Principles of Digital Communication
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Lecture – 04
Signal Spaces
Gram Schmidt Orthogonalization & Receiver Structures

In this lecture, we will be talking about Gram Schmidt orthogonalization procedure. But before that let us recap what we have learned in the first week.

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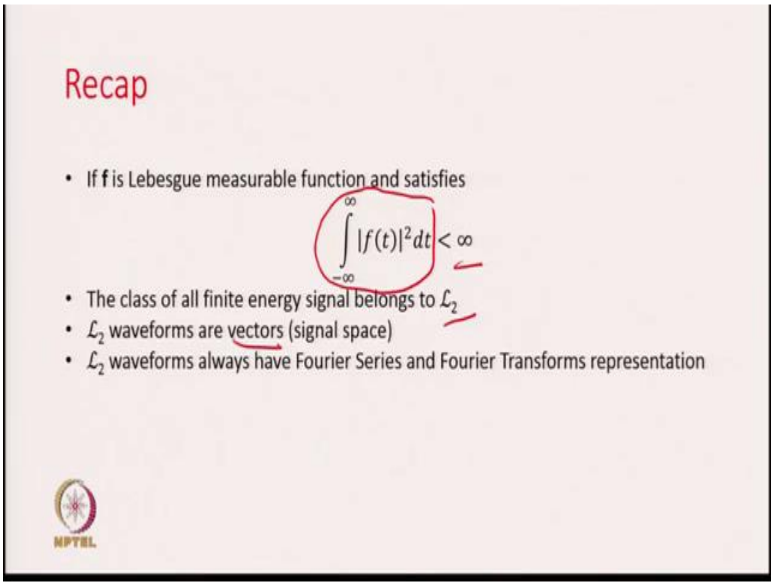

So, we have introduced the big idea of L_2 space (or L_2 waveforms or L_2 signals), and we have said that the class of all finite energy signal belongs to L_2 space. So, we know that the energy of the signal can be computed by the integration of mod square of that function, and if this quantity is finite, we say that the signal has finite energy and if the signal has finite energy the signal is in L_2 signal.

Now, we prefer these L_2 signals because L_2 waveforms are vectors, and L_2 waveforms always have Fourier series and Fourier transforms representation. So, this is the one big idea that we have learned.

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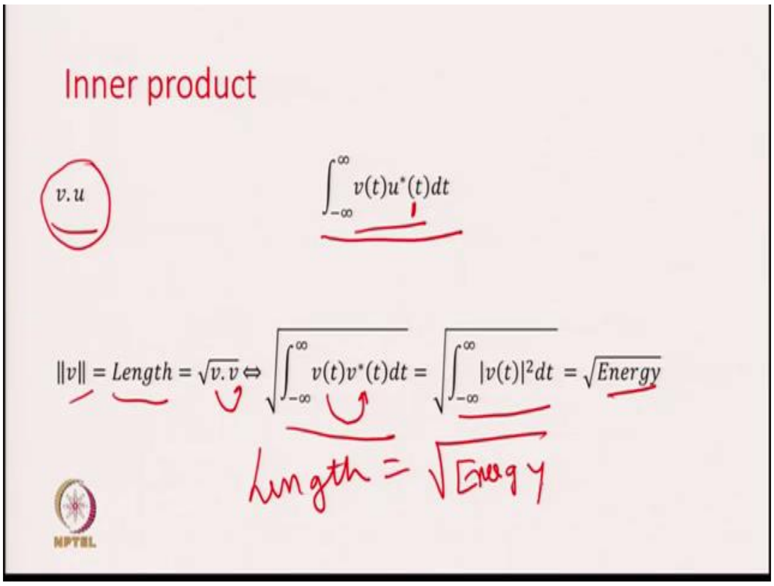

Second, we have learnt about the inner product. So, inner product is same as the dot product. So, in case of vectors we prefer this dot product, in case of signals we do this inner product. What is inner product? If you want to compute the inner product of the two signals you have to multiply the signal with a conjugate of another signal and then you have to integrate it from - ∞ to + ∞, and this is the inner product of the two signals. Physically the dot product and inner product have the same meaning.

In case of vectors we can compute the norm of a vector which is also the length of the vector and this could be obtained by square root of dot product of the vector with itself. In case of signals, we can also compute the norm of a signal by the square root of inner product of the signal with itself. This quantity is nothing but it is the energy of the signal, and hence norm of the signal is nothing but it is the square root of energy of the signal. So, we learn that length of the signal is nothing but square root of energy of that signal.

Carrying this idea forward, we can talk about the angle between the two vectors and you must have learned this identity before that cos of the angle between two vectors is given by the inner product of the two vectors divided by the length of the vectors.

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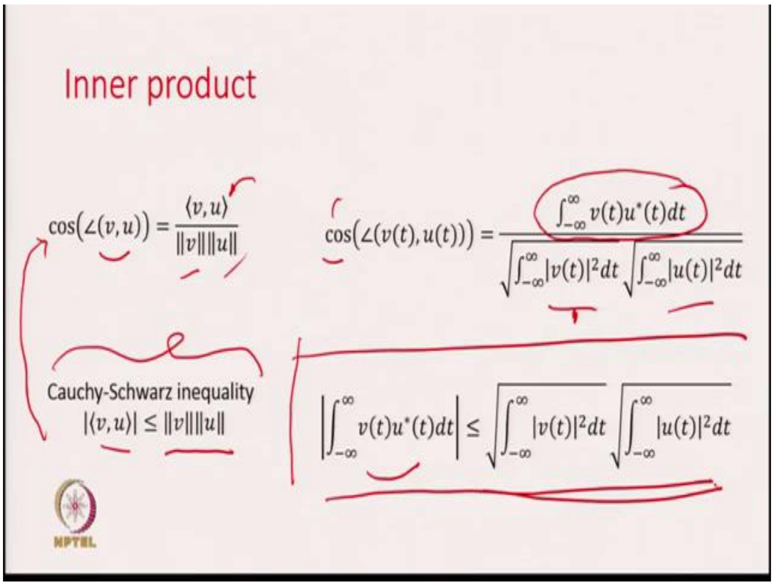

In case of signals, the picture remains exactly same. The cos of angle between the two signals is given by the inner product of the two signals. So, here we are computing the inner product of the two signals divided by the length of the signals. Remember this is the length of the signal because it is the square root of energy of the signals.

In case of vectors, Cauchy Schwarz inequality can be easily derived from this expression, and it states that mod of the inner product of the two vectors is less than or equals to the product of the lengths of the vectors. In case of signals exactly same thing follows that is the mod of the inner product of the two signals is less than or equals to the product of the lengths of the signals.

Now, using this analogy between signals and vectors, we have easily derived this inequality otherwise it would have given us tough time. So, that is why we prefer this analogy between signals and vectors: things becomes trivial and easy. Let us take this case forward and let us define what are known as orthogonal signals.

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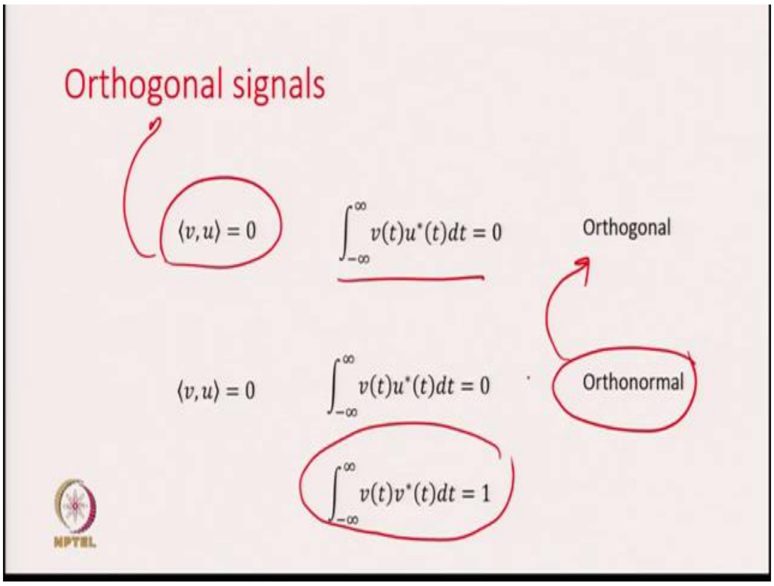

In case of vectors if you want to talk about the vectors, if the inner product of two vectors is zero then we call those vectors as orthogonal vectors or perpendicular vectors. In case of signals, again if the inner product of the two signals is zero, we call those two signals as orthogonal signals.

In case of vectors we have unit vectors, unit vectors have the magnitude of one. In case of signals, we have these orthonormal functions (or orthonormal signals). Orthonormal functions (or signals) are first orthogonal functions (or signals) and secondly, they should also have unit energy. So, if the signals are orthogonal and they have unit energy then we call those signals as orthonormal signals, orthonormal functions or orthonormal waveforms.

So, these are the three important ideas that we have learned in the last week.

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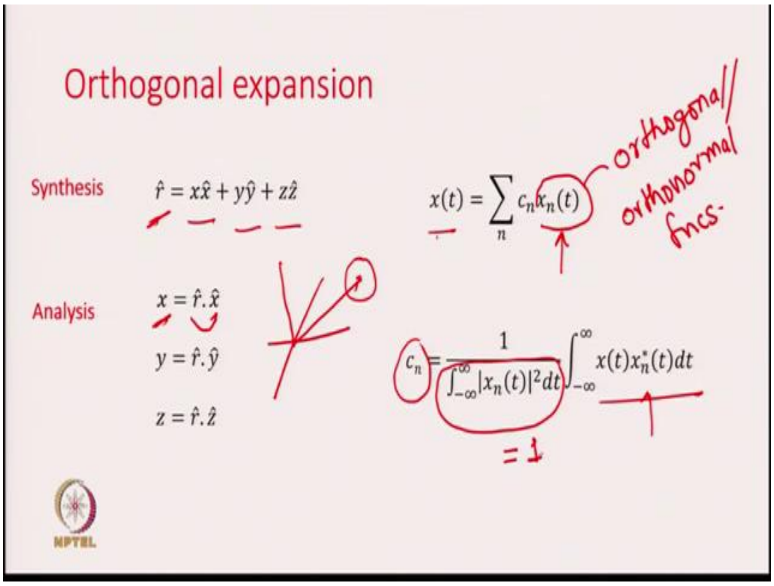

So, we have also seen about the orthogonal expansion of a signal which is easier to think in terms of a 3D vector. A 3D vector can be understood as it is built using components in the x, y and z direction. Similarly, a signal can be considered as if it is built using orthogonal or orthonormal functions. So, these $x_n(t)$'s are orthogonal or orthonormal functions.

