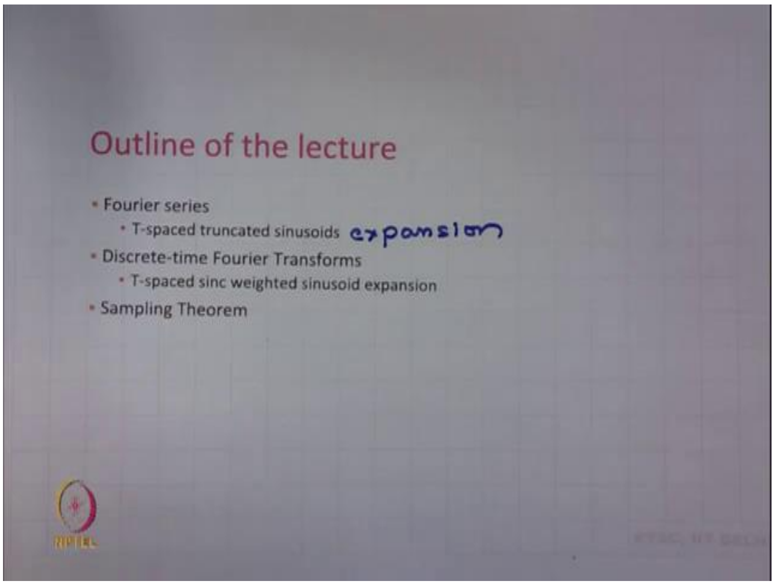
**Principles of Digital Communication**
**Prof. Abhishek Dixit**
**Department of Electrical Engineering**
**Indian Institute of Technology, Delhi**

**Lecture – 05**
**Signal Spaces**
**Fourier Series & Related Expansions**

In this lecture we will be talking about Fourier series and related expansions.

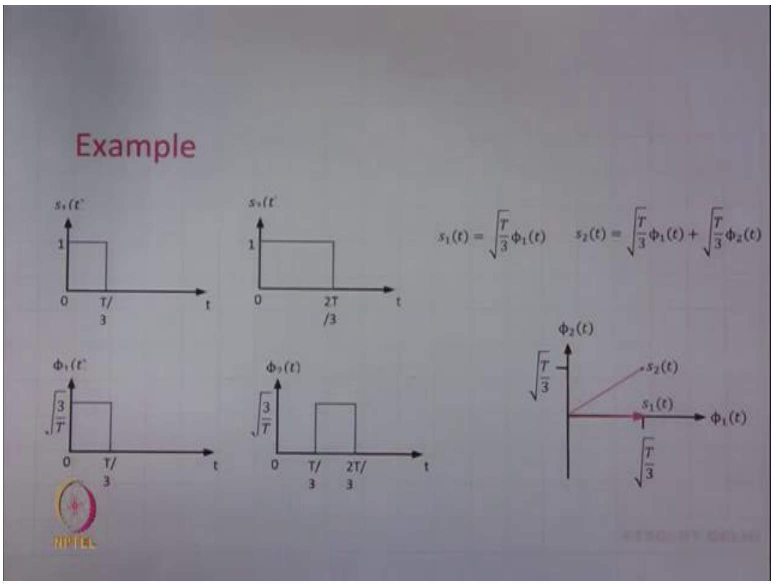
(Refer Slide Time: 00:31)




We will first talk about the Fourier series and from Fourier series we will learn about T-spaced truncated sinusoids expansion, then we will talk about discrete time Fourier transforms and from discrete-time Fourier transforms, we will learn about T-spaced sinc weighted sinusoid expansions and then, we will also learn in this lecture about sampling theorem.

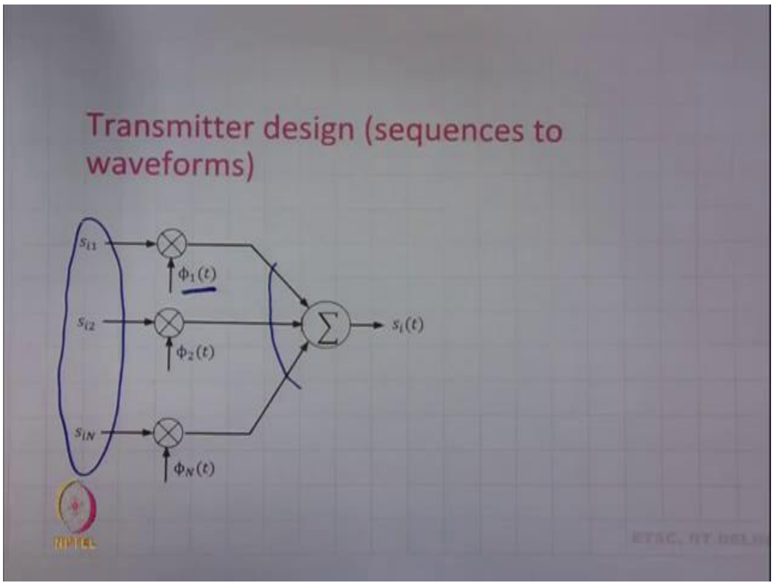
So, this is the outline of the lecture, but let us first recap what we have learnt in the last lecture.

(Refer Slide Time: 01:03)



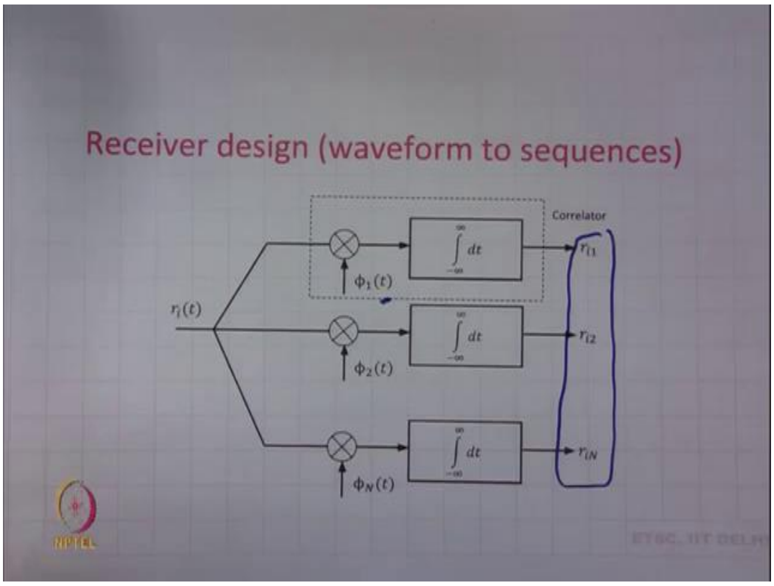

We have started with the idea of Gram Schmidt orthogonalization procedure. Using this procedure, we can convert signals into vectors by finding out the basis functions and we have to find the coefficients of these signals along those basis functions. Then we have also studied about the transmitter design.

(Refer Slide Time: 01:30)



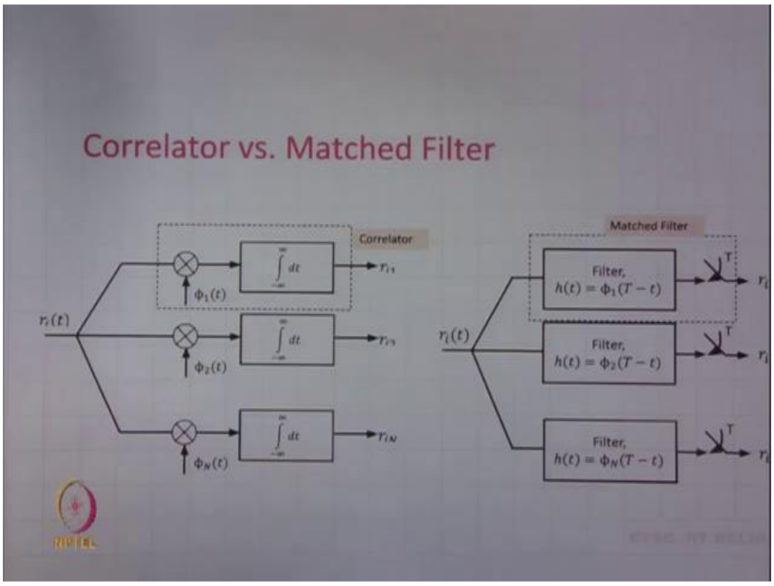

So, transmitter converts sequences to waveforms. If we have a sequence, we multiply each element of the sequence with orthonormal basis function and sum up all the products of this multiplier. So, from this we get a signal.

(Refer Slide Time: 02:07)



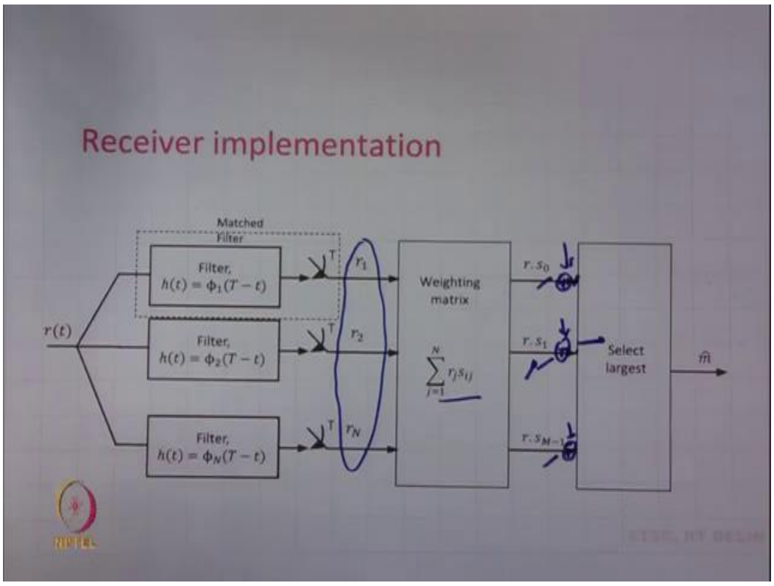

Similarly, we have looked into the receiver design. If we are receiving a waveform, we pass this waveform through what is known as correlator and at the output of the correlator, we have the projection of this waveform along this orthonormal basis functions. So, then we can obtain a sequence corresponding to the received waveform.

