obtained by dividing the time by T. So, this T_0 is the time scale in old universe and this is the time scale that we choose in a new universe. Thus, the pulse which was of a duration T in the old universe becomes a pulse of duration 1. So, we can simply replace this T with 1 and we can rather solve this equation because now T is 1 in the new universe and now we can find the value of B_n. Since we have changed the time scale in the new universe so the bandwidth will also change in the new universe.

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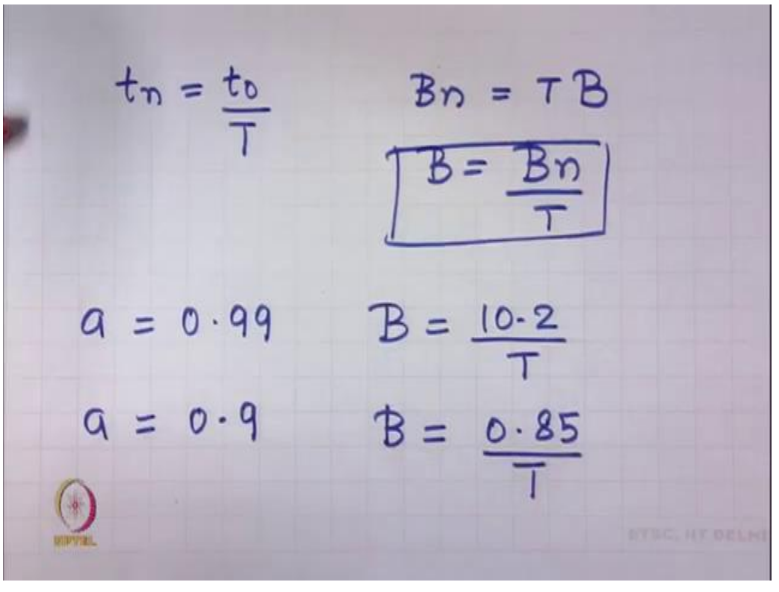

Since time and bandwidth follow a reciprocal relation, so bandwidth in the new universe would be T times the bandwidth in the old universe. So, bandwidth in old universe is B_n/T. So, we can get the value of B_n, which is the bandwidth in the new universe, and find out the bandwidth in the old universe, in which we were interested, by just dividing this bandwidth in the new universe by T. These are certain tricks that we can use. We could have solved this equation directly in computer.

Let me put out some values of bandwidth for values of a. So, if we choose $a = 0.99$, that means we want 99 % of energy to be confined, the bandwidth turns out to be 10.2/T and if we choose $a = 0.9$, the bandwidth that we need to have is 0.85/T. So, based on what fraction of energy containment we want to have, the bandwidth requirement varies. Of course, bandwidth should be a function of T.

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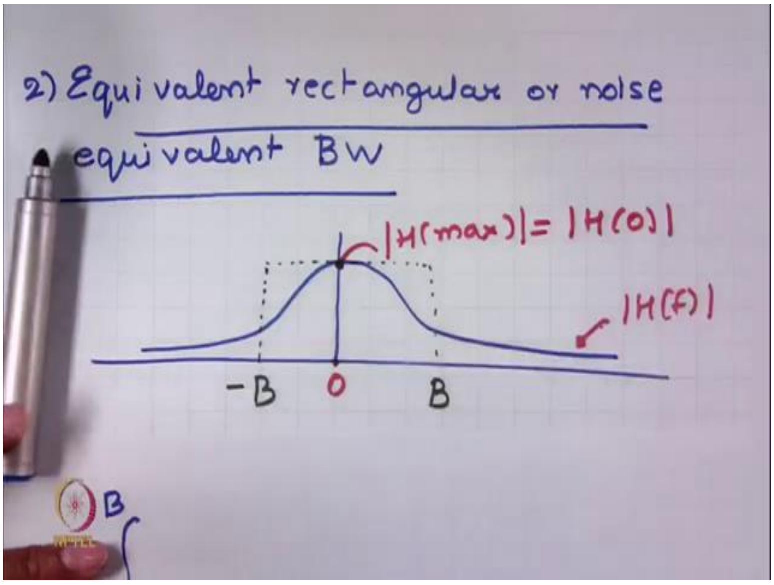

Let us take the second definition of bandwidth which is equivalent rectangular or noise equivalent bandwidth. Idea behind this definition is that if we have been given a particular $|H(f|$ then we can find the equivalent bandwidth of the system by thinking about putting a rectangular cap around $|H(f)|$ where the rectangular cap is centered around the peak value of $|H(f)|$. This rectangular cap takes a total support of 2B. The bandwidth is defined as the value of B for which the energy contained in this rectangular cap is the same as the energy contained in $H(f)$.