Again, if we want to think about how big is the component of this vector r in the x direction you can simply obtain this by dotting this vector r with the unit vector in the x direction. In the case of signals, if we want to find out the component of the projection of $x(t)$ in the direction of reference signal $x_n(t)$, you can simply obtain this by taking the inner product of the signal with the reference orthogonal or orthonormal function $x_n(t)$ and dividing by the energy of that orthogonal or orthonormal function. Of course, if this is orthonormal function then this quantity is going to be one.

So, we can understand about this orthogonal expansion by using the analogies between vectors and signals, and if we make use of this analogy then this is really simple and it is like a five-finger problem.

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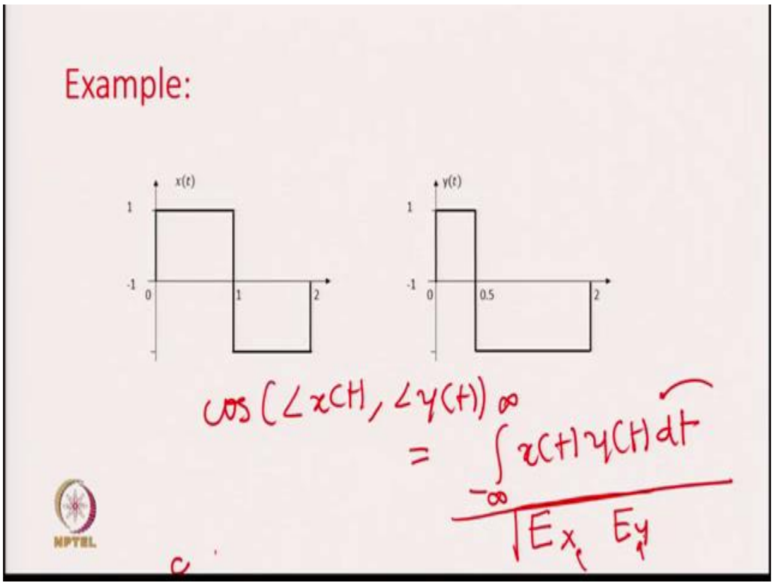

And let us do a question based on this. So, I have the two signals $x(t)$ and $y(t)$ and what I am interested in here is finding out the angle between the two signals.

Now, we have learnt that if I am interested in finding the angle between two signals, I can use this identity where I need to take the inner product of the signals involved. If they are real I do not need to take this conjugate and I have to divide it by the square root of energy of these two signals. So, I need to find out the inner product of the two signals involved and the energies of these signals involved.

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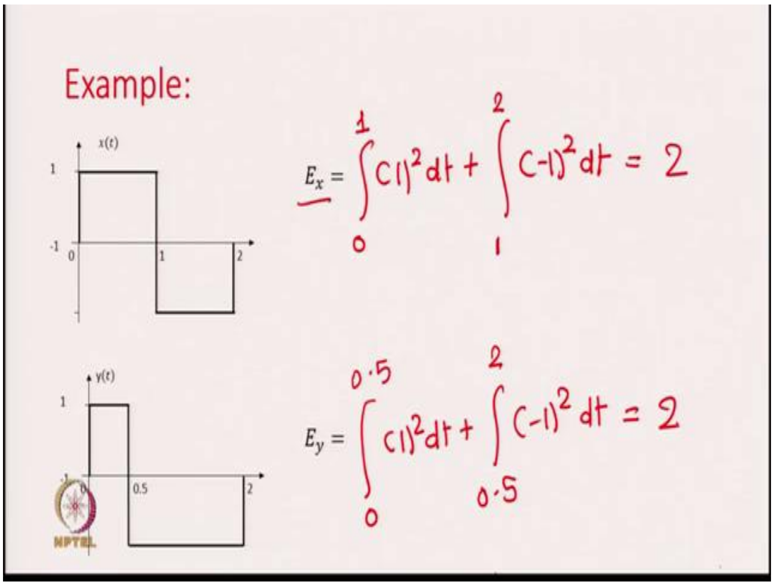

So, first let us derive what is the energy of signal $x(t)$ and this can simply be obtained by carrying out this integration and this will turn out to be 2. Similarly, I can find the energy of the signal $y(t)$ and this will also be 2. Then what I need to find out is the inner product between these two signals.

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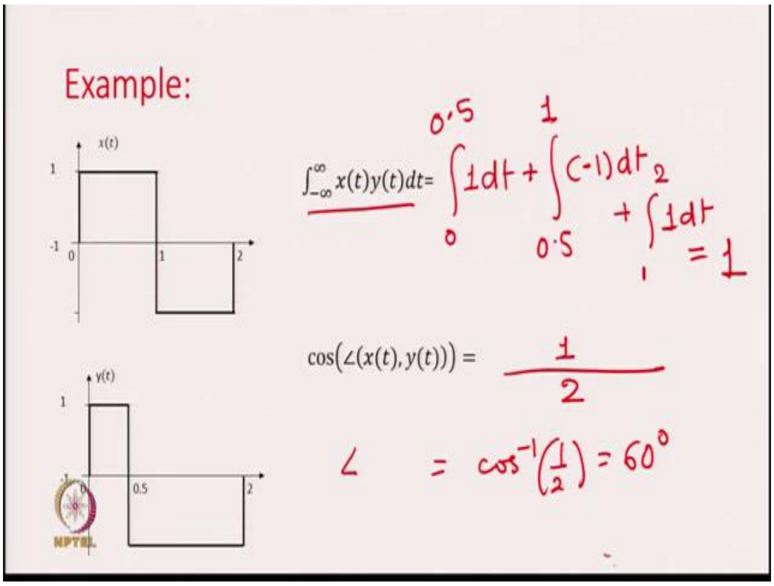

So, if I want to find the inner product between the two signals, I can break this integration in certain regions. So, first I am considering the area of this product of these two signals in the limit going from 0 to 0.5 and then this product is 1, then I can consider the integration from 0.5 to 1 where the product is -1 and then I can go from 1 to 2 where the product is 1. So, I will get 1.

So, cos of angle between $x(t)$ and $y(t)$ will turn out to be 1 / 2. So, angle will come out to be $\cos^{-1}\left(\frac{1}{2}\right) = 60°$. So, using some simple ideas we have been able to talk about the angle between the two signals.

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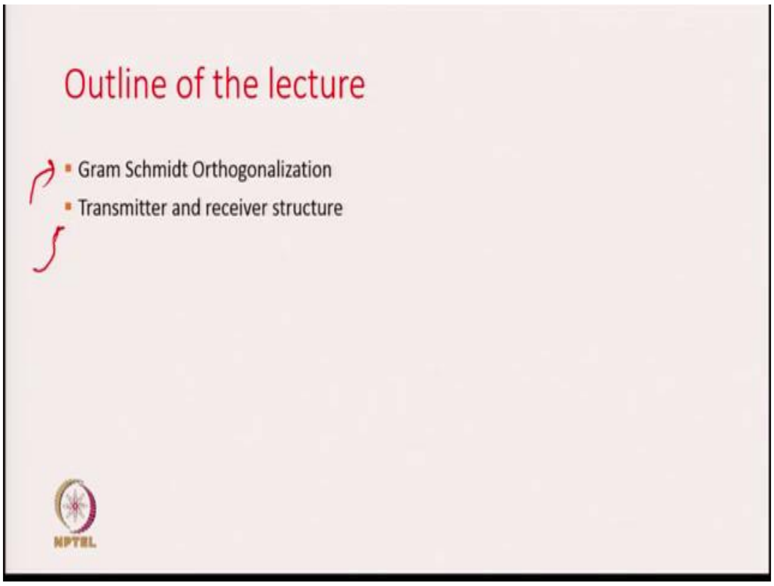

Let us now move on to the main theme of today's lecture where we will learn about Gram Schmidt orthogonalization procedure and then we will see the transmitter and receiver structure. So, let us get started with this Gram Schmidt orthogonalization procedure.

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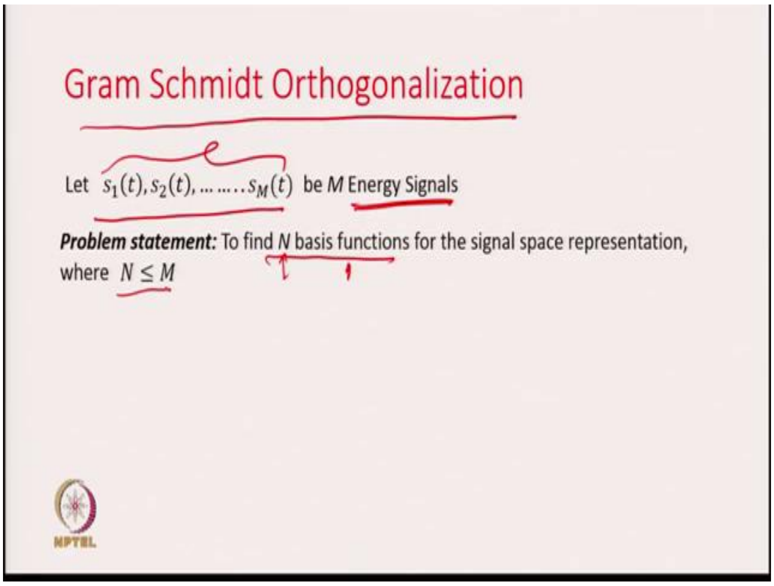

So, what is the objective of this orthogonalization procedure? Let us assume that we have M energy signals which means, the energy of these signals is finite. We want to find N basis functions for the signal space representation of these M energy signals. So, I want to convert these signals into vectors and I want to think about the signals in terms of basis functions. The number of basis functions that I have is N and we will find out that N ≤ M. Let us see how we do this orthogonalization procedure.