(Refer Slide Time: 02:39)




We have also said that instead of this correlator, which is made up of a multiplier and integrator, we can have a matched filter which is a filter whose impulse response is matched to orthonormal basis functions. So, we need to use a bank of matched filters or a bank of correlators. From theoretical point of view, they are similar. However, there are some practical differences.

(Refer Slide Time: 03:03)

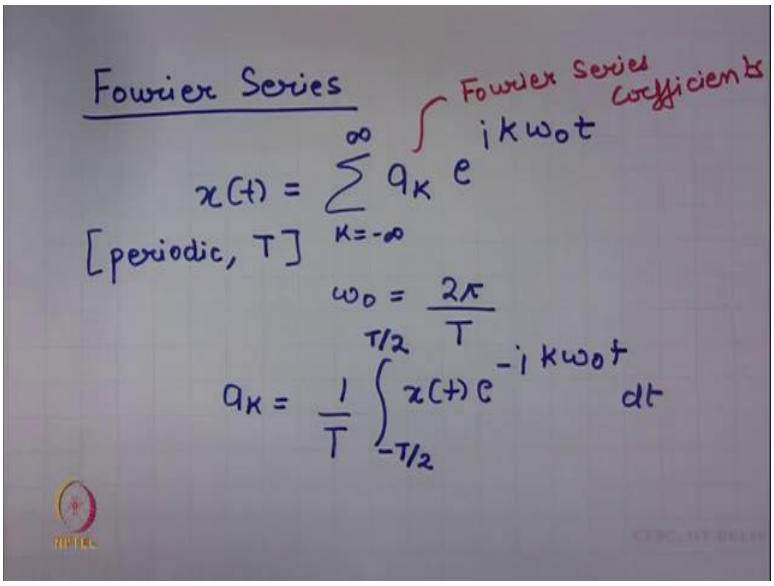



Then we have looked at the full blown receiver implementation where the receiver first produces the sequence corresponding to the received waveform and then, it does the inner product of this received sequence with the sequence of the signal set that it has. We have said if these signals are not of equal energy and they do not occur with equal probability, then normally you also have to add some bias.

So, whichever produces the largest output, receiver assumes that is the transmitted signal. For example, if I have the maximum output from this branch, then the receiver would select $s_1$ as the transmitted signal. So, this happens in a receiver implementation. Why do we add this bias? These things will become clear when we will go into the regime of detection. At this point, it is important to appreciate that at the receiver as well as the transmitter the important operations that happen is the inner product.

So, in today's lecture we will be talking about other forms of orthogonal expansions and these forms of orthogonal expansions can be arrived by using the principles of Fourier series or Fourier transforms. So, let us start with Fourier series.

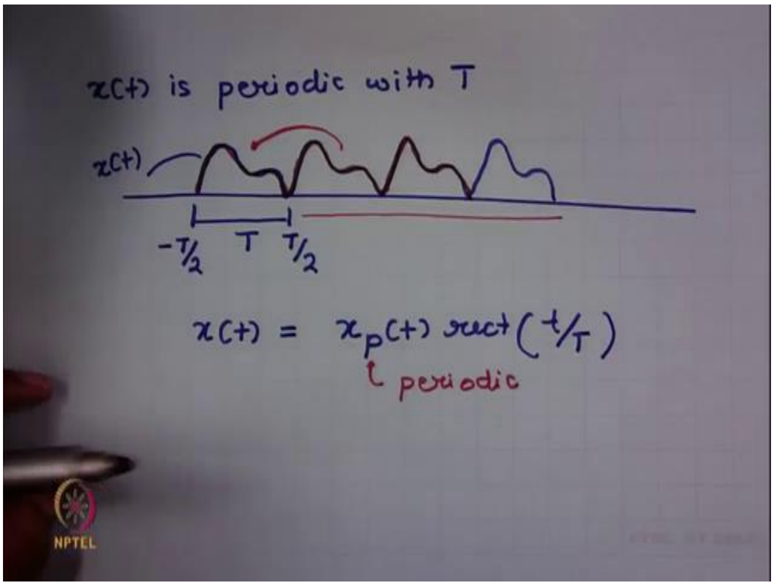
(Refer Slide Time: 04:49)




I would be just revising what you have learnt and maybe adding some flavor to Fourier series based on what is important in the context of digital communication.

So, what you must have read in Fourier series that if you have a signal $x(t)$ which is periodic with period T, then you can express this signal by using this expression. So, here $a_k$'s are known as the Fourier series coefficients and $\omega_0 = 2\pi/T$. We can find out this $a_k$ by carrying out this integration. We will also understand later on how this is related to orthogonal expansion.

(Refer Slide Time: 07:13).

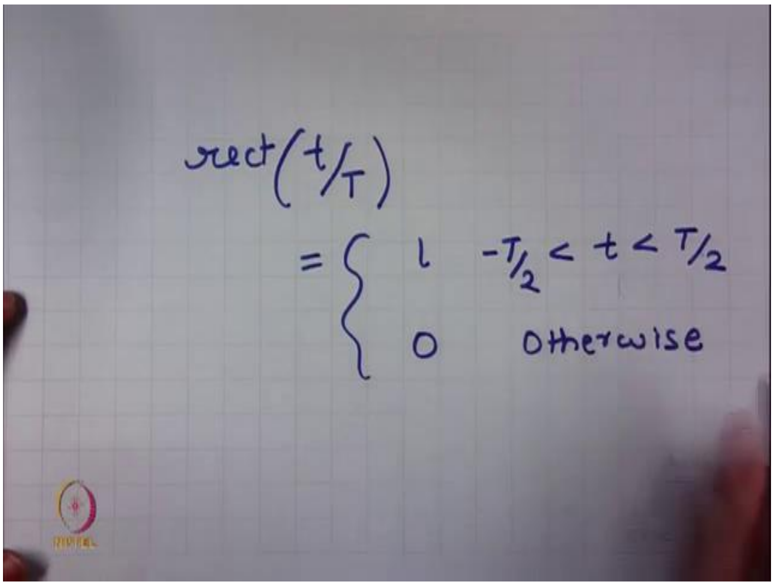


If $x(t)$ is periodic signal with period T, then this signal is useless signal from the point of view of communication because periodic signal repeats itself after a time duration of T seconds and if the signal repeats itself, then there is no information contained in that signal. So, such signals periodic signals are not important from the point of view of communication.

So, we would define Fourier series in a slightly different way than what you have studied in signals and systems. What we will be saying is: suppose I have a periodic signal, I am not interested in this part of the signal because this part of the signal has the same information as this part. So, I am interested in trying to find out the expression for a signal only contained in one time period. For example, I call this one part of the signal as $x(t)$. So, when I am saying $x(t)$, from now onwards it would mean that this is the signal that is present only in one time period which goes from $-$T / 2 to $+$ T / 2.

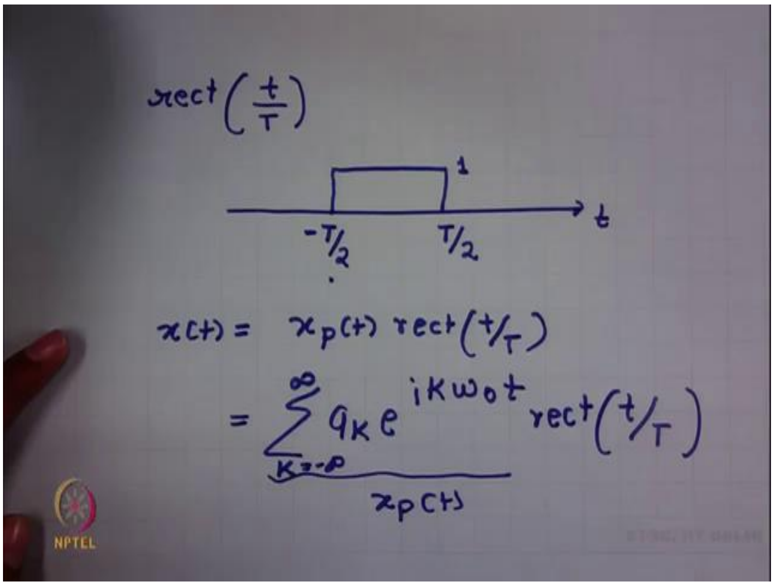
Now, this signal $x(t)$ can be obtained by multiplying $x_p(t)$ where p denotes that this is a periodic signal. So, $x_p(t)$ is the total periodic signal. I multiply this periodic signal with suitable gate i.e. a rectangular function.

(Refer Slide Time: 09:56)




So, rectangular function $rect(t/T)$ is 1 for $-T/2 \leq t \leq +T/2$ and it is 0 otherwise.