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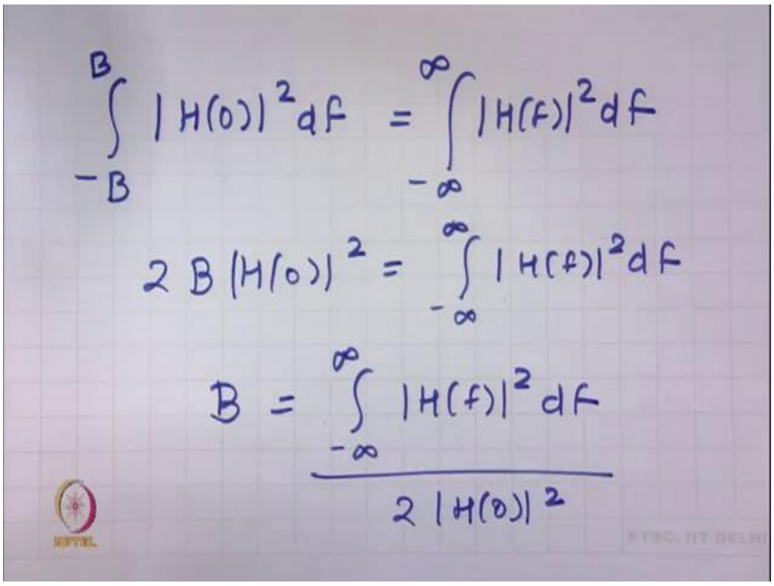

So, the rectangular cap goes from - B to + B and the height of this rectangular cap is same as H_{max}. For simplicity, let us assume that the peak value of this $|H(f)|$ happens at f = 0, so the magnitude of this is nothing, but $|H(0)|$ and the energy thus would be $\int |H(0)|^2 df$. This is the energy that would be contained in the rectangular cap and the total energy that would be contained in $H(f)$ can be evaluated using this expression. And because this quantity is independent of frequency, we can pull this out and from here, we can find the bandwidth. Let us try to do this for a sinc function because we did the previous examples with sinc.

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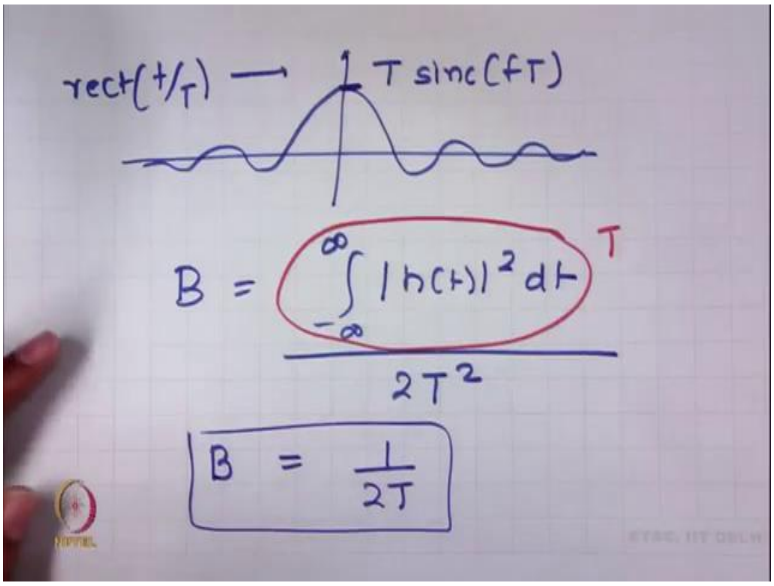

Let us again consider a sinc pulse. So, we have been using the same pulse for the comparison between the two definitions. And so, if we pluck this into this expression, again as we said finding this out directly will be very difficult. So, let us think this in term of this divided by $2H(0)$. Here the peak value is $H(0) = T$. So, this is $2T^2$ and this, as we evaluated in the previous case, is nothing but T. So, this will evaluate to T and the bandwidth with this definition turns out to be 1 / 2T. So, this is the equivalent rectangular bandwidth or noise equivalent bandwidth.