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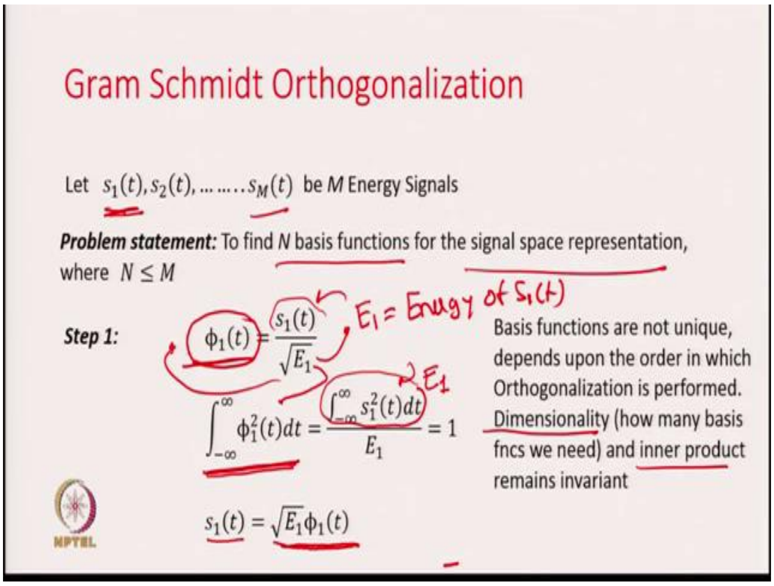

So, the first idea is let us start by picking a signal, I could have chosen any signal to start with, but let us start with this signal $s_1(t)$. Remember these basis functions that we will obtain will not be unique but will depend upon the order in which you do this orthogonalization. Since we have started with $s_1(t)$, we might get a different answer than what we would have got if we would have started with another signal $s_M(t)$. So, it really depends upon the order in which you are carrying out this orthogonalization procedure.

However, the dimensionality (that means how many basis functions we need for the signal space representation) would not change. It would be invariant to the order in which we are carrying out this orthogonalization procedure. So, dimensionality is invariant to the order and also the inner product between the two signals would also be independent of the order in which you carry out this orthogonalization procedure.

So, we have started by choosing the signal $s_1(t)$ and we are trying to find out the first basis function and we have said the first basis function is nothing but $s_1(t)$ divided by root of energy of signal $s_1(t)$. So, E_1 is the energy of $s_1(t)$. Why we have divided it with the root of energy of signal $s_1(t)$? Because we want this basis function to be an orthonormal basis function and for orthonormal basis function the energy of $\phi_1(t)$ should be 1.

So, energy of $\phi_1(t)$ can be obtained by carrying out this integration. We do not have mod in here because we are assuming that everything is real in this case, so all signals are real and the basis functions are also real. So, when we compute the energy of $\phi_1(t)$ you can simply plug in this value in this integration. So, as we have done here, and what you have in the numerator is the energy of $s_1(t)$. So, we have E_1 in the numerator, in the denominator also we have E_1. So, the energy of $\phi_1(t)$ turns out to be unity or 1. Hence, the first thing is clear that, this $\phi_1(t)$ basis function has a unit energy. Then $s_1(t)$ can be obtained in terms of $\phi_1(t)$ or rather $\phi_1(t)$ in terms of $s_1(t)$.

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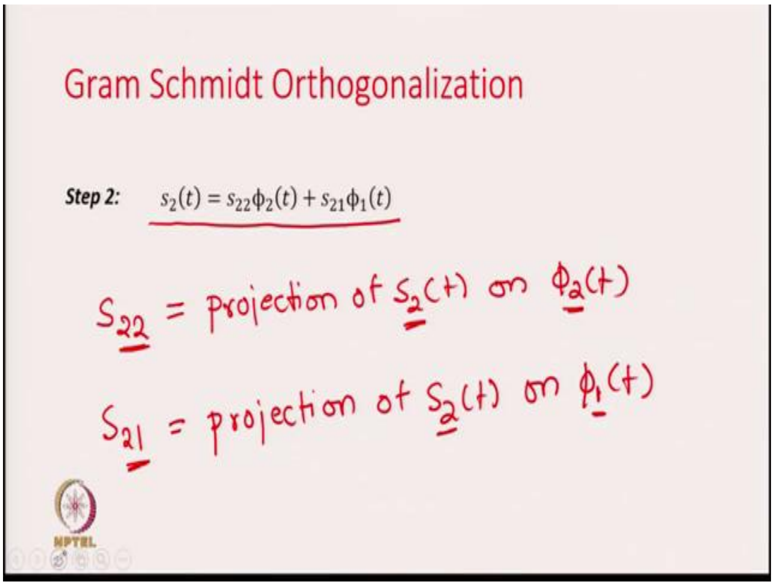

Let us move on, and let us try to obtain $\phi_2(t)$, but before finding out $\phi_2(t)$ let us understand this expression.

So, what is the meaning of these terms s_{22} and s_{21}? So, s_{22} is the projection of $s_2(t)$ on $\phi_2(t)$ which explains this number 22. Similarly, s_{21} will be projection of $s_2(t)$ on $\phi_1(t)$. So, this explains this subscript. So, what do we mean with this after having understood the nomenclature?

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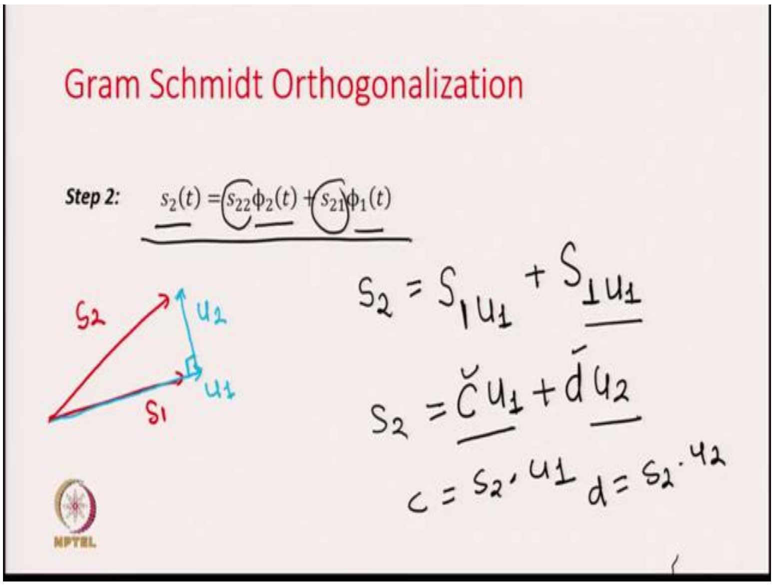

So, let us consider two vectors first. So, if I have vectors s_2 and s_1. And let us assume that I have considered a unit vector in the direction of s_1, and then there is a unit vector which is perpendicular to u_1. And we have already learnt, in one of the previous lectures, that if you have a vector you can decompose this vector in terms of a parallel component. So, I have parallel and perpendicular components to vector u_1.

So, this I say that maybe I have some constant times a vector u_1, and I have some constant times vector u_2 because vector u_2 is perpendicular to vector u_1. So, I can think about a vector decomposition in terms of two unit vectors, where these two unit vectors are perpendicular to each other. And what remains is finding out these coefficients c and d. If you want to find c it is quite straightforward: c can be obtained by $s_2.u_1$ and d could be obtained by $s_2.u_2$.

So, these vector operations are easy and we are using the same ideas, but for the case of signals. So, I am saying that a signal can be decomposed in terms of two orthonormal functions and s_{22} and s_{21} give me the projection of $s_2(t)$ along those orthogonal or orthonormal functions.

So, after having understood this basic idea of how we are writing out this expression, let us try now to find out $\phi_2(t)$.

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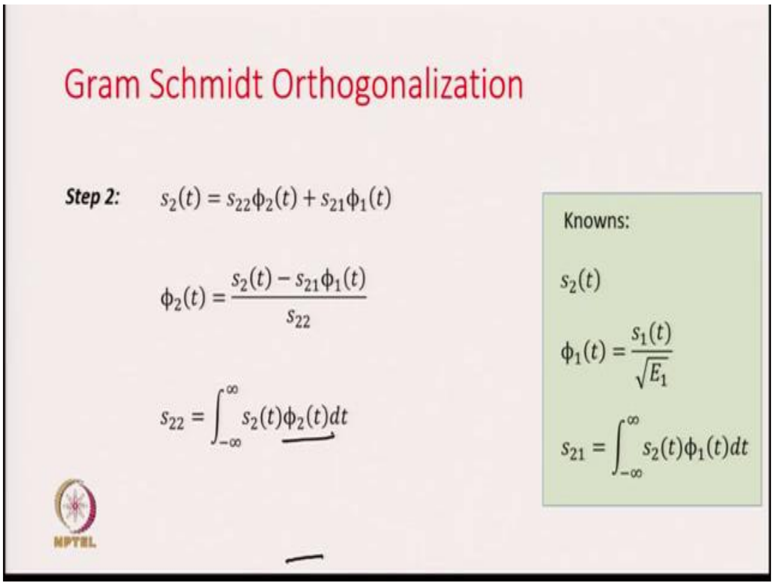

So, let us see if we have all the knowledge available to construct this $\phi_2(t)$. In this equation, we already know $s_2(t)$ because it must be given. We have already obtained $\phi_1(t)$ in terms of $s_1(t)$; s_{21} is also calculated by taking the inner product of $s_2(t)$ with $\phi_1(t)$. Only s_{22} is unknown. Now, s_{22} is also a function of $\phi_2(t)$ and hence there is a chicken and egg problem: until and unless we know s_{22} we cannot find $\phi_2(t)$, and until and unless we know $\phi_2(t)$ we cannot find s_{22}. So, instead of directly finding $\phi_2(t)$ what we can do is, we can define an intermediate function $g_2(t)$ which is the difference of $s_2(t)$ and $s_{21}\phi_1(t)$.