(Refer Slide Time: 10:25)

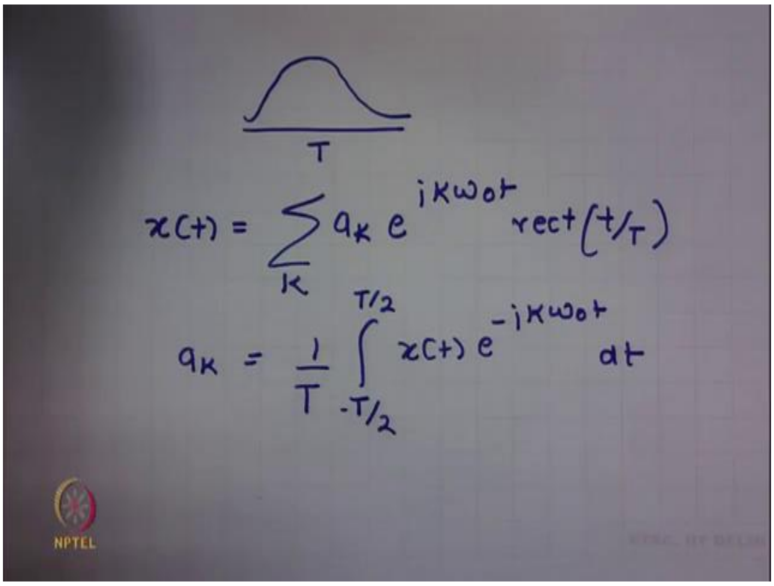



Let me draw the picture of rectangular function. It has a value 1 for time duration between $-$T / 2 to $+$ T / 2 and everywhere else this will be 0. So, once I multiply $x_p(t)$ with this rectangular function, it will only give me the signal contained in this time duration or in just one period where the period spans from $-$T / 2 to $+$ T / 2. So, $x(t) = x_p(t) rect(t/T)$. Thus I can obtain the Fourier series for this $x(t)$ where $x(t)$ is now an aperiodic signal. So, I can write the Fourier series expansion for $x(t)$ as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} rect(t/T)$$

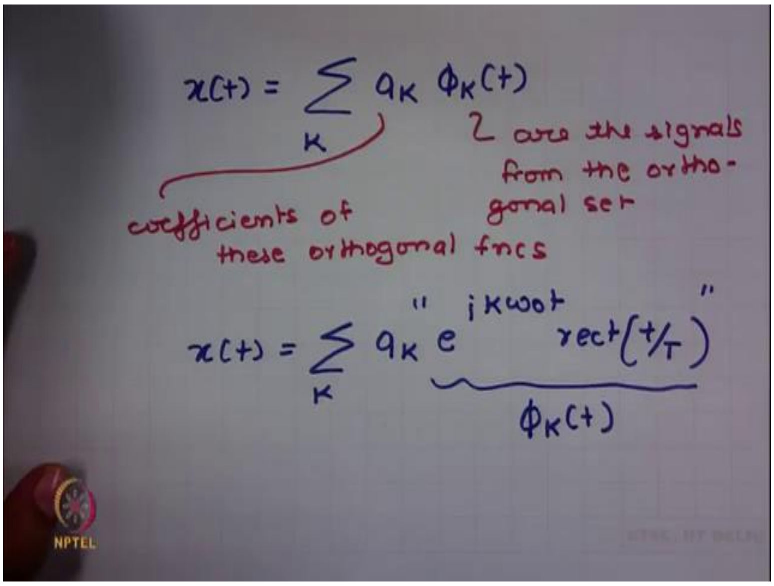
Sometimes I may not write that the limit of k is from - ∞ to + ∞ and then, you should understand it as that k takes all possible values.

(Refer Slide Time: 12:20)



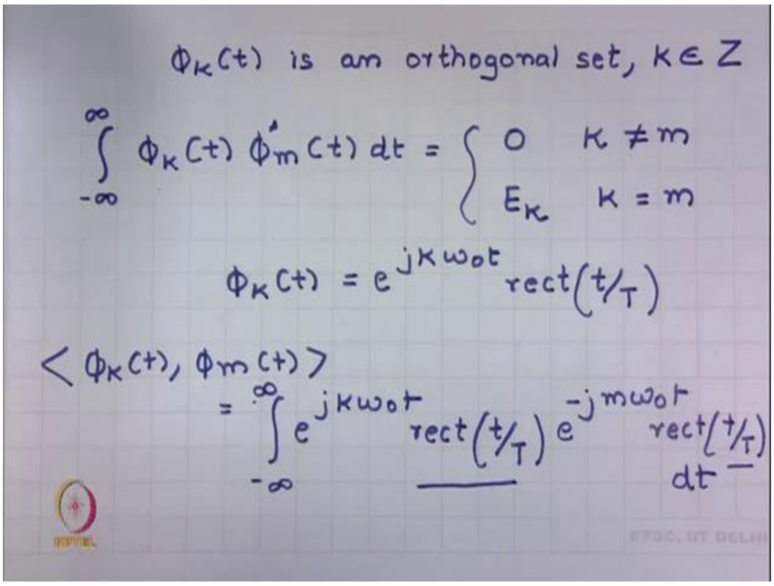

So, what I am saying is because the communication engineers are only interested in aperiodic signals and if I have an aperiodic signal which is limited to the time duration of T, I can obtain the Fourier series for this signal by using this expression for all values of k. Also, $a_k$ can be obtained as before. The expression of $a_k$ would not change. Let us now try to understand this from the orthogonal expansion idea that we have just seen.

(Refer Slide Time: 13:20)




So, I can write $x(t) = \sum_k a_k \phi_k(t)$, if $\phi_k(t)'$s are orthogonal functions and $a_k'$s are the coefficients of these orthogonal functions. Now, the Fourier series looks exactly similar to this. So, if I consider $\phi_k(t) = e^{jk\omega_0 t} rect(t/T)$, these two expressions look exactly same. The only thing that now I have to prove is that this $\phi_k(t)$ or the set corresponding to $\phi_k(t)$ is an orthogonal set, where k belongs to set of integers.

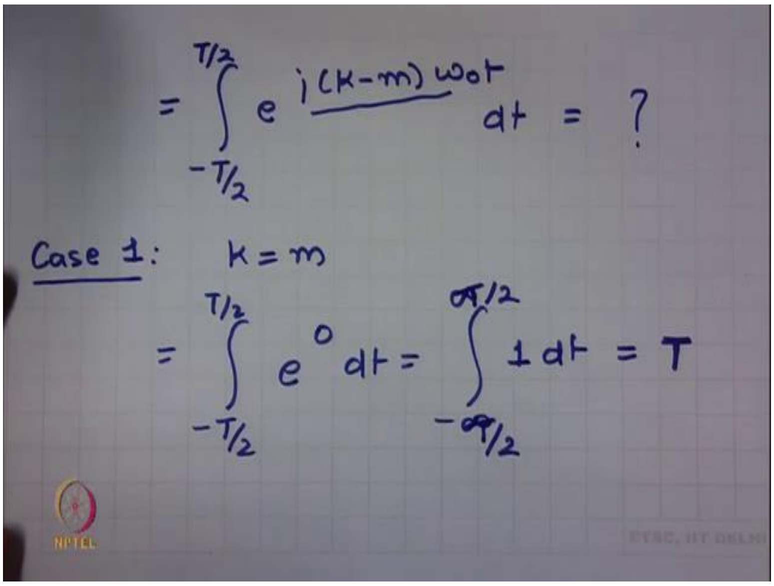
(Refer Slide Time: 15:18)



If we prove that this is an orthogonal set for k belonging to set of integers, then we can say that Fourier series is nothing but the orthogonal expansion of a signal. And how do we prove that something is orthogonal? We take the inner product. So, if I take inner product of $\phi_k(t)$ with let us say another signal from the set which I call as $\phi_m(t)$ and if I can prove that the inner product of these two signals is 0 if k ≠ m, then we can say that these signals are orthogonal signals. Of course, if k = m, then this is the inner product of the signal with itself and it will not be 0, but it will correspond to the energy of the signal. So, $\phi_k(t) = e^{jk\omega_0 t} rect(t/T)$. If I take the inner product of $\phi_k(t)$ with $\phi_m(t)$, first I have to have $\phi_k(t)$, multiplying this with conjugate of $\phi_m(t)$dt. So,

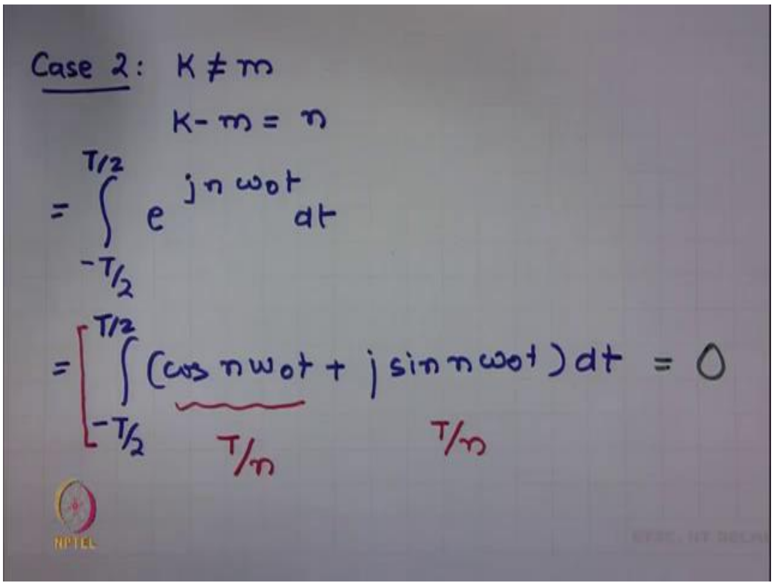
$$< \phi_k(t), \phi_m(t) >= \int_{-\infty}^{+\infty} e^{jk\omega_0 t} rect(t/T) e^{-jm\omega_0 t} rect(t/T)\, dt$$

(Refer Slide Time: 17:49)