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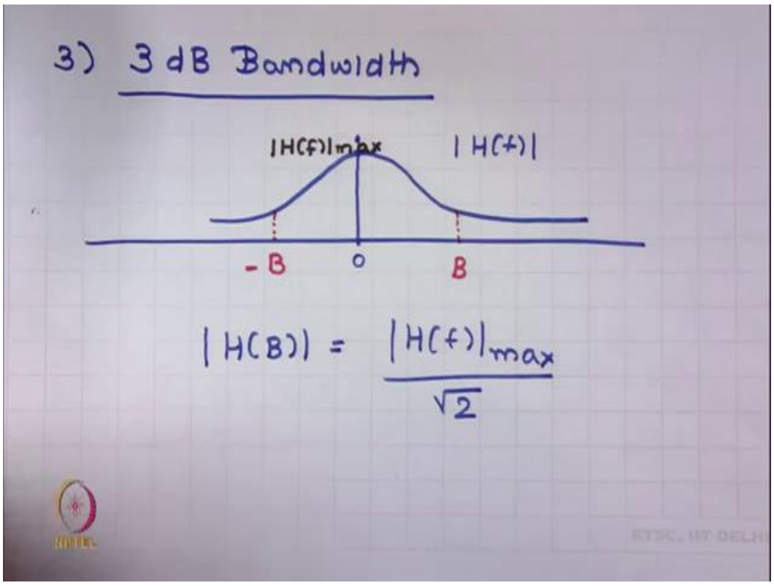

Now, let us look at the third definition that is 3 dB bandwidth. 3 dB bandwidth is defined as the range of frequencies for which $|H(f)| = |H(f)|_{max}/\sqrt{2}$. This is the most commonly used definition of bandwidth that is adopted in measurement instruments or the function generators.

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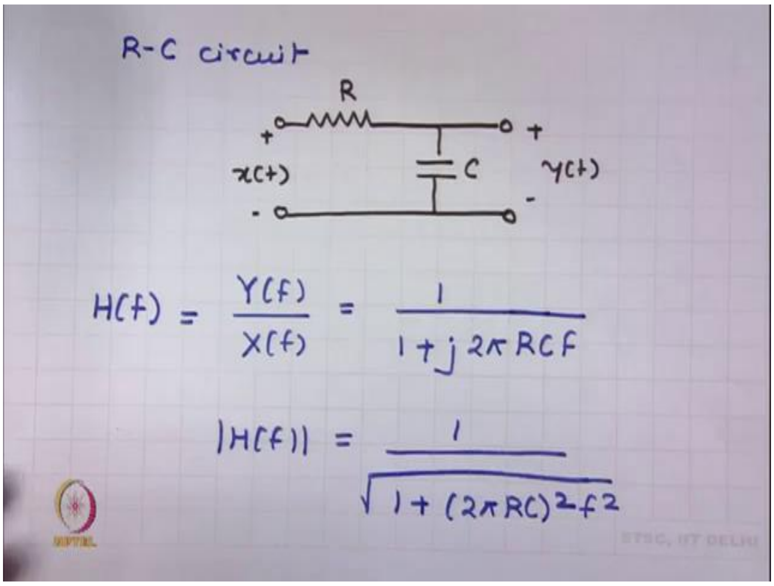

So, let us try to understand this 3 dB bandwidth with a common example from electrical circuit world. So, here we have the example of RC circuit. So, you must have seen that the frequency response of an RC circuit is given by $1/(1 + j2\pi RCf)$. If we find the mod of this frequency response, it can be easily obtained by this expression. Now what is the maximum value of this $|H(f)|$? This will have the maximum value when this denominator achieves the smallest value and this will be the case when f = 0 and hence the maximum value of $|H(f)|$ is 1.