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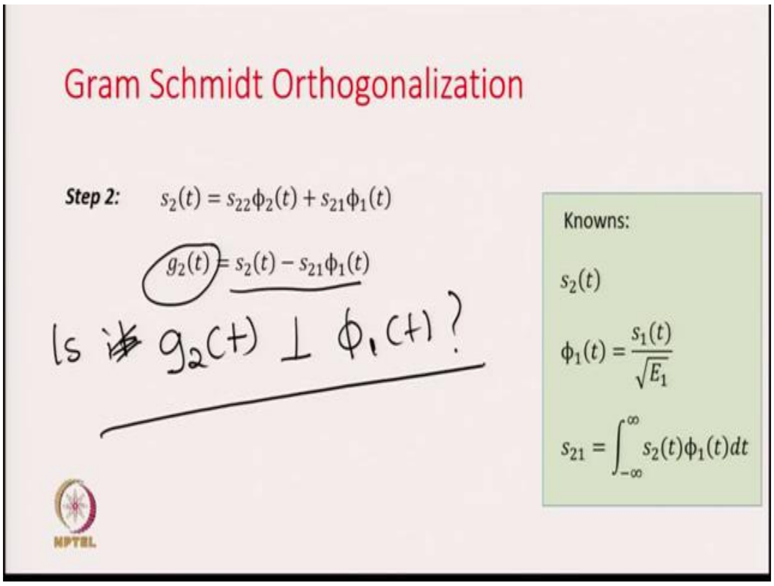

Now, let us first check whether $g_2(t)$ is perpendicular or orthogonal to $\phi_1(t)$? We can check this out by finding out the inner product of $g_2(t)$ with $\phi_1(t)$, and if that inner product turns out to be zero, we will be sure that $g_2(t)$ is perpendicular to $\phi_1(t)$.

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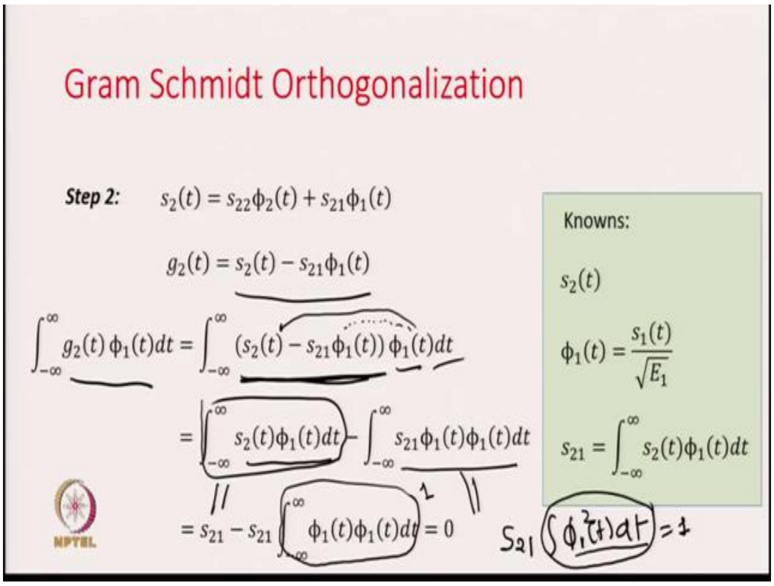

So, we are trying to compute the inner product of $g_2(t)$ with $\phi_1(t)$. So, here we can plug in the value of $g_2(t)$ and multiply with $\phi_1(t)$. Again there are no conjugates because everything is real and then we can work out this integration.

So, taking $\phi_1(t)$ here we get this integration and then multiplying $\phi_1(t)$ we get this expression. And now by definition this quantity is s_{21} which is a constant. So, it could be pulled out of this integration and because $\phi_1(t)$ is orthonormal so its energy will be 1. So, essentially what we have is s_{21} - s_{21} and this is zero. So, we have proved that $g_2(t)$ is perpendicular or orthogonal to $\phi_1(t)$.

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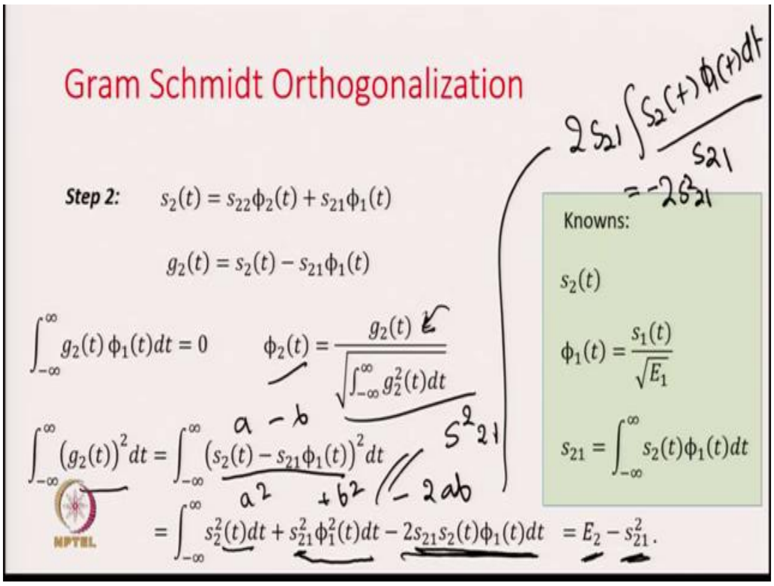

So, now, after having understood that $g_2(t)$ is perpendicular to $\phi_1(t)$ because its inner product is zero, we can quickly define $\phi_2(t)$ as $g_2(t)$ divided by the length of $g_2(t)$ and length of a function is the square root of energy of that function. We have this $g_2(t)$ and we divide by the square root of energy of $g_2(t)$. If we do this then we know for sure that $\phi_2(t)$ has a unit energy and it is an orthonormal function and that is why we are doing this scaling operation.

Energy of $g_2(t)$ could simply be obtained by finding out the integration of a square of $g_2(t)$. So, plugging in the value of $g_2(t)$ here, then using the identity $(a-b)^2 = a^2 + b^2 - 2ab$. So, this quantity is E_2 which is the energy of the second signal in consideration. We already know how to work this integration out this even by look you should be able to tell that this should be nothing but s_{21}^2 and this expression is $2s_{21}\int s_2(t)\phi_1(t)dt$, and you know that this is s_{21} by definition. So, this will turn out to be - $2s_{21}^2$. So, finally, we have $s_{21}^2 - 2s_{21}^2 = -s_{21}^2$ and we can say that $\phi_2(t)$ is nothing but $g_2(t)$ divided by $\sqrt{E_2 - s_{21}^2}$.

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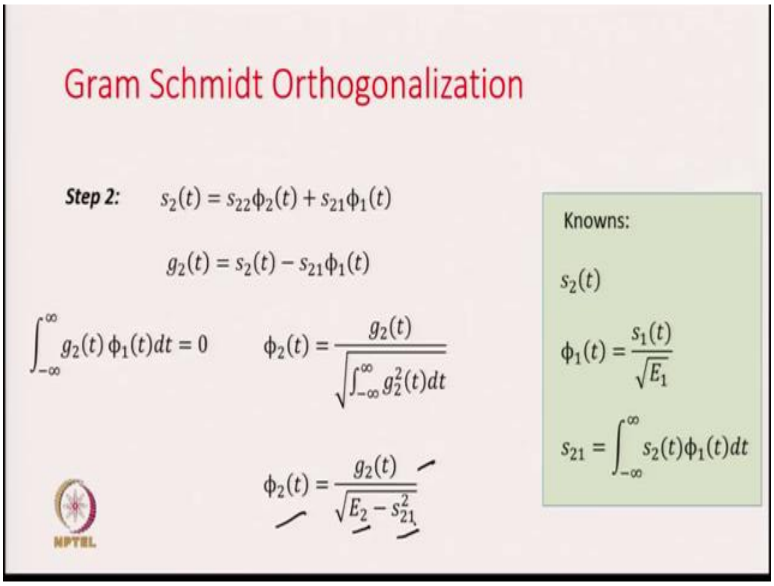

So, we have obtained $\phi_2(t)$, and similarly we can carry out this orthogonalization procedure and we can obtain other orthonormal functions as well. Let me just do one more.

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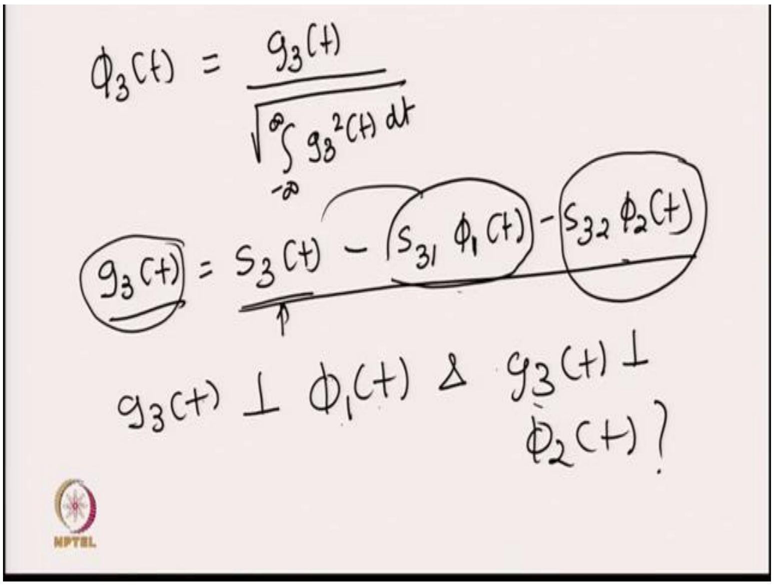

So, we can start by computing $\phi_3(t)$ which can be conveniently obtained as $g_3(t)$ divided by the square root of energy of $g_3(t)$. Let us assume my third signal is $s_3(t)$, and if I subtract from this $s_3(t)$ the projections of this $s_3(t)$ in the directions of $\phi_1(t)$ and $\phi_2(t)$, then we will get $g_3(t)$.