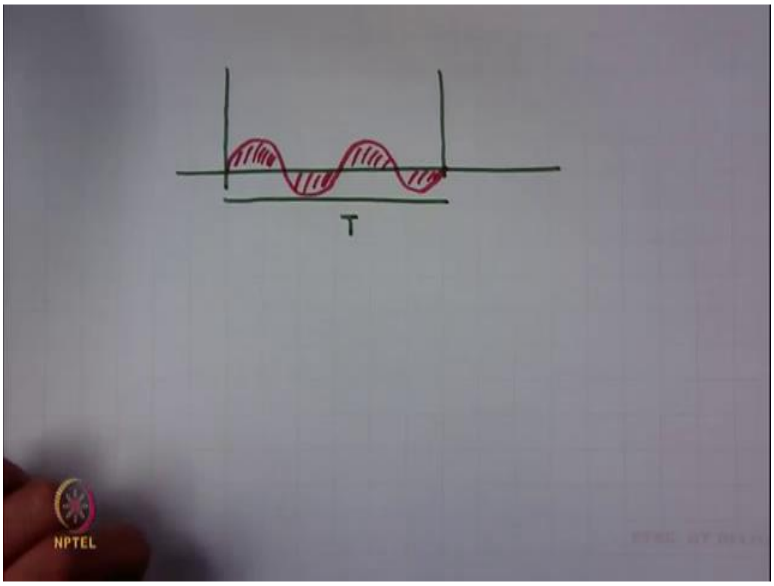

I can absorb $rect(t/T)$ in the limit of integration and the integration goes now from $-$T / 2 to $+$ T / 2. We start thinking about this by considering the easy case first. So, we have Case 1: k = m. When k = m, we have $e^{0t}$ which is 1. So, going from $-$T / 2 to $+$ T / 2, we have T. So, when k = m, the inner product of $\phi_k(t)$ with $\phi_m(t)$ is T. Now, let us investigate the second case, Case 2: k ≠ m.

(Refer Slide Time: 18:52)



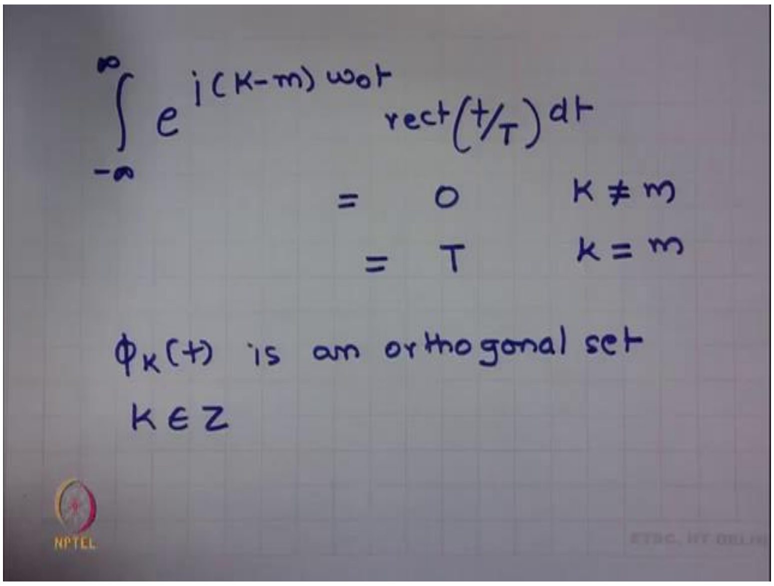

Let us assume that k - m is another integer, n. So, this will become $e^{jn\omega_0 t}dt$. Now, instead of thinking this in terms of complex exponential, we use Euler's theorem and break it into sinusoids. So, I can write this as (cos n$\omega_0$t + j sin n$\omega_0$t) dt. Now, the fundamental periods of both these signals cos $n\omega_0 t$ and sin $n\omega_0 t$ is T/n. So, in this limit of integration, this cos and sin will make n cycles.

(Refer Slide Time: 20:44)



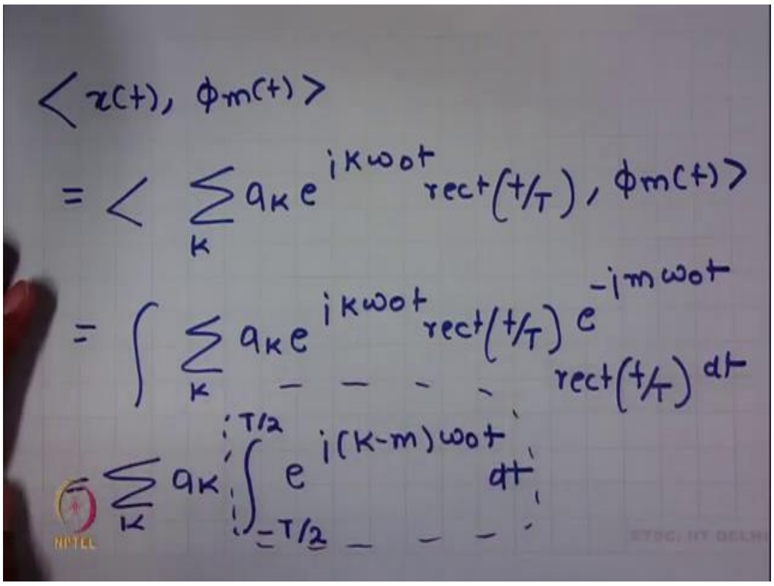

In this time duration T period, these sinusoids will complete integer number of cycles. For example, this sine will make integer number of cycles. In this case this integer number is 2, but what we have to appreciate is, if a sinusoid function makes or completes an integer number of cycles in the period of integration, then the area under this function is always going to be 0 because the positive part will always cancel out the negative part. So, if I am integrating a sinusoid function and if that sinusoid function completes integer number of cycles in that period of integration, then the integration of that sinusoid function would be 0. So, we can now understand that this integration would give us 0.

(Refer Slide Time: 22:06)




We can summarize that if I am carrying out the inner product of $\phi_k(t)$ with $\phi_m(t)$, that means I am interested in carrying out this integration, then this will be 0 when k ≠ m and when k = m, this is going to be T and hence, we can say that the set $\phi_k(t)$ where k belongs to set of integer is an orthogonal set.

(Refer Slide Time: 23:07)

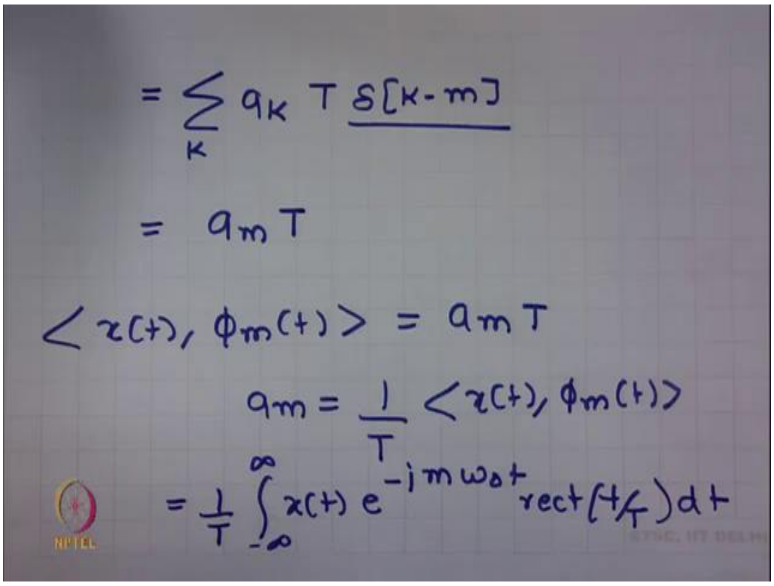



Let us carry out the inner product of $x(t)$ with $\phi_m(t)$ and this gives us

$$\sum_k a_k \int_{-T/2}^{+T/2} e^{j(k-m)\omega_0 t} dt$$

I can absorb these rectangular functions again in the limit. We have already said this is non zero only when k = m.

(Refer Slide Time: 24:44)

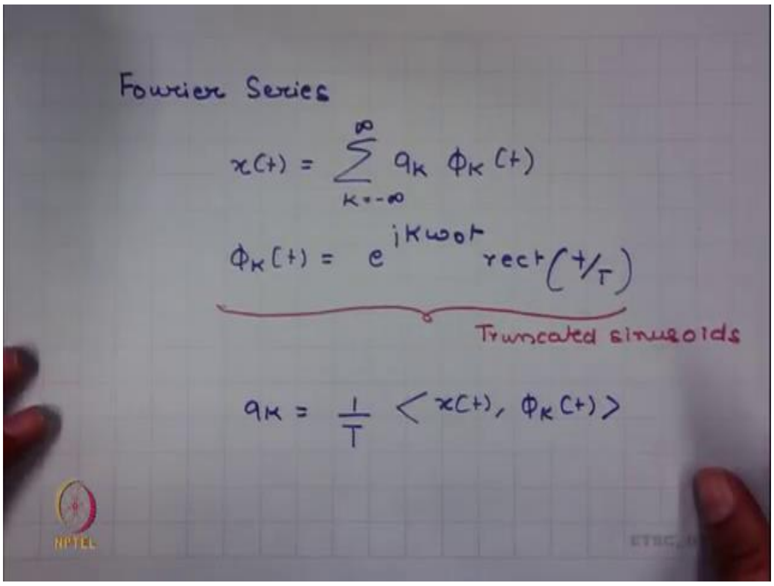



So, we can write this as $\sum_k a_k T\delta(k-m)$ where $\delta$ is an impulse function. So, this will mean that this is only non zero when k = m. When k = m, it is 1 and it is multiplied by T and when k ≠ m, this will give you 0. So, I have only $a_m T$ left because this will make all coefficients other than the coefficient $a_m$ to be 0. Thus, $< x(t), \phi_m(t) >= a_m T$ and from this we have

$$a_m = \frac{1}{T} < x(t), \phi_m(t) >= \frac{1}{T}\int_{-\infty}^{+\infty} x(t) e^{-jm\omega_0 t} rect(t/T)\, dt$$