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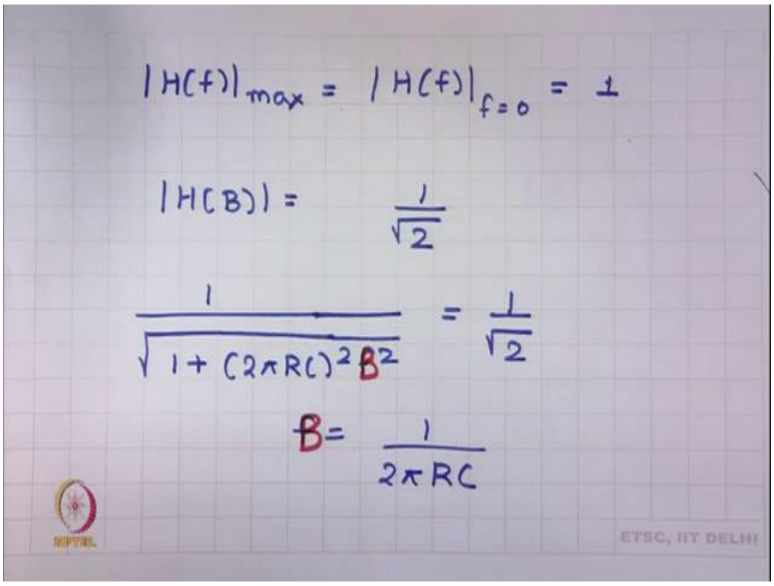

So, we want B for which $|H(B)| = 1/\sqrt{2}$. Now substituting $|H(B)|$ in this expression we can find out the value of B. From this we can find that $B = 1/2\pi RC$ which is the 3 dB bandwidth for an RC circuit. There is a last definition that we have to look into and that is the peak to first null bandwidth.

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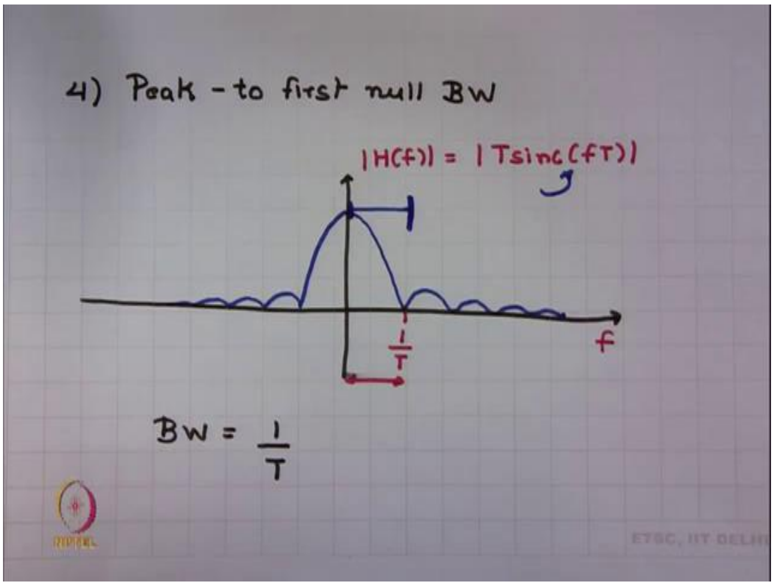

So, we have plotted the $|H(f)|$of our good old sinc pulse, and for this definition of bandwidth, we have to take the difference between the first null and the peak position. And this difference is very conveniently obtained as $1/$T. So, if we consider the bandwidth of the sinc pulse based on this idea of peak to first null bandwidth, the bandwidth of the sinc pulse becomes $1/$ T.

So, there are different definitions of bandwidth that we have considered and not surprisingly, these different definitions of bandwidth have given us different results and thus the choice of definition of bandwidth depends upon its context.

