Principles of Digital Communication
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Detection
Lecture – 36
Performance of Orthogonal Modulation Schemes & Bit – Level Demodulation

Good morning, welcome to a new lecture in detection and in today's lecture we will be talking about Performance of Orthogonal Modulation Scheme and then we will look into this Bit-Level Demodulation ok.

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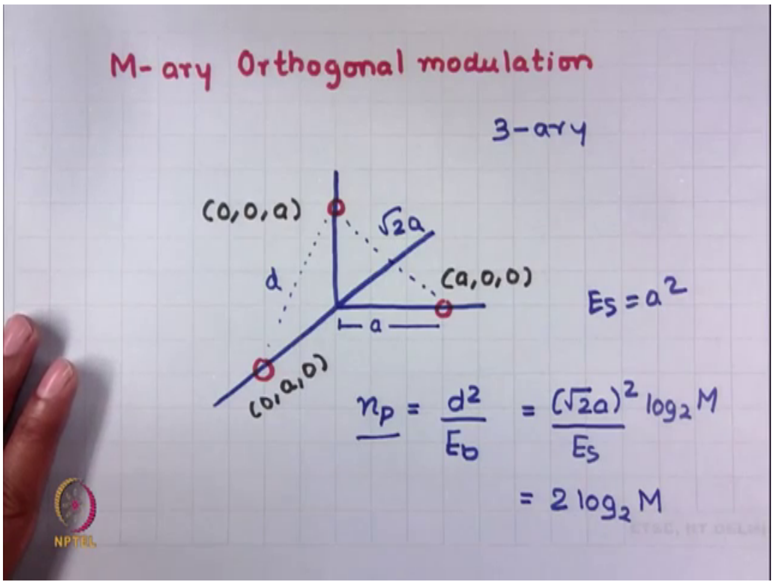

So, let us get started and as I have said we will talk about this M-ary orthogonal modulation scheme. So, in this orthogonal modulation scheme we have one signal per orthogonal dimension ok. So, for example, this is the case of 3 array orthogonal modulation a scheme.

So, underlying orthogonal modulation scheme can be fsk, can be ppm, can be the orthogonal modulation realized using Walsh Hadamard codes. So, whatever may be the kind of orthogonal modulation scheme it does not matter you can use the same framework to analyze these various modulation schemes in AWGN channel ok. So, what we notice here is we have 3 symbols and these 3 symbols have equal energy right. So, they are at the same distance from the origin. So, they will have the same energy. And secondly, we also notice that these symbols are at the same distance from each other alright.

So, let us say if the distance between the symbols is d then every symbol is at a distance of d from each other right. So, the first thing that we want to define when we are talking about this modulation schemes is the power efficiency right, we have started looking into the power efficiency of various modulation schemes like com, psk and pam in the last lecture. And today we will be looking into this power efficiency of orthogonal modulation schemes. So, what is power efficiency? It is d square by E b; and what is d in this case?

If this distance is a d will be simply root 2 a; and what is E b? E b as always is E s by log 2 M ok. So, power efficiency we have got is root 2 a square times log 2 M divided by E s. And what is this E s a symbol energy? E s is simply a square ok. So, what we will get is the power efficiency is 2 times log 2 M and this power efficiency is very different from the power efficiency that we got in the case of com, pam and psk. What happened in the case of power efficiency there? There the power efficiency decreased with them, here the power efficiency increases with them, does as you increase the order of modulation. The modulation schemes becomes more and more power efficient ok.