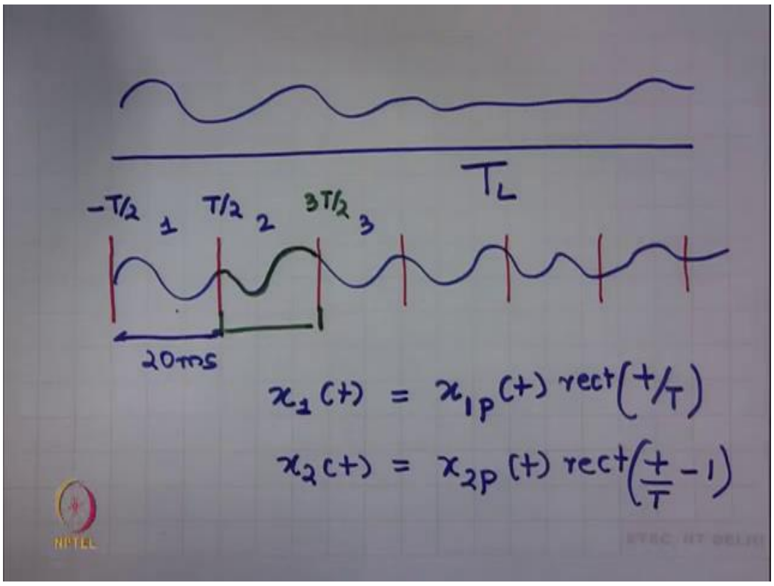
So, we have learnt about the Fourier series and let us now recap the major ideas that we have developed.

(Refer Slide Time: 26:41)




So, what we are saying is we can express a signal $x(t)$ in terms of orthogonal functions $\phi_k(t)$. We have proven that this is a set of orthogonal functions if k belongs to the set of integers and these orthogonal functions are this and in literature this is also referred to as truncated sinusoid. This is a sinusoid and this truncates the sinusoid to one time period. So, these $\phi_k(t)$'s are also referred to as truncated sinusoids. You can obtain the $a_k$'s just by carrying out the inner product of $x(t)$ with $\phi_k(t)$ and dividing it with the energy of $\phi_k(t)$ and that is T. So, now we have seen that this Fourier series is the orthogonal expansion of a signal. Is this of any use? Let us see it now. So, in practice, the signals in which we are interested will be of a pretty long duration.

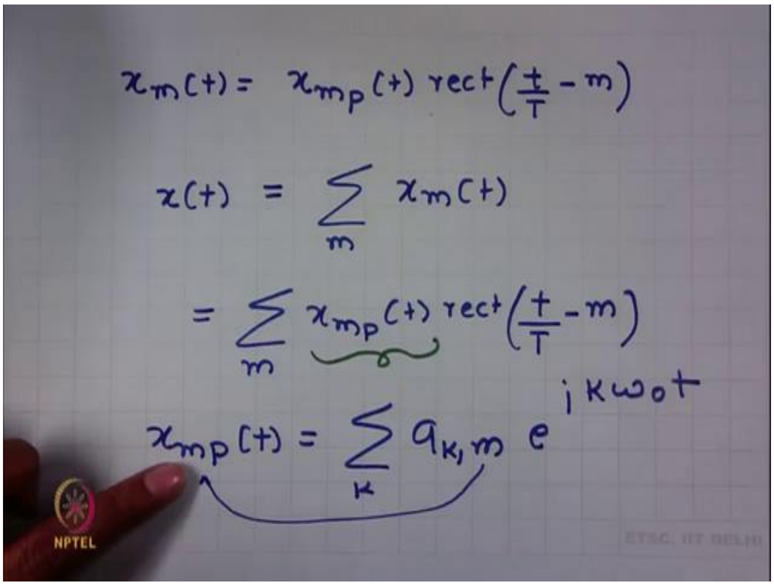
(Refer Slide Time: 27:57)




Let us say this signal expands a pretty long duration and we say this as $T_L$. Now, if you have to wait to process the signal or if you have to wait to receive the signal for a time duration of $T_L$, then only you can calculate these $a_k{}'$s and then only you can transmit this $a_k{}'$s. This will create lot of delays in the communication system. So, one idea is that if you have a signal instead of waiting for the entire reception of the signal, you can break the signal into segments. So, I can take a segment of these signals. For example, in voice applications, the duration of this segment is typically chosen to be 20 ms and then, I find the coefficients corresponding to a segment of a signal. So, this is a segment 1, 2, 3 and so on.

So, if I pull out a segment let me call this as $x_1(t)$. This segment $x_1(t)$ can also be thought as it is constructed from the periodic signal $x_{1p}(t)$. So, $x_{1p}(t)$ is a periodic signal. If you multiply this periodic signal with a gate, we can get this segment. Now, similarly I can think about the second segment which again I think in the same lines as if it is constructed out from the second periodic signal and by multiplying this second periodic signal with a gate, but now this gate would be shifted by T units because this second segment is T spaced from the first segment. So, you need to choose a gate which is non zero in this duration, so, in a duration between T/2 to 3T/2.

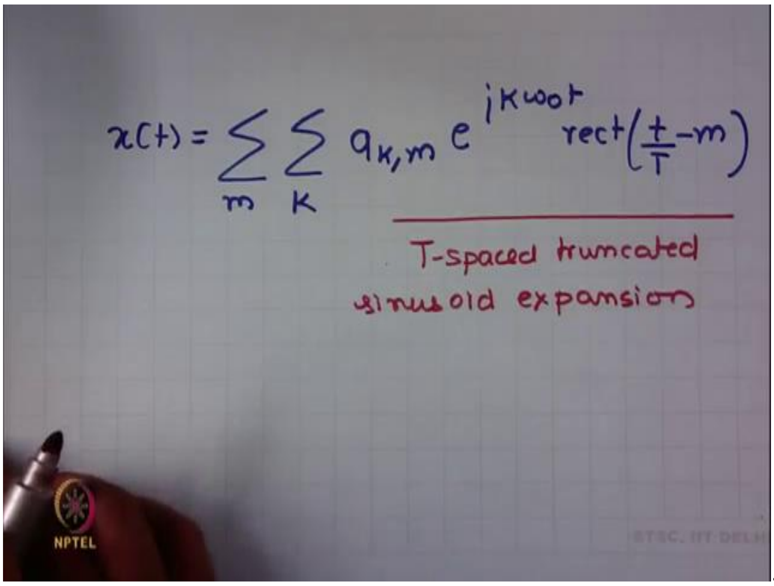
(Refer Slide Time: 30:43)



I can think about the m^th^ segment from the m^th^ periodic signal and multiply this with the gate, send it spaced by mT units. So, $x(t)$ is built up of these segments. The total signal is composed of these segments where m takes in all possible values, then I can substitute the value of $x_m(t)$ and I can get the expression of $x(t)$ and each periodic signal can use Fourier series representation. So, if it is a periodic signal I can think about this periodic signal by the Fourier series expansion.

First I had only $a_k$'s, but now there is an additional subscript m and this m tells me that these are the coefficients corresponding to m^th^ periodic signal. So, each periodic signal will have different k coefficients. So, to tell about for which periodic signal these coefficients belong, we need to have this additional subscript.

(Refer Slide Time: 33:03)

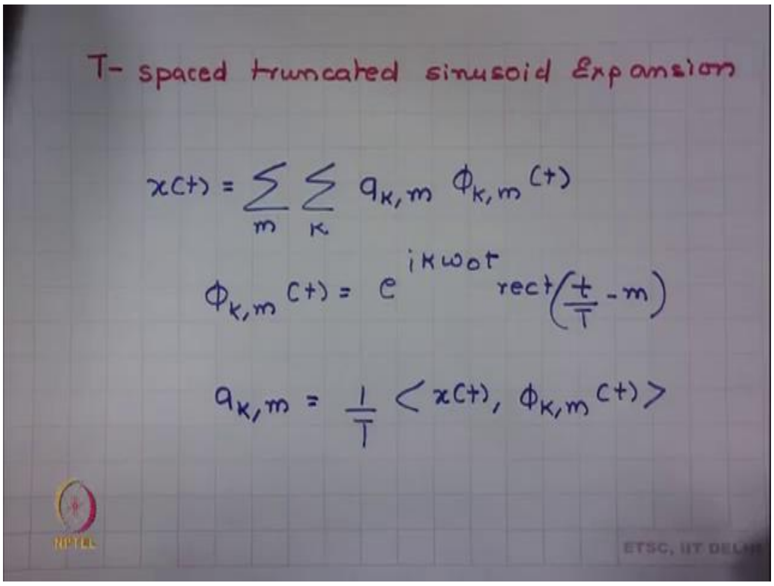



So, plugging in this value of $x_m(t)$ in this expression, we get

$$x(t) = \sum_m \sum_k a_{k,m} e^{jk\omega_0 t} rect\left(\frac{t}{T} - m\right)$$

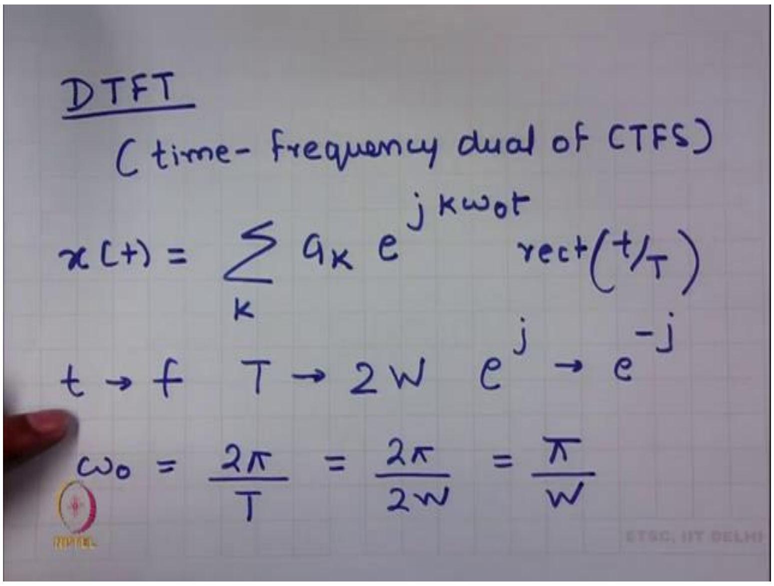
and this is known as T-spaced truncated sinusoid expansion. So, first of all these sinusoids are truncated by this gate function, but these sinusoids are also T-spaced truncated. So, this gate that has truncated the sinusoids is T-spaced.