Now, it should be clear from inspection that $g_3(t)$ will be perpendicular to $\phi_1(t)$ and it will be perpendicular to $\phi_2(t)$. Why is this? Because we have already subtracted from $s_3(t)$ whatever projections it had along $\phi_1(t)$ and $\phi_2(t)$, that means, there should not be any energy left in $s_3(t)$ along $\phi_1(t)$ and $\phi_2(t)$ directions and hence the inner product of $g_3(t)$ with $\phi_1(t)$ should be zero and inner product of $g_3(t)$ with $\phi_2(t)$ should also be zero. You can work out the proof yourself. Here we are just presenting the intuitive idea behind why should this $g_3(t)$ be perpendicular to $\phi_1(t)$ and $\phi_2(t)$. And we can generalize this procedure.

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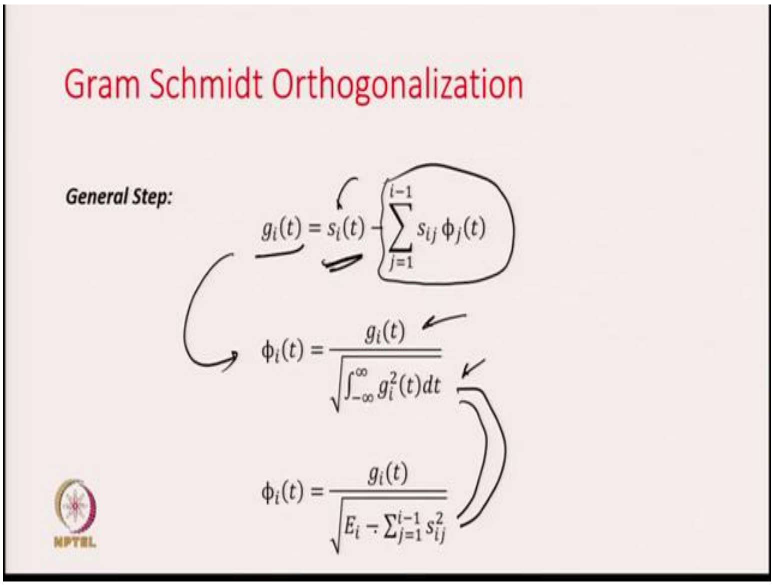

First, you can obtain $g_i(t)$ by subtracting (from the i^{th} signal) the projections of this i^{th} signal on all previous basis functions, and from this $g_i(t)$ you can obtain this $\phi_i(t)$ by just dividing $g_i(t)$ with the proper scaling factor, and the proper scaling factor is nothing but the square root of energy of $g_i(t)$. And as we have worked out, this expression is also same as this quantity.

So, using this idea of orthogonalization you can obtain various basis functions for the orthogonal expansion of a signal. The basis functions that we have calculated are orthonormal basis functions. So, let us make this more concrete with one example.

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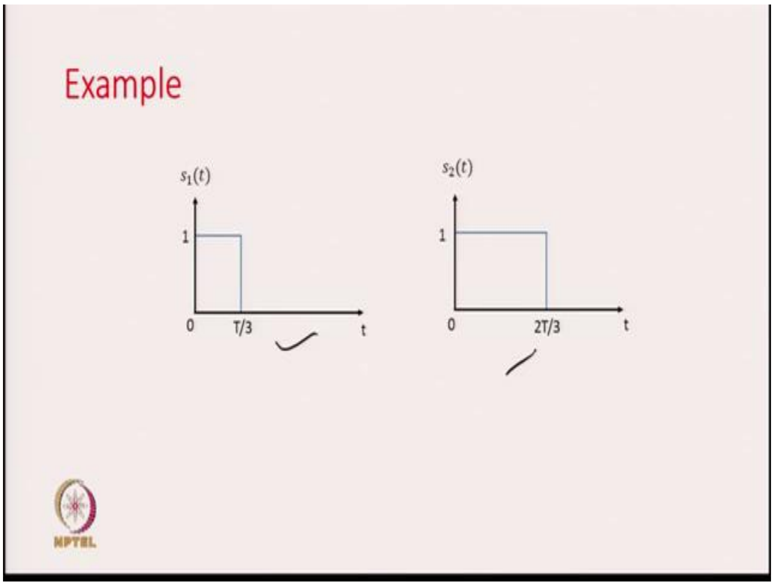

So, let us consider two signals, $s_1(t)$ and $s_2(t)$ as given here, and we want to find the signal space representation for these two signals. So, let us start by defining $\phi_1(t)$.

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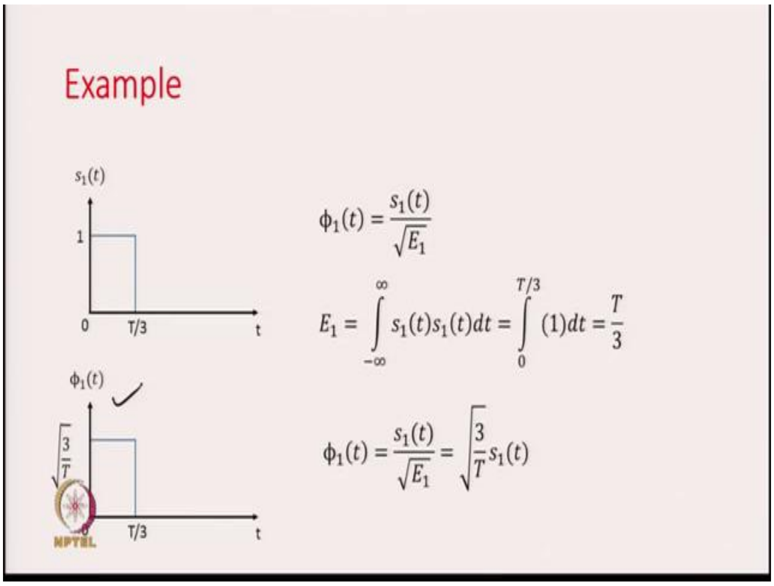

So, $\phi_1(t)$ can be easily obtained by dividing $s_1(t)$ by the square root of energy of $s_1(t)$ or rather dividing it by the length of $s_1(t)$. Energy can be simply obtained by taking the inner product of the signal with itself and you know that instead of running out this integration from - ∞ to + ∞ you can simply run this integration between 0 to T / 3 because everywhere else the product would be zero and in this limit the product would be 1. So, we can easily obtain energy of $s_1(t)$ as T / 3.

So, once this energy is known we can easily obtain this $\phi_1(t)$ by dividing $s_1(t)$ by the proper scaling factor and the scaling factor is root of energy. So, $s_1(t)$ divided by $\sqrt{T/3}$ and we can also draw this. So, if $s_1(t)$ is known, then $\phi_1(t)$ is exactly the same thing only the magnitude changes and instead of 1 it becomes $\sqrt{3/T}$. Let us now go to the second step i.e. finding $\phi_2(t)$.

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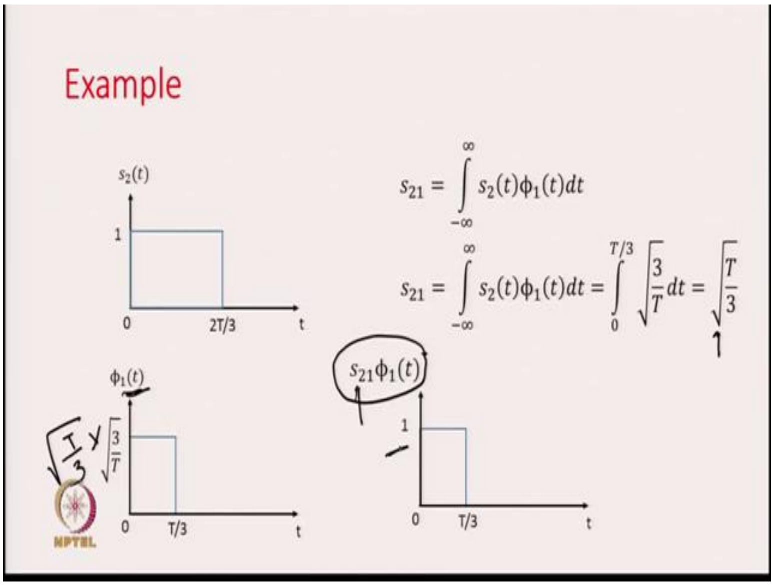

So, once we know this $s_2(t)$ and $\phi_1(t)$, we can first obtain s_{21} which is the projection of this $s_2(t)$ on $\phi_1(t)$. So, you can work it out yourself and find out that $s_{21} = \sqrt{T/3}$. Now, we can also find out the signal $s_{21}\phi_1(t)$. So, you just have to multiply $\phi_1(t)$ with $s_{21} = \sqrt{T/3}$ and the product is nothing but simply 1.

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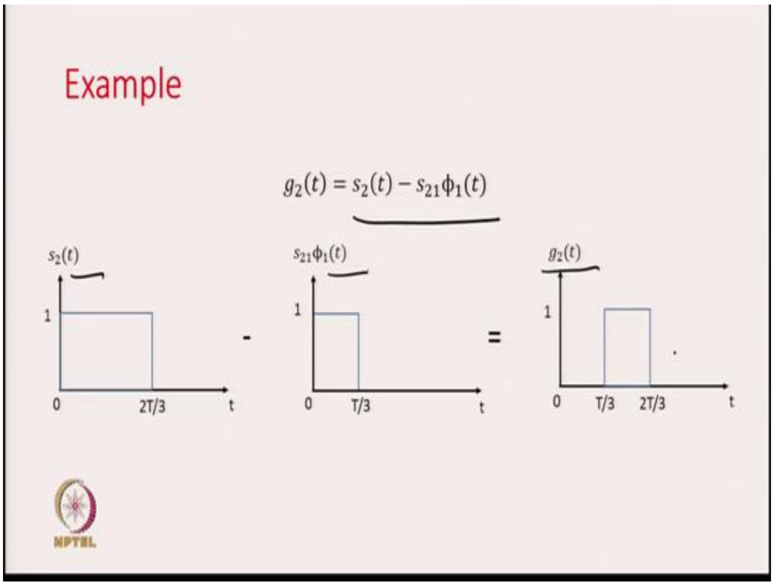

And thus, you can easily obtain $g_2(t)$ which is nothing but the difference between these two signals. Here, $s_2(t)$ is already known and $s_{21}\phi_1(t)$ we have already obtained. If we take the difference between these two signals, $g_2(t)$ can simply be obtained like this. So, finding $g_2(t)$ is easy.