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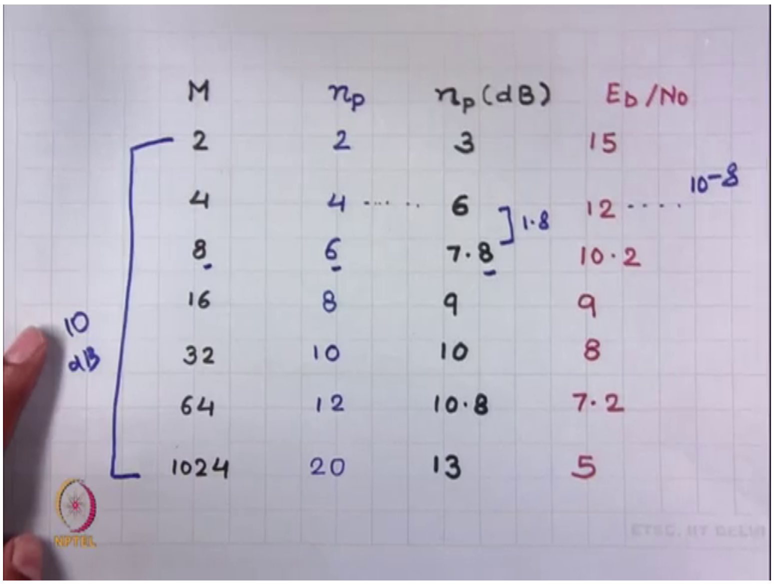

And that is why we like these orthogonal modulation schemes to be used in the power efficient channel ok. So, let us look at some kind of numbers. So, if you double M. So, increase M from 2 to 4 to 8 to 16, 32, 64 and 1024 power efficiency increases by 2 ok. So, for every double increase in M power efficiency increases by 2 and you can also express this power efficiency in dB scale and this power efficiency in dB scale will help you to quickly calculate the E b No requirement for a given symbol error rate performance.

For example, we have seen that in case the power efficiency was 4 the E b No requirement was 12 dB at a bit error rate of 10 to the power minus 8. So, now, I can easily calculate the E b No requirement if my orthogonal modulation is 8 array orthogonal modulation. If I am using M as 8 my power efficiency is 6, in dB it is 7.8, thus the E b No requirement will decrease by 1.8 dB. So, it will go from 12 to 10.2 of course, this is not very precise, but these numbers will help you arrive at quick estimates of E b No requirements ok.

So, you can see that if you increase this M to 1024 you can have a 10 dB gain when you go from 2 array orthogonal modulation scheme to 1024 array orthogonal modulation scheme and this will reduce the E b No requirement also by 10. So, if for 2 array orthogonal modulation scheme you require 15 dB E b No for 1024 array orthogonal modulation scheme you will typically require somewhere around 5 dB ok. What is the main message in here? The main message is the calculation of this power efficiency is really trivial its easy right. It is simply 2 log to M. So, if you know the E b No requirement for a certain array orthogonal modulation scheme from that you can derive the E b No requirement for any other array orthogonal modulation scheme and that is why we will have this power efficiency ok.

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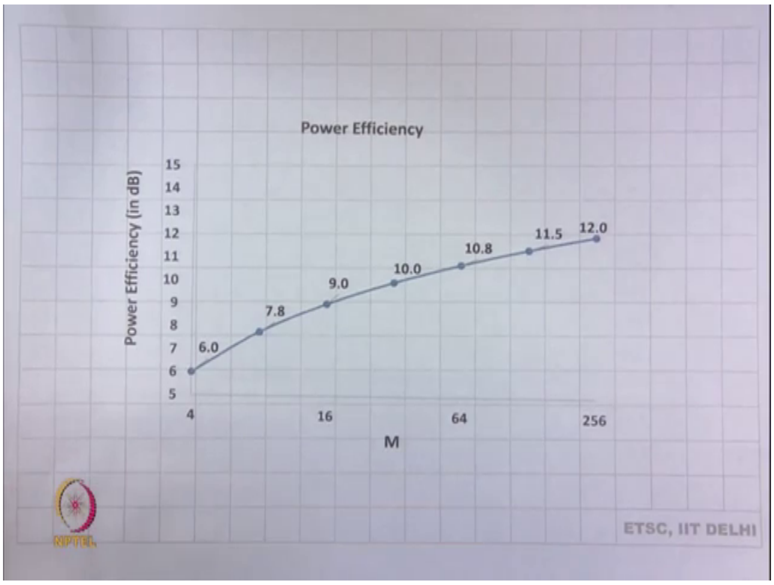

So, it will also be helpful to show you how does this power efficiency in dB a scales with M ok, it increases logarithmically alright.