(Refer Slide Time: 34:10)



We have just finished learning about T-spaced truncated sinusoid expansion where we have been able to express a signal as a double sum of orthogonal expansion. These orthogonal functions are T-spaced truncated sinusoids and this $a_{k,m}$, can be simply obtained by taking the inner product of $x(t)$ with these orthogonal functions and dividing by the energy of these orthogonal functions which is T. Ok, let us now learn about discrete time Fourier transforms or DTFT.

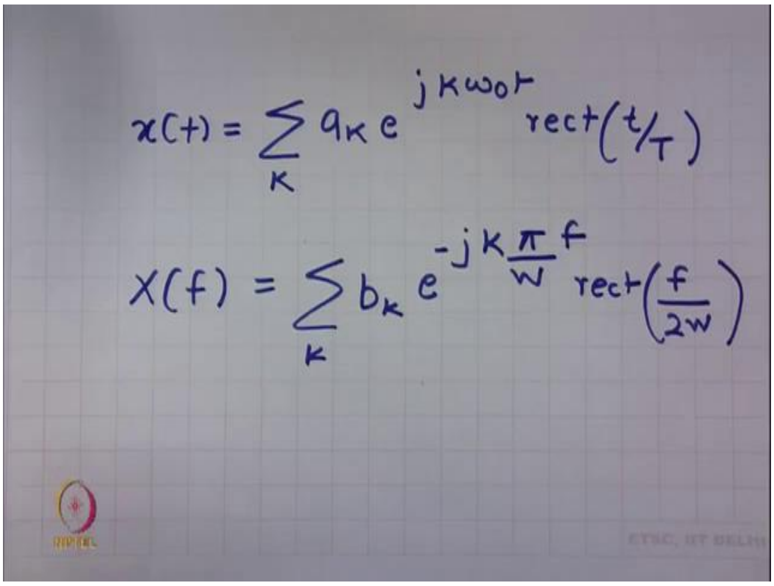
(Refer Slide Time: 34:55)




Discrete time Fourier transform is time frequency dual of continuous time Fourier series that we have just seen. So, we know that continuous time Fourier series expression can be conveniently obtained, as we have already seen, by having the Fourier series coefficients multiplied by truncated sinusoids.

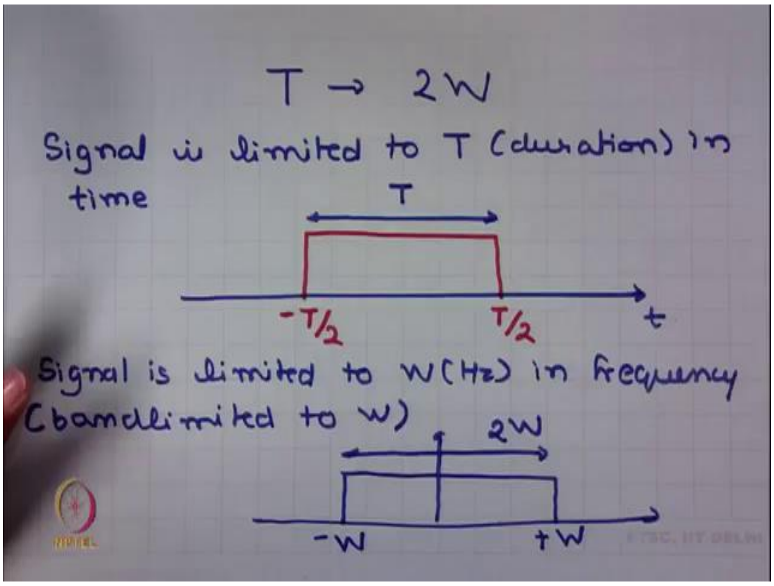
So, this is the expression of continuous time Fourier series and we can obtain the expression of discrete time Fourier transform by using this idea of time-frequency duality and what is that? So, because of this property of time-frequency duality I can replace t by f, T by 2W and $e^{j}$ by $e^{-j}$. By carrying out these replacements, I can go from continuous time Fourier series to discrete time Fourier transform. Also we can see that because $\omega_0 = 2\pi/T$ and we have already said that T should be mapped to 2W. So, $\omega_0 = \pi/W$. So, let us see what would be the expression of discrete time Fourier transform.

(Refer Slide Time: 37:04)




So, first let me write again the expression of continuous time Fourier series and let us now carry out the replacements. So, changing t to f, we go from a time domain signal to frequency domain signal. The $a_k$'s are replaced by $b_k$'s. Let us assume that for DTFS or DTFT, the coefficients are $b_k$, then changing j to - j, k remains k, $\omega_0$ gets mapped to $\pi/W$, t is mapped to f, rectangular function remains rectangular function, t is mapped to f and capital T is mapped to 2W and this is the expression of discrete time Fourier transform.

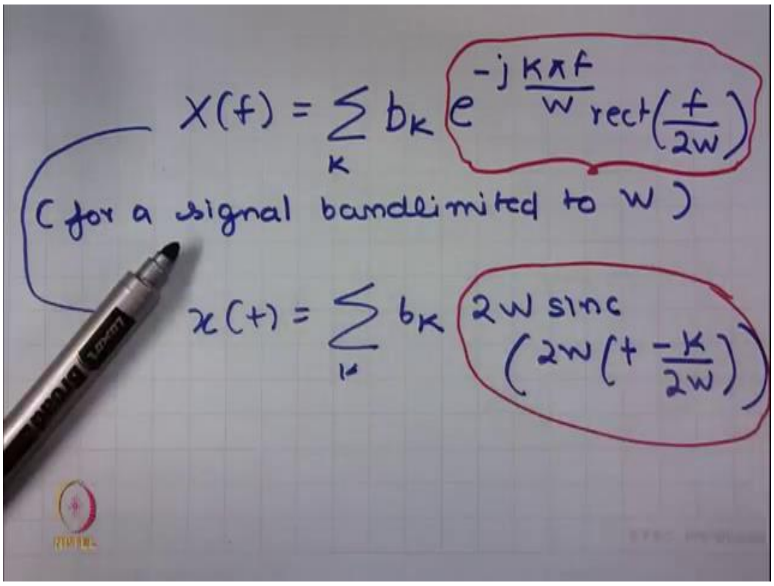
(Refer Slide Time: 38:28)



So, let us first understand the obscurity when we are saying that T is getting mapped to 2W and why there is a factor 2 here. So, for example when I say that a signal is limited to T duration in time, it can mean that signal exist from $-T / 2$ to $+T / 2$ and the total duration of the signal is T. However, when we say signal is limited to W Hz in frequency, other way to say this is signal is band limited to W, it means that the signal occupies W Hz only in the positive side of the spectrum.

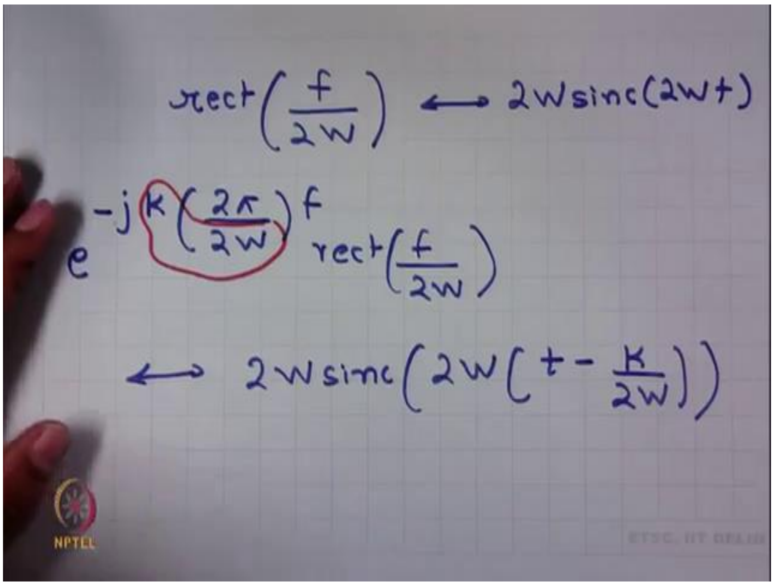
So, W only denotes the occupancy of the signal in the positive side of the spectrum and the total support of the signal is 2W and hence, when you say that signal is limited to T duration in time, it denotes that the total duration of the signal is T in time, but when you say that the signal is limited to W Hz in frequency, it means that the total duration of the signal is 2W in frequency including positive and negative side of the spectrum. Hence because of these notations, T is getting mapped to 2W.