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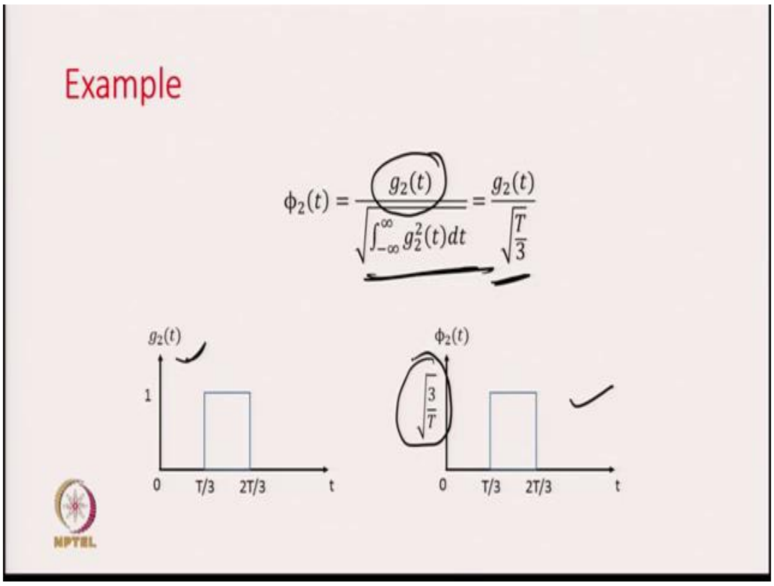

If we know $g_2(t)$, we can find $\phi_2(t)$ by dividing $g_2(t)$ by the length of $g_2(t)$. Length of $g_2(t)$ again is nothing but the root of energy of $g_2(t)$. You can simply obtain that this is again $\sqrt{T/3}$. So, $\phi_2(t)$ is same as $g_2(t)$ only we need to change the amplitude.

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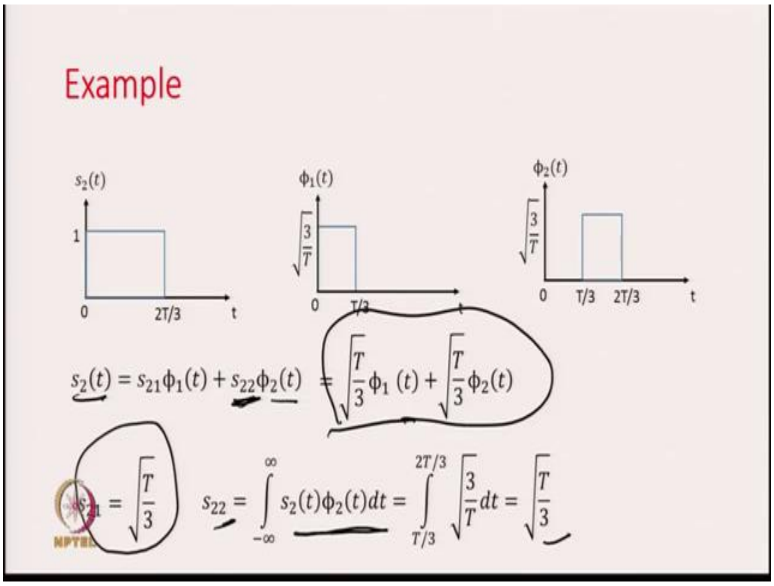

Now, what remains is we have yet to find s_{22}, but that should also be trivial because we have been able to find out $\phi_2(t)$ and s_{22} is nothing but it is the projection of $s_2(t)$ in the direction of $\phi_2(t)$. So, I can write $s_2(t) = s_{21}\phi_1(t)+s_{22}\phi_2(t)$ and s_{22} again can be simply obtained by carrying out this integration. It is trivial and I leave it to you to do this and again it comes out in with the same form that this is $\sqrt{T/3}$. So, finally, the expression for $s_2(t)$ is as shown above.

Let us now try to wind up this issue. So, we started with $s_1(t)$ and $s_2(t)$, we easily derived the expressions for $\phi_1(t)$ and $\phi_2(t)$ and we have defined $s_1(t)$ and $s_2(t)$. Now, very conveniently we can draw the signal space representation of these two signals.

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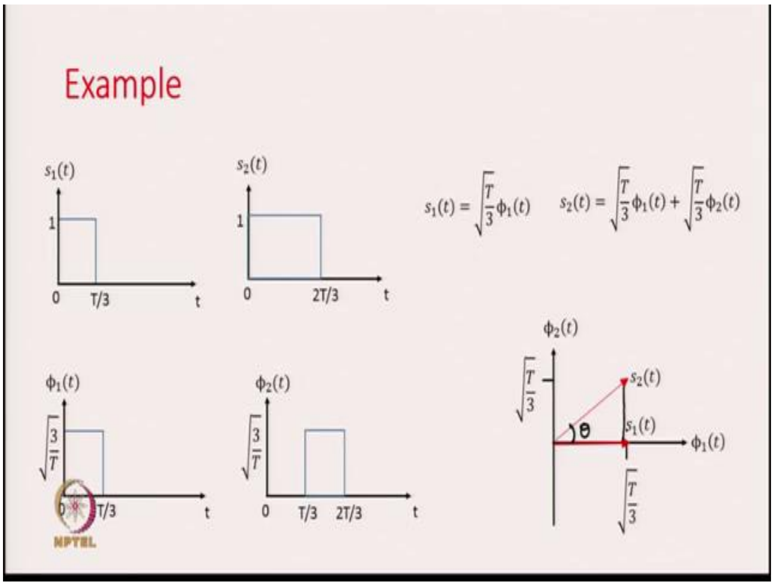

So, we can assume that we have two vectors $\phi_1(t)$ and $\phi_2(t)$. These vectors are perpendicular vectors. They are orthonormal vectors, so they have to be orthogonal, so they have to be perpendicular. And these vectors come out with unit energy and $s_1(t)$ is this vector. So, see that this only has a component of $\phi_1(t)$ and this has got this value along $\phi_1(t)$. So, I have to consider a point along $\phi_1(t)$ with this value, and $s_2(t)$ contains both $\phi_1(t)$ and $\phi_2(t)$ and it has the value along $\phi_1(t)$ of root t by 3 and also along $\phi_2(t)$ of root t by 3. So, this is this vector.

Now, what you can see very clearly from here is that now we can talk about the angle between these two vectors and you can easily obtain the distance between these two vectors. So, we have been able to convert the signals into a vector space, and this is the idea of signal space representation. If I would have had more signals and again nothing changes conceptually. You just have to do more algebra and you can convert the set of time signals into a convenient vector space or signal space representation.

Now, let us move to this transmitter and receiver structure because it makes use of the ideas that we have just developed while going through this signal space. So, the transmitter and receiver structure, follows directly from the ideas of signal space. We want to maximize what we have learnt and thus it is a good idea to introduce this already.

The second important point that you can note is when we confront our self with noise and other non-idealities, these will not change the receiver and transmitter architectures or structures. So, it is a good idea to think about these transmitters and receivers structures, already from today because they form the substratum of what is digital communication.

So, let us proceed by first looking into the transmitter design.

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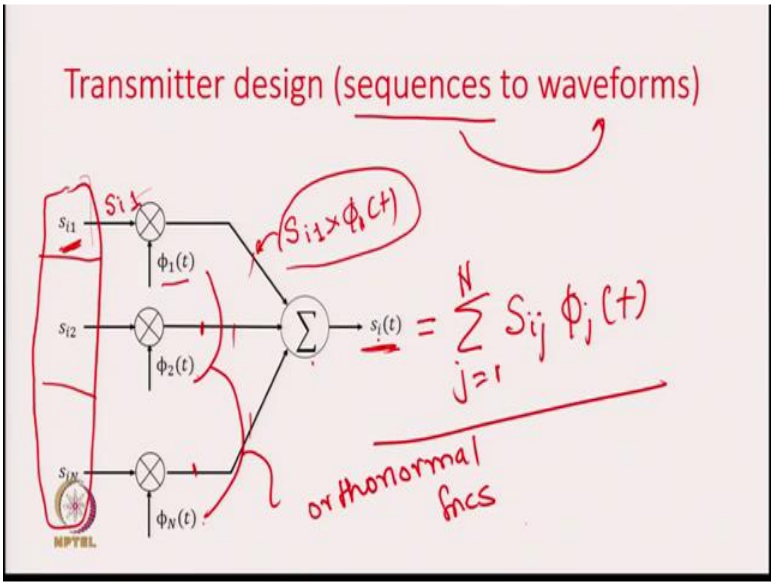

So, transmitter design solves a simple problem it converts these sequences to waveforms. So, we have a sequence and as I have already said a sequence can be understood as a bunch of numbers. So, I have a bunch of numbers here, these numbers can be real numbers or these numbers can be quantized real numbers. We call these quantized real numbers as symbols.

So, I take in a number and let us first focus on what the subscripts of this number denote. So, here i denotes that this is a sequence corresponding to signal $s_i(t)$ and 1 denotes that is the first number of the sequence. So, I have this number s_{i1}, I multiply this number with $\phi_1(t)$ and what I get here is $s_{i1}\phi_1(t)$.

So, at the output of these multipliers, I will be going to get similar products and this summer collects all such products and we get $s_i(t)$. So, $s_i(t)$ would be the sum of these products where j spans from 1 to N. And this is exactly what we have said that if you want to think about a waveform, a waveform is nothing but it is a linear combination of orthonormal functions. And $\phi_1(t)$, $\phi_2(t)$ and $\phi_N(t)$ are orthonormal functions.

So, what is the transmitter design? You have a bunch of numbers, you multiply these bunch of numbers with orthonormal functions, you collect all products by using a summer and you get to a waveform. So, this is as easy as it can get.