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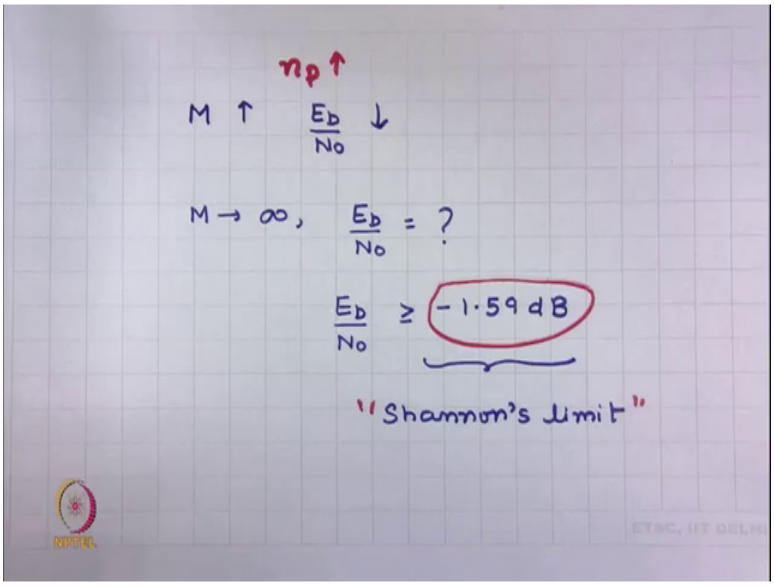

Now, the question is so we now know that if M increases power efficiency increases, power efficiency increases E b No requirement decreases. So, what happens if I increase this M to infinity right?

What will be the E b No requirement in that case? And the first thing that we will see in this lecture is that E b No requirement will drop to somewhere around minus 1.59 dB if you have M as infinity. So, what is we are saying is we will show in this lecture as M tends towards infinity, the E b No requirement drops to as a small as minus 1.59 dB and actually this is the Shannon's limit that we have seen before.

So, Shannon's predicted that if you have infinite bandwidth in the channel then the E b No requirements can be made as small as minus 1.59 dB and we will see today that these orthogonal modulation schemes also requires an E b No of minus 1.59 dB if you have M tending to infinity ok. Let us see how this number can be derived using Shannon's idea.

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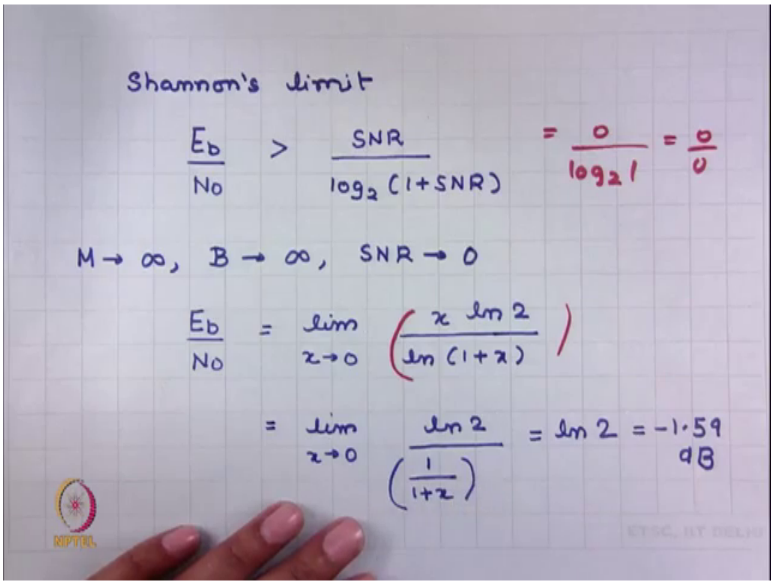

So, we have seen this equation before I guess when we first time looked into this orthogonal modulation schemes we said that E b No should be greater than SNR divided by log 2 1 plus SNR when M tends to infinity these orthogonal modulation schemes begins to require infinite dimensions. Thus they will have infinite bandwidth requirement, when bandwidth tends to infinity the SNR approaches 0 because noise power becomes infinite as well and there is to think about this.