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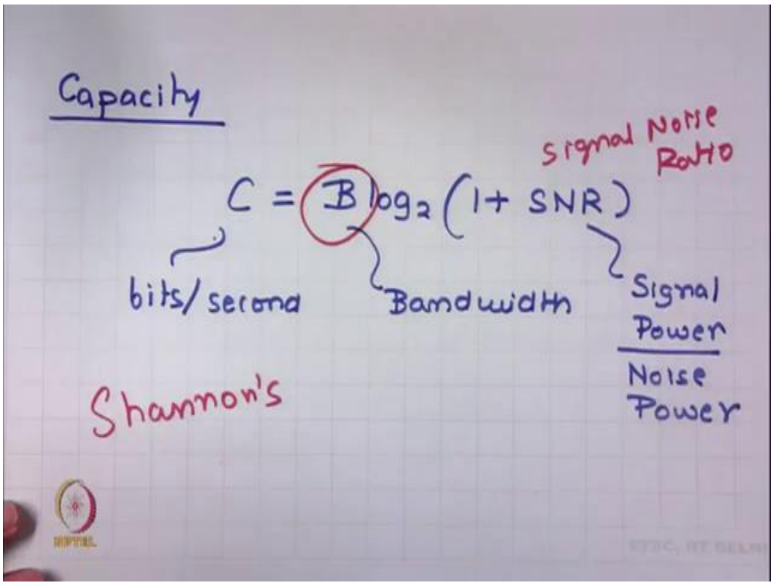

Lastly, we have to look at the capacity which illustrates why this bandwidth is so important. So, if you look at the capacity of a communication system which is in bits per second. So, how fast your internet can be, depends upon this factor. This capacity is $B\log_2(1+\mathrm{SNR})$, where B is the bandwidth of the communication system and SNR represents signal to noise ratio, i.e., signal power divided by noise power and this expression came from Shannon's celebrated work on communication system.

Essentially, it says that you can increase the capacity by either increasing the bandwidth of the communication system or by increasing the signal power. So for a higher capacity, we want to have a larger bandwidth and this bandwidth is thus very important.

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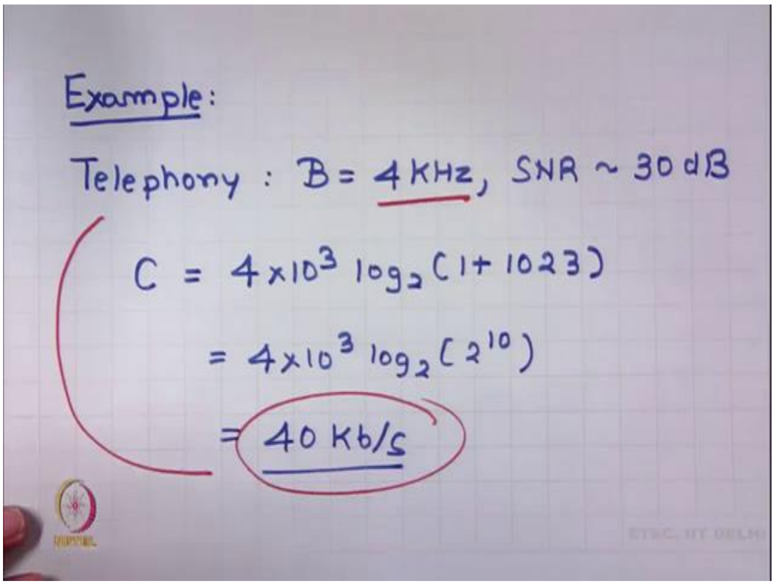

Let us look at what are the typical numbers for bandwidth and signal to noise ratio for a telephone system. So, the bandwidth of a telephone system is restricted to this 4 kHz, signal to noise ratio is typically around 30 dB and thus you can find that the capacity of a telephone system is around 40 kbps.

So when communication system started, people were trying to make communication systems which can achieve this theoretical value of capacity. This is a theoretical value of capacity meaning that this is the best that you can have. A communication systems capacity is upper bounded by this value of C. So, when you are designing a communication system your objective is to reach to this value of upper bound of the capacity that is given by the Shannon's equation.

So, with this we have completed this unit and this lecture. In this lecture today, we have learnt about what is the bandwidth, degree of freedom and we have also seen T-spaced sinc weighted sinusoid expansion. In the next unit, we will start by looking into noise because as you can see that the capacity and the performance of a system depends upon this noise power and hence it is very important to understand what noise is and what are the typical values of noise power. So, we will start with this in unit 2 from next lectures.

Thank you.