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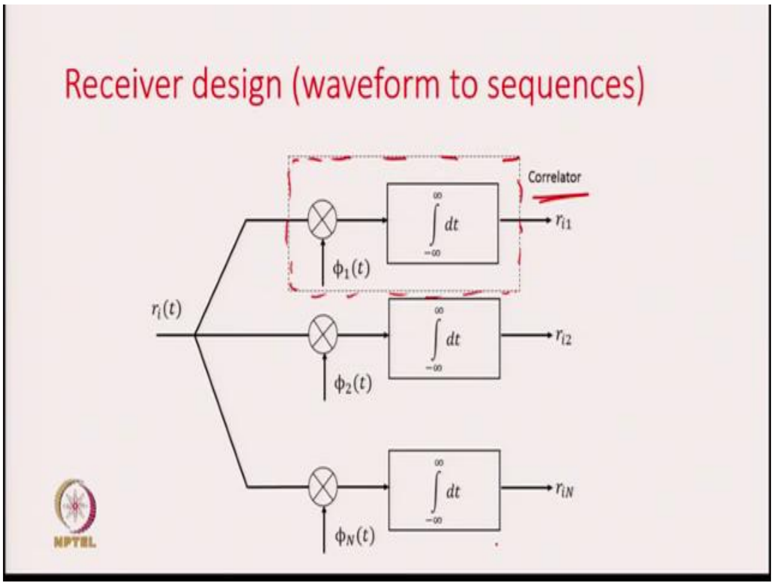

Let us now move to the receiver design. So, the receiver solves the problem of converting a waveform into a sequence. So, I have a waveform which I call as $r_i(t)$. I pass this waveform through a multiplier. Again this multiplier is fed with orthonormal function $\phi_1(t)$. So, at the output of the multiplier I am going to have $r_i(t)\phi_1(t)$. And then I pass it through an integrator which integrates this product from - ∞ to + ∞. I get the inner product of $r_i(t)$ with $\phi_1(t)$, and it gives me the coefficient of projection of $r_i(t)$ on $\phi_1(t)$ and I call this as r_{i1}.

Similarly, I can pass this $r_i(t)$ through other orthonormal functions and I can get the coefficients of projections of this $r_i(t)$ on other orthonormal functions. In this way, what I get is a sequence and here the numbers in the sequence corresponds to the projection of $r_i(t)$ or the received waveform on various orthonormal functions. So, again in this way using the simple design, I can convert a received waveform to a sequence.

Let us now look something more on the slide. This block which consists of a multiplier and integrator is known as a correlator. So, in the receiver we have a bunch of correlators or we have a bank of correlators.

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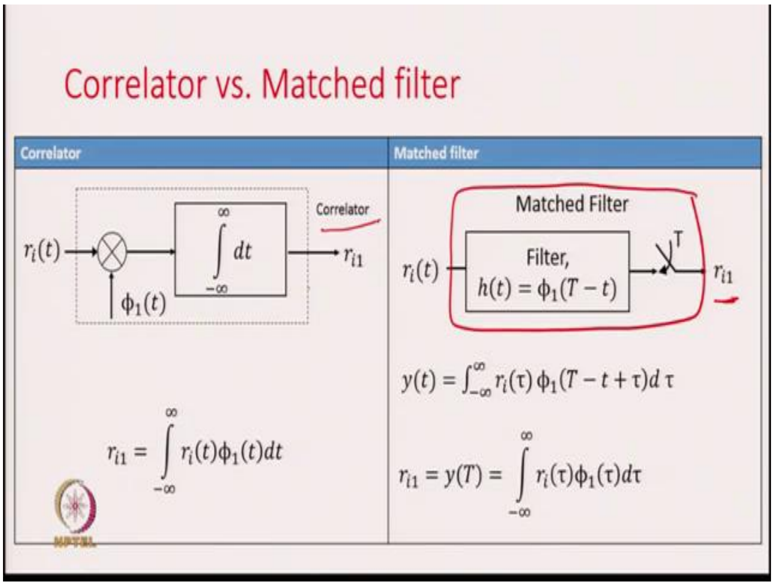

Let us now try to look into what is this matched filter? So, matched filter is same thing as correlator, however, from practical point of view, its implementation is different. So, correlator has a multiplier and has an integrator, on the other hand a matched filter has a filter (a linear time invariant filter) and a sampler. So, let us see what is the result or output of this block.

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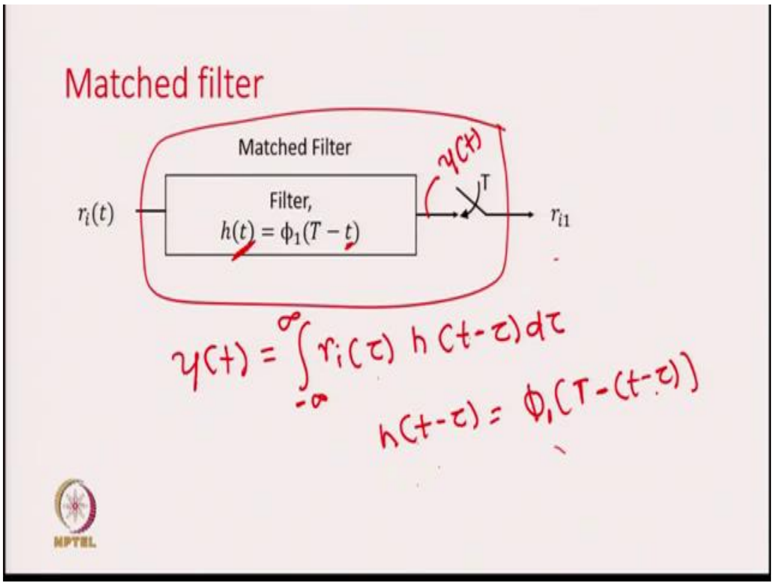

So, suppose I have a matched filter which has an impulse response like this. Let us try to investigate what is the output of this system.

So, let us first concentrate on the output $y(t)$ at this point. So, $y(t)$ can be given by convolution of input with impulse response which you must know is this. So, this is $y(t)$ output of this filter, which is obtained simply by carrying out the convolution operation between the received waveform and impulse response of the filter. So, we replace t by t - τ.

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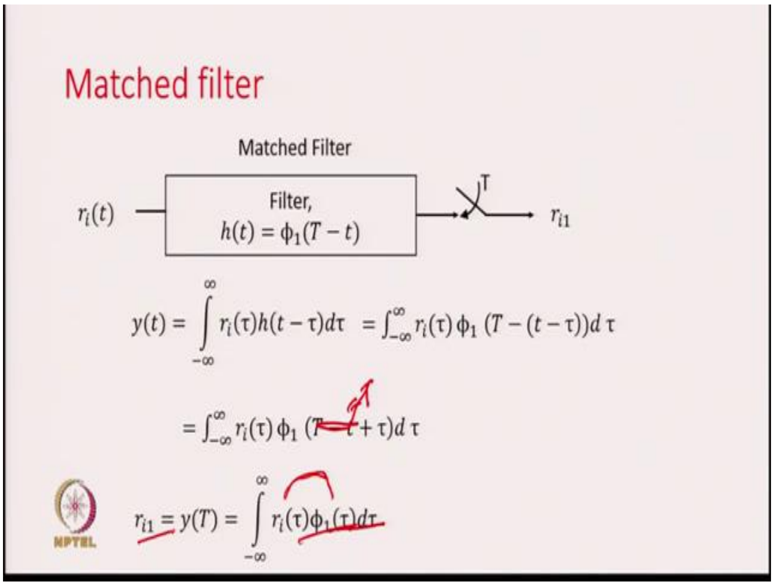

As I have said $y(t)$ is obtained by the convolution of the received waveform with impulse response of the filter, and because I already know that the impulse response of the filter takes in this form, so I can substitute this value of impulse response and I can get an expression of $y(t)$. Now, we have calculated $y(t)$ and the idea is suppose I sample this $y(t)$ at T, I get r_{i1}. So, I want to look this $y(t)$ only at the time instants, where t = T. So, when t = T, then this becomes zero and I have $\int r_i(\tau)\phi_1(\tau)d\tau$. This is the inner product of this received waveform with this orthonormal function $\phi_1(t)$ and this gives the coefficient of projection of $r_i(t)$ on $\phi_1(t)$ which is r_{i1}.

So, this system has the same output as this system and hence matched filter gives me exactly what a correlator gives. So, matched filter theoretically is same thing as the correlator.

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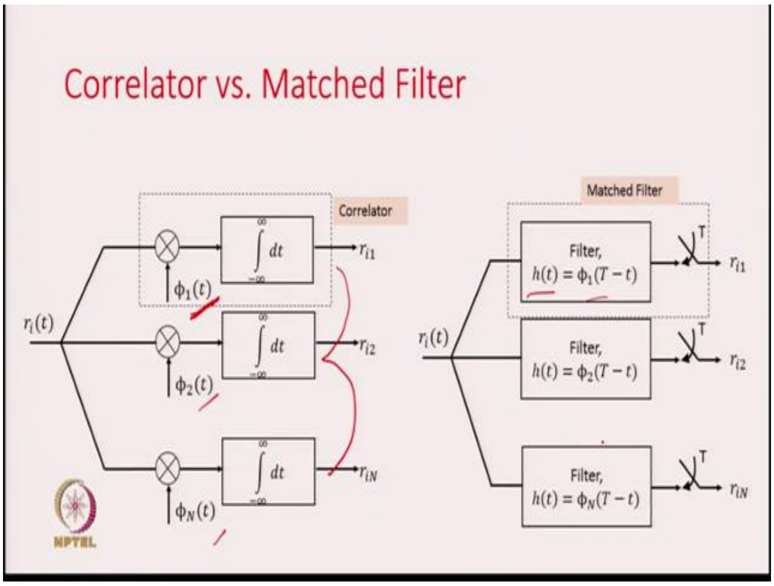

So, now, we know that in a receiver you have this bunch of correlators. Each correlator chooses a multiplier where one of the multiplying factors is the orthonormal function. In the case of the matched filter, we choose various filters, and samplers, and the impulse response of these filters is matched to the orthonormal functions.