So, when SNR tends to 0 this is 0, log 2 1 which is 0. So, this is 0 by 0 form and you have to arrive at this quantity in the limiting sense and you can use L'Hopital ok. So, if I express SNR as x and I want to write log 2 as l n divided by ln 2. So, actually this is this and then I can take the limit as x tends to 0 when x tends to 0 I have to differentiate both numerator and denominator. When I differentiate numerator I get ln 2, when I differentiate denominator I get 1 upon 1 plus x. Now I can substitute x as 0 and I get E b

No requirement as ln 2 which is minus 1.59 dB ok. So, this follows from Shannon's basic equation.

And we will prove in today's lecture that orthogonal modulation schemes also requires the E b No of minus 1.59 dB if you assume that M is tending to infinity ok.

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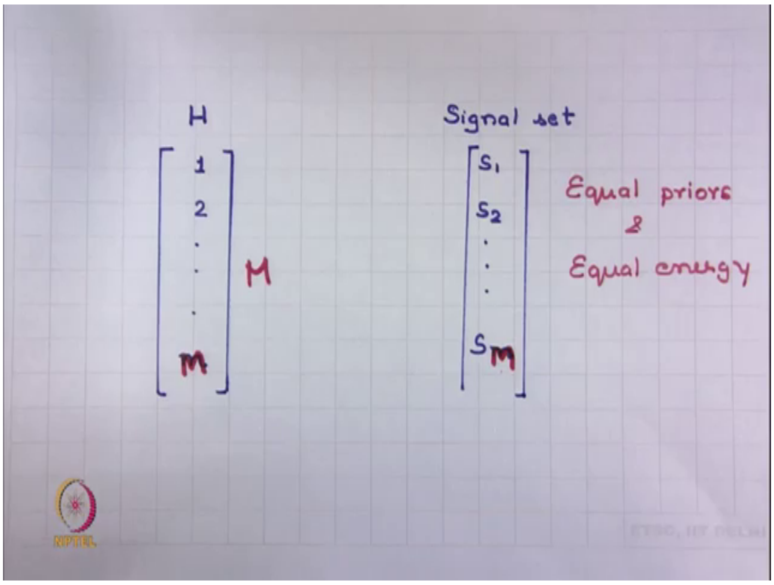

So, let us start looking into the performance of these orthogonal modulation schemes. So, what we assume is that we have M hypothesis. So, we can use the capital letter M instead of small letter M. So, we have M hypothesis and we have a signal set where these signals are assumed to be of equal energy and because we want to use this ml detection rule we also assume that these signals have equal priors ok. So, this signal s 1 corresponds to hypothesis 1, signal s 2 corresponds to hypothesis 2, signal s M corresponds to hypothesis M. So, we have M hypothesis and we have M signals ok.

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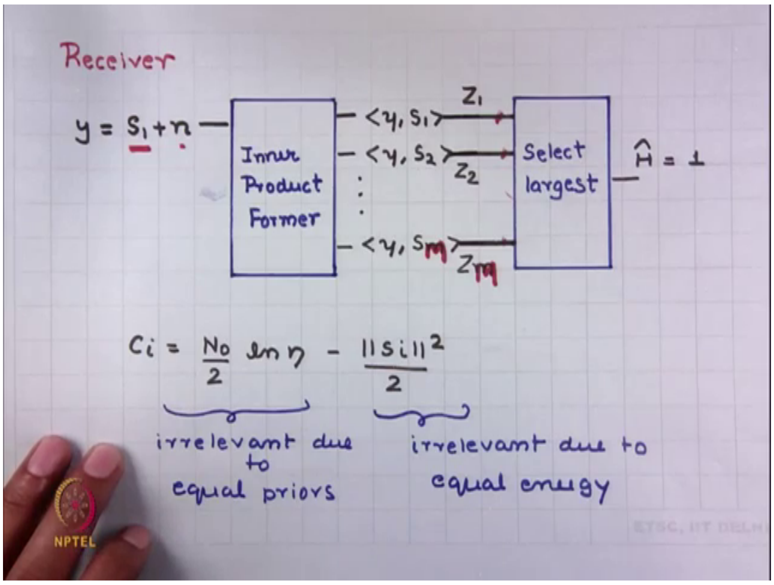