(Refer Slide Time: 41:04)




So, we have said that in discrete time Fourier transform, you have an expression like this and remember that this expression is valid for a signal which is bandlimited to W. So let us start taking the inverse Fourier transform of this DTFT expression. So, $X(f)$ will become $x(t)$, and then, we have to take the inverse Fourier transform of this. So, let us first try to see what is the inverse Fourier transform of this expression using the properties of signals and systems.

(Refer Slide Time: 42:23)

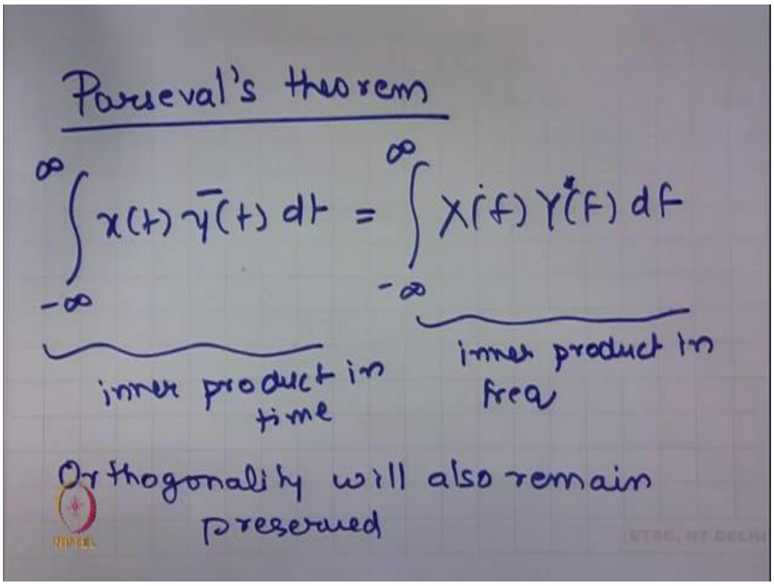



You must know these relationships, for example, the inverse Fourier transform of a rectangular pulse is a sinc pulse and if I multiply this rectangular pulse with a rotating complex exponential, then the independent variable in another domain gets shifted by a certain quantity. For example, in this case, this would be shifted by k/2W. So when you multiply a function with a rotating complex exponential then the variable in another domain gets shifted.

So, here the variable t will get shifted by a factor of k/2W. So, we have been trying to obtain the inverse Fourier transform of this expression and now I can complete it like this. So, this is a set of orthogonal functions using the same approach as we have used in the case of Fourier series. This is actually truncated which is a set of orthogonal functions.

Now, if this is a set of orthogonal function can we say something about the set of these signals where the set is obtained by changing the value of k's and remember that k belongs to the set of integers.

(Refer Slide Time: 45:20)

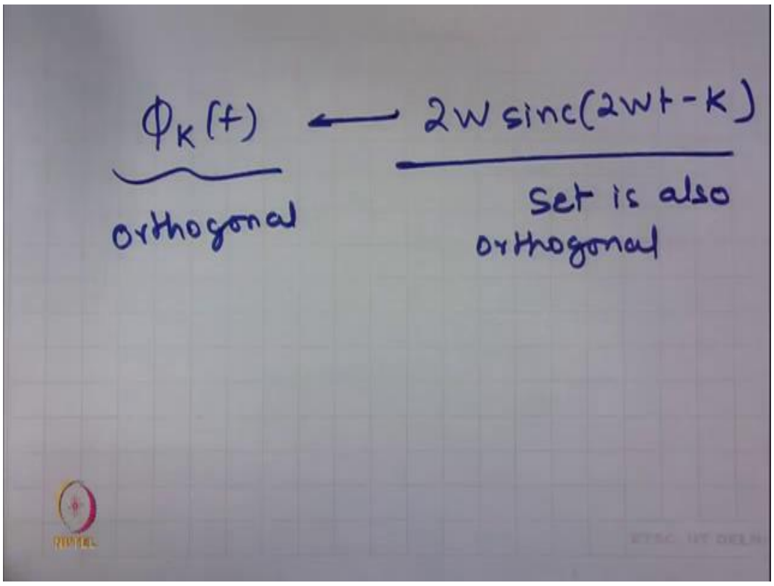



Let me first write Parseval's theorem. So, Parseval's theorem states that

$$\int_{-\infty}^{+\infty} x(t) y^*(t)\, dt = \int_{-\infty}^{+\infty} X(f) Y^*(f)\, df$$

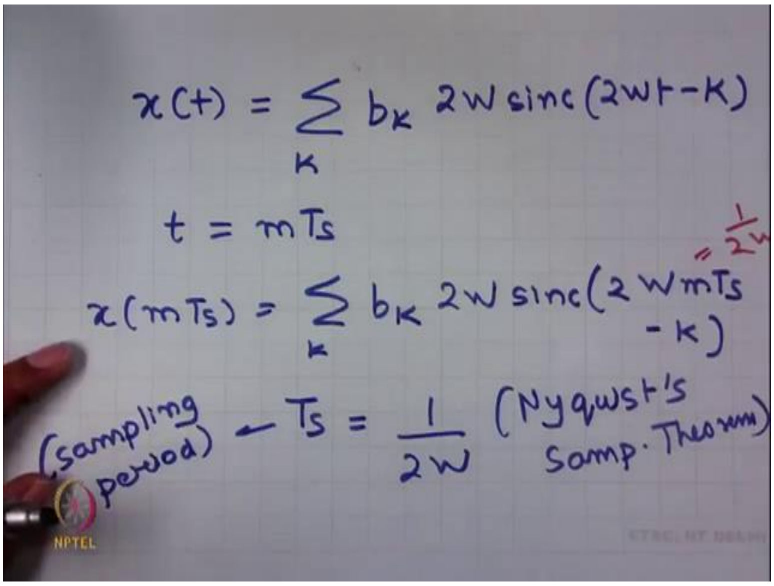
This is the inner product of the signals in time domain on LHS and this is the inner product of the signals in frequency domain on RHS. Here $X(f)$ is the Fourier transform of $x(t)$ and $Y(f)$ is the Fourier transform of $y(t)$. So, when you are transforming the signals from time domain to frequency domain, the inner product between the signals do not change. Inner product is preserved. If inner product is preserved, it will also imply that orthogonality will remain preserved.

(Refer Slide Time: 47:01)




So, $\phi_k(f)$ was the set of orthogonal signals used for orthogonal expansion in the case of DTFT. If we take the inverse Fourier transform of these functions, we get a set 2W sinc(2Wt – k) and because $\phi_k(f)$ is an orthogonal set, so this set of functions is also orthogonal set because of Parseval's theorem. This is because when you go from time domain to frequency domain or frequency domain to time domain, the inner product remains preserved and thus the orthogonality of the functions also remain preserved.

(Refer Slide Time: 48:14)

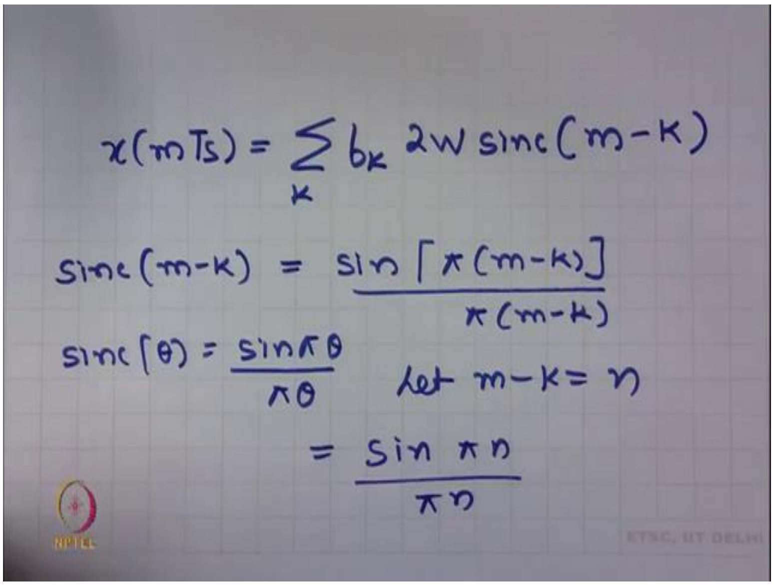


So, let us now rewrite the expression for inverse Fourier transform of DTFT. Let us substitute t as $mT_s$ and see what happens. So, this $x(t)$ becomes

$$x(t) = \sum_{k} b_k 2W \operatorname{sinc}(2Wt - k)$$

Here t becomes $mT_s$ and as I have said, we want to map this $T_s$ to $1/2$W. That means we this $T_s$ is also known as sampling period. We are choosing a sampling period same as $1/2$W and this follows from Nyquist sampling theorem. So, why is the sampling time like this we will see in a while, but let us just see that what happens if you choose the sampling time as $1/2$W. So, if I choose $T_s$ as $1/2$W, we can also see in here, this 2W cancels with this 2W.