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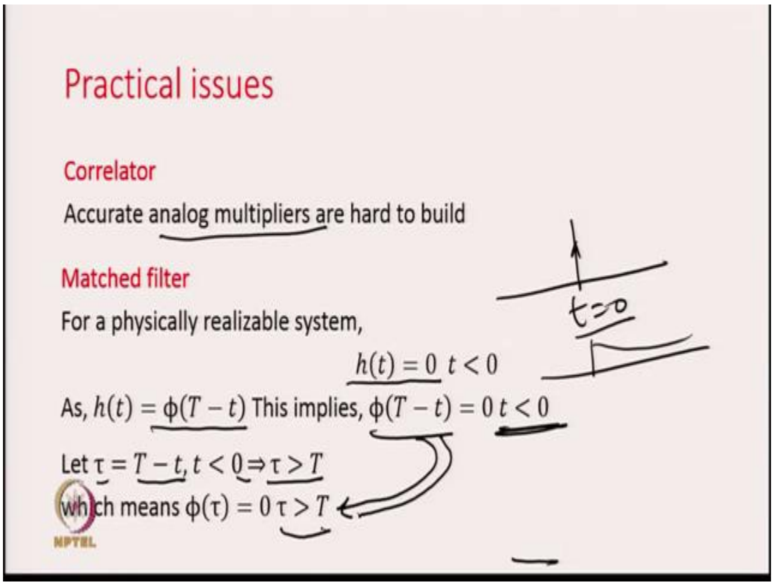

So, the question is which one should we prefer? Accurate analog multipliers are hard to build and that is why this matched filter design was preferred. It is easier to make filters rather than accurate analog multipliers. But for this matched filter there is an issue of physical realizability of the system.

So, we you must have learned in the course in signals and systems that we can only realize systems which are causal systems. Causal system means the systems which are non-anticipatory systems that means, these are the systems which cannot anticipate an input. So, if you want to realize a system it will definitely be a causal system. It will not be able to anticipate which input is going to come in future and such a system is known as causal system.

One property of a causal system is that the impulse response of the causal system is zero for time $t < 0$. And why is this? Because if a system is causal system, it can only have the response to an impulse after an impulse is applied. Impulse response is the response of the system to an impulse. So, if a system is causal, it can only respond to an impulse after an impulse is applied, and impulse is applied at $t = 0$. Impulse signal is a signal which exist or has its effects at around $t = 0$.

So, the response of the system can only come after $t = 0$ whatever that response is. It cannot proceed before an impulse is applied because it is a causal system or a non-anticipatory system. So, the impulse response of a causal system is zero for time $t < 0$, and this is also the condition for the physical realizability of a system. Now, we know what the impulse response of the filter is. This implies that $\phi_N(T - t) = 0$ for $t < 0$ because that is the condition for the system to be a causal system.

Now, if we substitute $\tau = T - t$, then $t < 0$ implies that $\tau > T$. So, one way I in which I can write this expression is that this condition translates to the condition that $\phi_N(\tau) = 0$ for $\tau > T$. That means, you can only have orthonormal functions which live for a duration of T. So, it cannot live from $-\infty$ to $+\infty$. So, the physical realizability of matched filter put some restrictions on the kind of orthonormal functions that the system can deal with.

Let us move ahead and let us see now the complete full blown receiver implementation. Maybe it is too farfetched because we have not yet studied noise and so on so forth, but let us see how the full blown receiver looks like and we will refine this when we study detection and other things.

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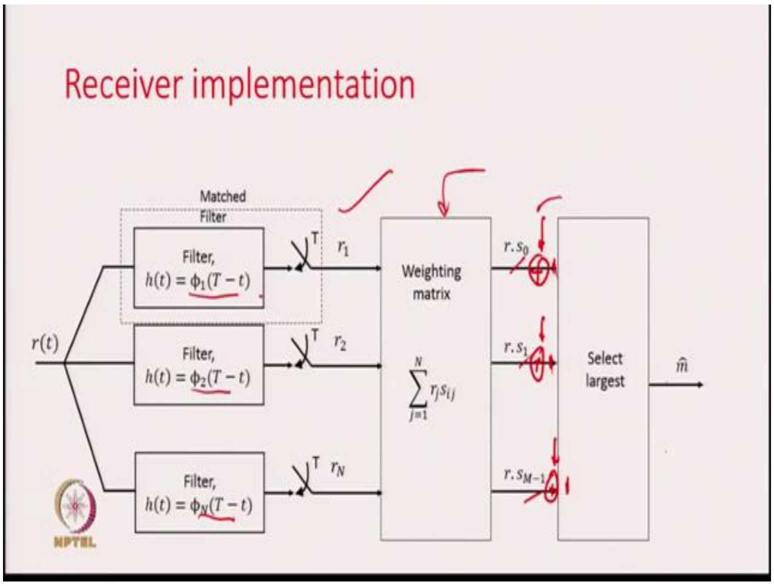

So, the first part of the receiver is taking this received waveform and produce a sequence corresponding to that received waveform and then these sequences goes to this weighting matrix which computes the inner product between test sequence, the received sequence and the sequence available from the signal set.

So, remember receiver knows the signal set and the sequences corresponding to that signal set, so this weighting matrix just computes the inner product between this received sequence and the sequence of the signal set. Of course, we are assuming everything to be real in this case, so there are no conjugates and so on so forth.

Now, there is a mental exercise that you can do and think about that this operation wherein you are computing the inner product between the sequences is same thing as the operation where you calculate the inner product between the waveforms. That means, if I take the inner product between the two waveforms. this inner product is the same thing as inner product between the corresponding sequences. So, this is a simple exercise and I request you to do it yourself.

So, this weighting matrix actually gives the inner product of the received waveform with all signals, assuming that there are M signals involved here. Now, based on these output, receiver selects the possible transmitted signal. So, it decides in the favor of the signal which has the largest output for this operation. Namely, for example, if $r.s_1$ is largest out of all these inner product operations then the receiver selects s_1 as the possible transmitted signal. So, this happens in the receiver implementation, it receives a waveform and from that received waveform it has to decide for the possible transmitted signal.

Now, I have little bit simplified this picture and this picture will be true if I assume that all these signals happen with equal probability and these signals have equal energy. If this is not the case then what we need to add a bias here, and based on these outputs then the decision happens. So, this bias takes care if the signals are not equi-probable signals or these signals come up with different energies then this bias takes care of that.

So, how to choose this bias and why we are choosing this bias - all these things will become clear when we do detection. At this point it is important to appreciate that the decision that happens in a receiver is basically motivated by a bunch of inner product operations.

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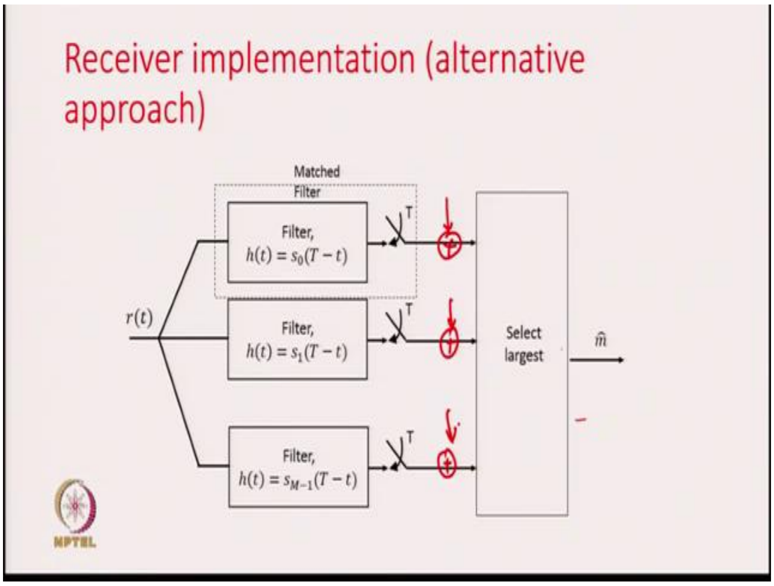

One another issue of course, you can have implemented this receiver in a slightly different way. So, in the previous case we had these matched filters whose the impulse response were matched to orthonormal functions. In this case, we are matching the impulse response directly to the signals. This would directly give me the inner product between the received waveform and the signals.

Now, once I directly get these inner products what you can appreciate is that there is no requirement of this weighting matrix and hence this architecture looks simpler. So, the question is which one should I prefer? Number of filters required here is N, and number of filters that are required in this case is M and because M >> N so this architecture would require more number of filters than this architecture. And hence this is a preferred design in most cases. Of course, here as well, we have to add bias if the signals are not of equal energy and they do not occur with equal probability before taking the decision.

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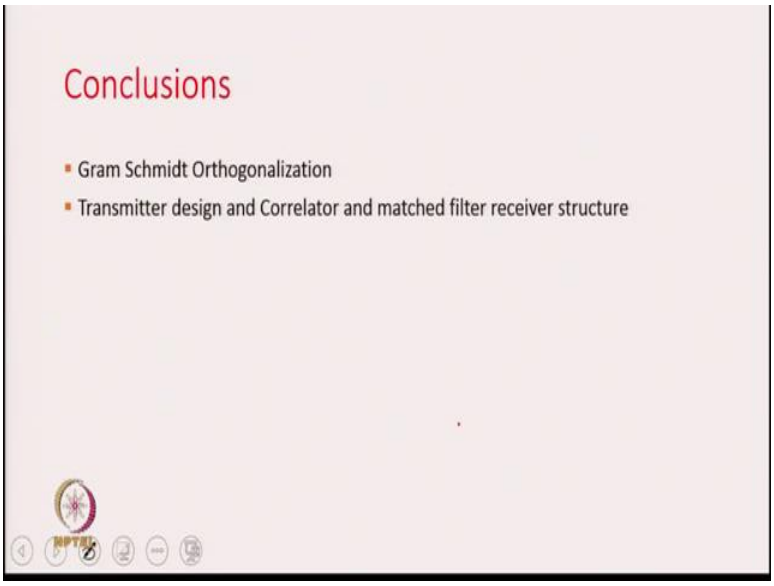

So, with this we have come to the conclusions of today’s lecture. In this lecture we have studied about Gram Schmidt orthogonalization. We have studied about transmitter design, and we have looked into the receiver design as well and we have looked into two important receiver designs, one design we refer to as correlator and the other designing architecture is referred to as matched filter. So, in the next lecture, we will start thinking about other ways in which we can expand our signal.

Thank you.