Now, let us look at this receiver structure that we will require to detect these m orthogonal signals and receiver structure would be same as the receiver structure that we have seen in the case of M array detector. So, the receiver structure that we have seen when we looked into this M array detector can be used for any modulation scheme whatsoever it is. So, it can be used for pam, com, fsk, ppm whatsoever that was a generalized receiver structure. So, the receiver structure will be same. So, you receive a signal which in this case is s 1 plus n. So, we are assuming that the first hypothesis is transmitted first hypothesis has a signal s 1 and then what we do in this receiver?

We take the inner product of the received quantity with all signals in the signal set and then once you have computed this inner product you have a block which select the hypothesis which has the largest resultant at this point and what we also show here is that at these points we have M random variables Z 1, Z 2 and Z M ok. So, look at this receiver structure again. So, we are assuming that the signal that is arriving is s 1 plus n then we compute the inner product of this received value with all signals in the signal set and then we decide upon the hypothesis where we have the largest output.

So, if in this branch we have the largest output we select the hypothesis s 1. So, when we discuss this M-ary detector we have this bias also at this point, but now we have got rid of this bias, why is that? So, remember that this bias looked something of this form, this term in the bias dealt with the signals which are with different priors and because we are assuming all signals have the equal a priori probability this term is 0 and we if we assume that all signals have equal energy then this term is also relevant. So, we do not have to use any bias in the case we have signals with equal energy and equal probability. So, we should not be using bias and that is why for this M-ary orthogonal modulation scheme we are not using this bias.

Other than that this receiver structure is exactly similar to the receiver structure that we discussed in the case of M-ary detector ok.

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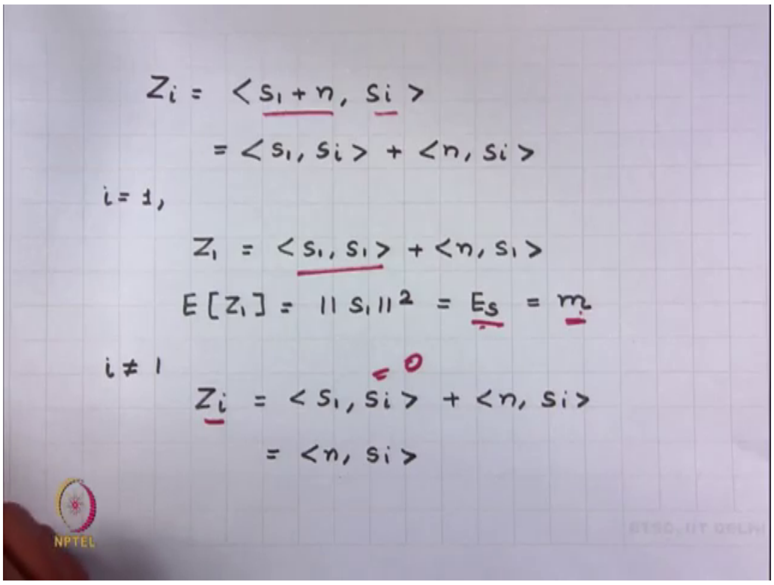

Now, before we can dig our teeth into this probability of error formulation let me revise some basic stuff ok. So, what we need to do is we need to find out the mean and variance of these random variables. So, these random variables that we have in here we have to find the mean and variance of these random variables, what will be the probability density function of these random variables? The probability density function of these random variables will be Gaussian because any linear functional of a Gaussian noise is a Gaussian random variable; is not it? So, at this what we are having is a linear functional of a Gaussian noise.