(Refer Slide Time: 50:29)




And the expression that we get is

$$x(mT_s) = \sum_{k} b_k 2W \operatorname{sinc}(m - k)$$

The definition that is normally used in the courses on communication system is

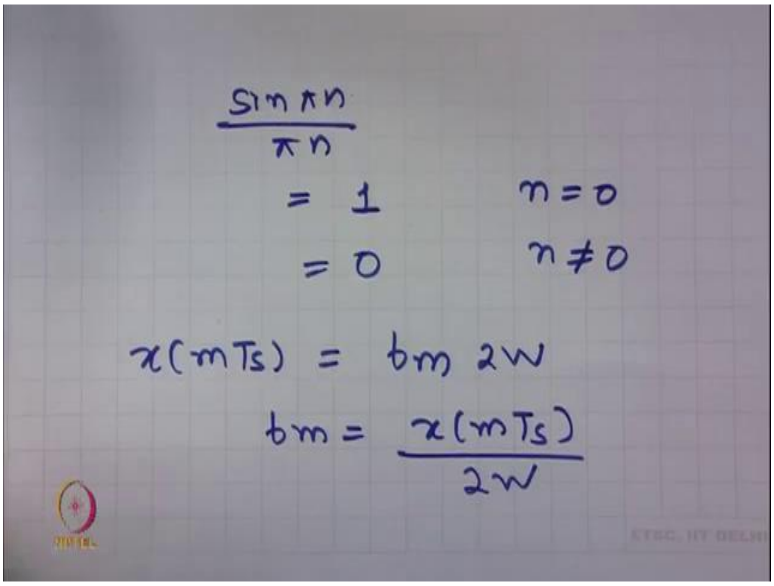
$$\operatorname{sinc} \theta = \frac{\sin \pi\theta}{\pi\theta}$$

This is also known as normalized sinc function. So, this is known as the normalized definition for sinc function. In that case,

$$\text{sinc}(m-k) = \frac{\sin[\pi(m-k)]}{\pi(m-k)}$$

Let us now substitute this m - k as n, m is an integer, k is an integer and this n is also an integer.

(Refer Slide Time: 52:13)

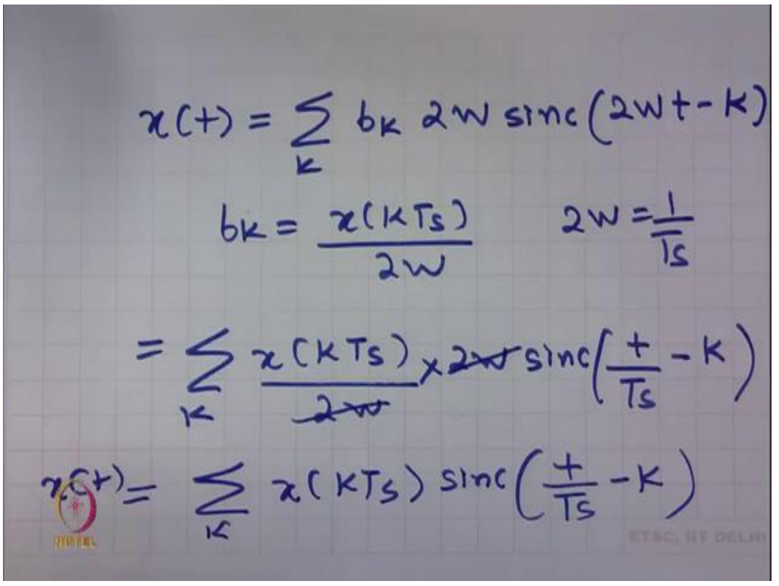



We can use L'Hospital rules and obtain that if n = 0, then $\frac{\sin \pi n}{\pi n}$ will be 1 and if n ≠ 0, then this is going to be 0. So, we see that when m ≠ k, this term will vanish and there will be no contributions in the summation of the coefficients other than when k = m. Hence, we can simplify the summation to just one term where the contribution will be only from $b_m$ because all other coefficients will become 0 because of this multiplication by sinc. So,

$$x(mT_s) = b_m 2W \Rightarrow b_m = \frac{x(mT_s)}{2W}$$

So, now let us substitute this value of $b_m$ in the original expression.

(Refer Slide Time: 53:59)




So, we have said that

$$x(t) = \sum_k b_k 2W \mathrm{sinc}(2Wt - k)$$

and substituting

$$b_k = \frac{x(kT_s)}{2W}$$

the expression that we get is

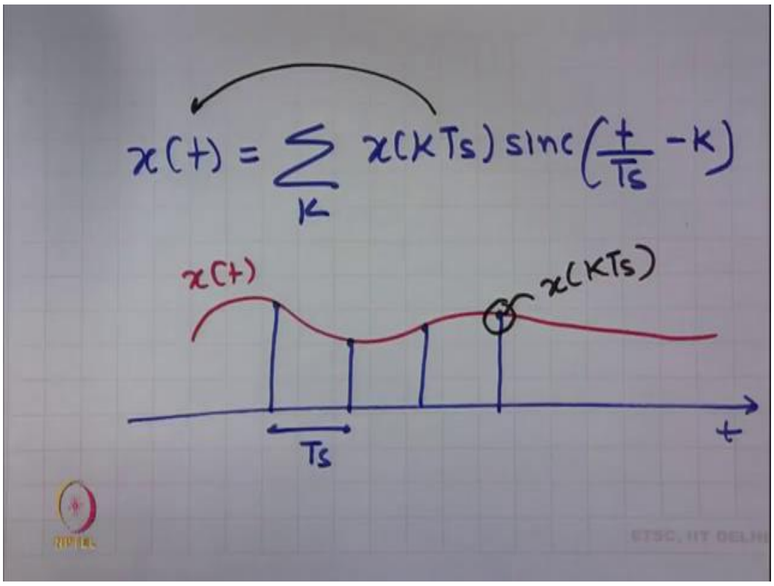
$$x(t) = \sum_k \frac{x(kT_s)}{2W} 2W \mathrm{sinc}(2Wt - k)$$

and we can also substitute this 2W= $1/T_s$. So, we get the resultant expression as

$$x(t) = \sum_k x(kT_s) \mathrm{sinc}\left(\frac{t}{T_s} - k\right)$$

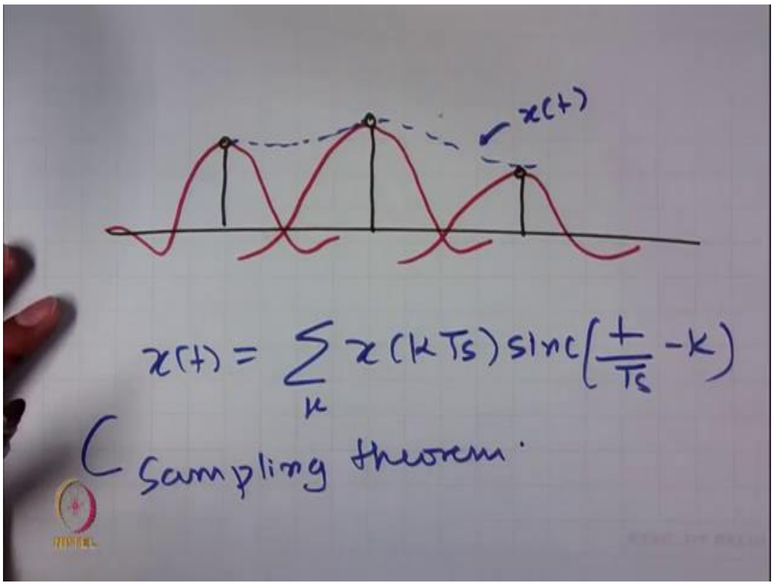
and this is a very celebrated expression. It goes by the name of sampling theorem. So, let me write this expression again.

(Refer Slide Time: 55:13)




Now, let us try to understand what this expression means. For example, let me assume that I have a signal$x(t)$. These $x(kT_s)$ are the samples of this function collected at an interval of $T_s$. So, I can go from samples to the signal by putting sinc caps around the samples. For example, let us assume that I have got certain samples.

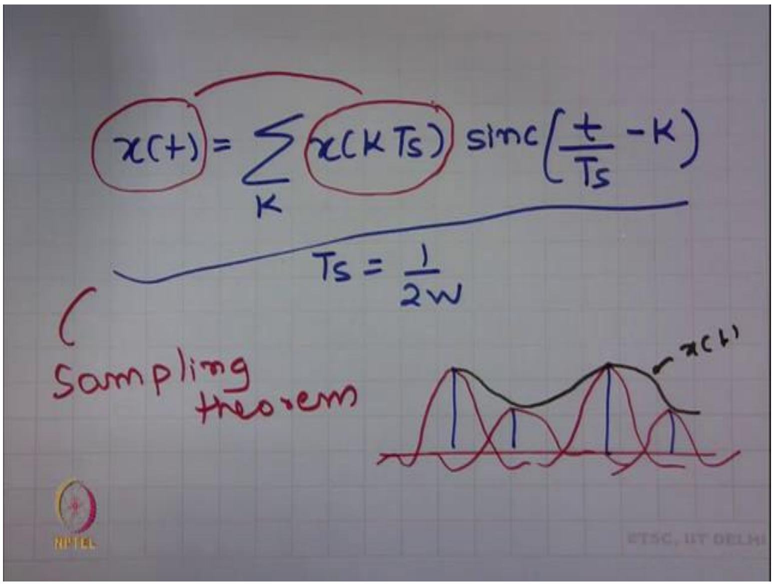
(Refer Slide Time: 56:40)




And if I start putting this sinc caps around these samples and I sum up everything then I hope to get a continuous function. This function is $x(t)$, these black lines are the samples and these red functions are the sinc caps.

So, this expression states that if you put the sinc caps around the samples, you get the waveform and this is a very important result and it goes by the name of sampling theorem. When we will see the random processes and noise, we will see that you can express noise again exactly in the same way. If you start putting the sinc caps around random variables, then you get two random processes. There also we use this idea of sinc expansion of a signal.

(Refer Slide Time: 58:08)




When we will go to the regime of modulation, there also we see that you can start thinking about the waveforms by putting these sinc caps around these samples of the waveform. So, this equation is pretty useful and we will use lot of it in random processes and modulation. So, next we will be trying to understand double sum orthogonal expansion using DTFT which is exactly same thing as we have done in the case of Fourier series. So, we will see about this double sum expansion in the next lecture.

Thank you.