So, all these random variables will be Gaussian random variable and thus what remains is to identify what is the mean and what is the variance ok. So, this we have to see, you must be an expert in finding out mean and variance by now, but let us do it ok. So, what we are assuming that we are receiving s 1 plus n and we are finding out the inner product with the signal s i and that is giving us this random variable Z i ok. So, you know that in a product is bilinear operation. So, you can compute this by computing the inner product of s 1 with s i plus inner product of noise with s i, let us see what happens when i is 1. So, we will get Z 1 and Z 1 is simply the inner product of s 1 with s 1 plus inner product of noise with s 1.

And what is the mean of this random variable? Mean of this random variable corresponds to simply norm square of s 1, this term does not have any mean this mean is 0 because the mean of noise is 0 and hence mean of Z 1 is simply norm square of s 1 which is simply the energy of a signal or a symbol and which we call as M just for simplicity. So, remember the mean of this random variable is simply E s or symbol energy. What happens when i is not 1? When i is not 1 we have Z i which is the inner product of s 1 with s i plus inner product of noise with s i and this is 0 because the signals are orthogonal signals ok.

So, what we are left with is inner product of n with s i and what is the mean of this? Mean of this is 0 because noise has 0 mean ok.

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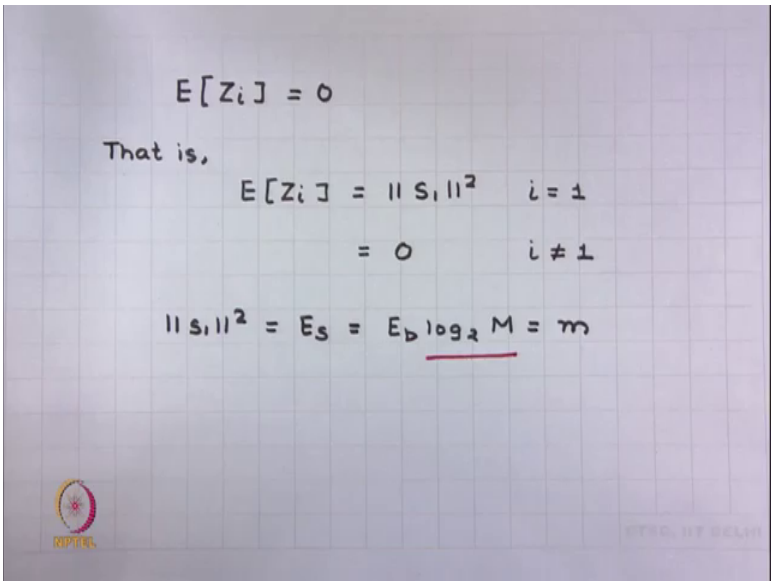

So, in short we can conclude that mean of Z i is energy of a signal if i is 1 and for all other is its simply 0 and also remember because we will be using the shortly that norm square of s 1 is E s symbol energy and symbol energy is always bit energy times number of bits in a symbol and number of bits in a symbol is log to M and we call this thing as m ok. Now, what remains is to find out the variance of these random variables.

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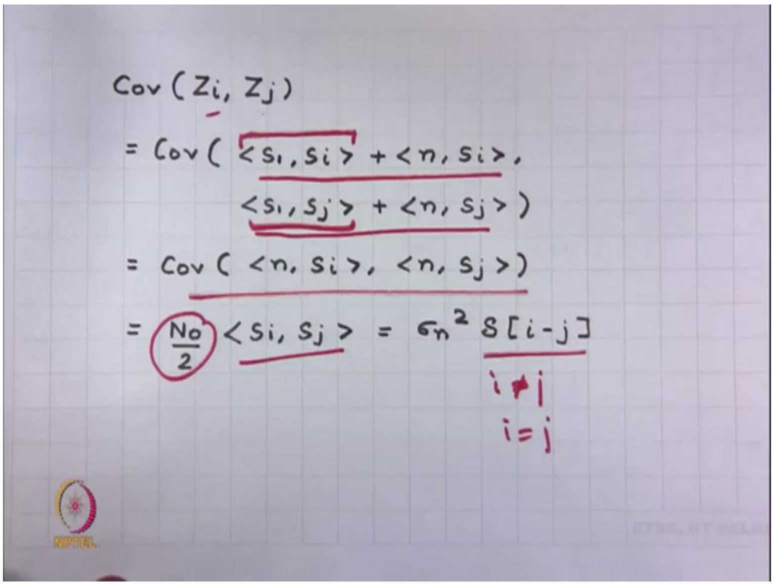

And to think about variance we can start thinking about the covariances and here we are taking the covariance of Z i with Z j and you know that you have to write first Z i and Z j and you have to take the covariance.

Covariance we have seen is not influenced by constants; this is a constant term ok; this is a constant term. So, you can better remove this constants and thus this covariance reduce to this covariance. So, here we are finding the covariance of inner product of n with s i with inner product of n with s j and we have solved this several times in this course. This is simply N naught by 2 if I assume that the one sided power spectral density of input noise is N naught and if it is white Gaussian noise because we always assume input noise to be white Gaussian noise with 1 plus sided power spectral density of N naught and so is here.

So, this covariance is simply N naught by 2 times inner product of s i with sj and we know that this will be 0 if j is not same as i. And thus to have this effect we have an impulse function here which is 0 if i is not same as j and it is 1 if i is same as j. So, we are saying that this thing is simply sigma n square times an impulse function ok, we will see what is the sigma n square now.

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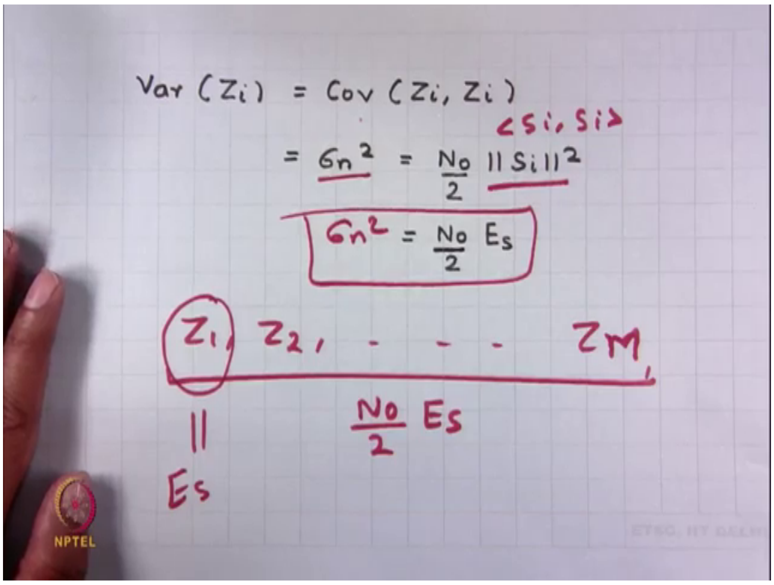

So, from this we can find out the variance of Z i which is covariance of Z i by Z i and we have seen that this will be sigma n square delta impulse function is 1 if j same as i. So, this is just sigma n square and this will be N naught by 2 inner product of s i with s i and inner product of s i with s i is norm square of s i. So, we have got sigma n square is N naught by 2 times E s energy of the signal ok.

So what we have got is we had this bunch of Gaussian random variables and we have seen that all these random variables have the variance of N naught by 2 times E s. And what we have seen is that this random variable has got a mean of E s and all other random variables have got a mean of 0 and now we can easily calculate the probability of error for this orthogonal modulation scheme ok; let us get it started.

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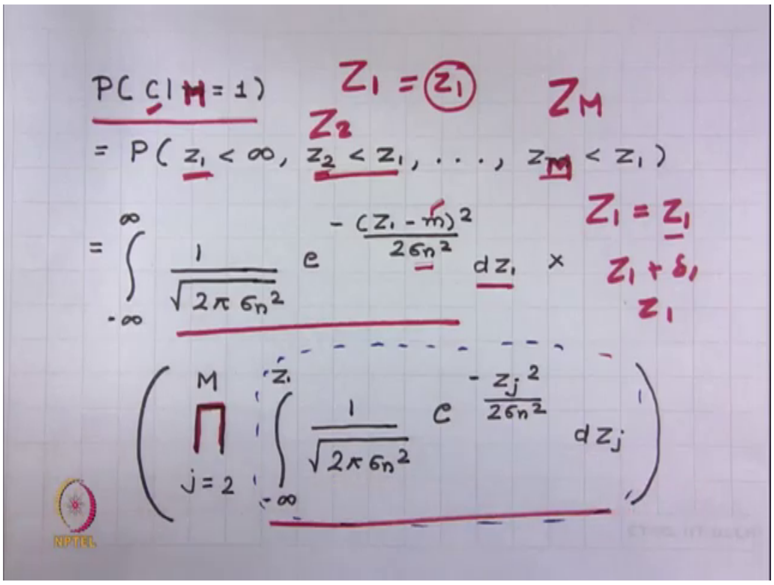

So, here what we are trying to do is we are trying to evaluate probability of being correct when hypothesis 1 is transmitted ok. So, let us use H instead of M. So, in this we are evaluating what is the probability of being correct if first hypothesis is transmitted ok.

And from here we can easily derive what is the probability of being correct over alright, because all these signals that we have are the signals with equal energy, all the signals have the same distance from all other signals. And thus there is no reason why noise will favor one signal over another. All signals will see the same error probability and thus on average the error probability that you will see will be the error probability corresponding to one hypothesis being transmitted. So, that is the big plan that we are having here ok.

So, we simply calculate what is the probability of being correct a first hypothesis was transmitted and from here we can find out the average probability of error ok. So, given that we have transmitted hypothesis 1 what would you like to have? You would like to have that the numerical value corresponding to random variable Z 1 should be the largest; because when the numerical value corresponding to the random variable Z 1 will be the largest your detector will choose the hypothesis to be the first hypothesis ok.

So, that is even corresponds to the probability of being correct. So, what we want is if we assume that this random variable has a numerical value a small z 1 then what you want to have is that this Z 1 is the largest. So, what you want to have is this random variable takes in a numerical value small z 1 and all other numerical values corresponding to all other random variables should be a smaller than z 1 ok.

So, z 2 which corresponds to the numerical value of random variable z 2 should be less than z 1, Z M which corresponds to the numerical value of random variable Z M should be less than Z 1 and so on so forth ok. And what is the condition on numerical value Z 1? This can be any number from minus infinity to plus infinity ok.

So, if this condition is satisfied the receiver will make a correct choice ok, let us see how can we solve this up. So, first we have to find what is the probability that random variable Z 1 takes in a value around a small z 1 it is not exactly z 1 because that probability is 0 right. If you have a continuous random variable probability that a random variable will pick on a specific value is 0. So, we do not talk about probability of this random variable taking a numerical value z 1, but what we want to find out what is the probability that this random variable takes in a value around z 1. That means, between z 1 plus delta and z 1 ok, delta being very small and this probability you can find easily by multiplying the pdf of this random variable Z 1 with a small delta which in this case is d Z 1.

And how do we find the pdf of z 1? Is simple, it is a Gaussian random variable just you have to plug in the mean, mean is M and the variance we have calculated the sigma n square ok. So, this will give us the probability that random variable Z 1 will take in a value around z 1. And now to this we have to multiply the probabilities of all these events and all these events are independent events and thus we have to simply multiply the probabilities of these events ok.

So, you have this multiplier which simply multiplies the probability corresponding to each event M minus 1 time, is not it? And what is the probability of each event? So, let us focus on let us say this event. So, here we have to find out what is the probability that random variable Z 2 takes in a value less than Z 1 and this can be obtained by integrating the pdf of Z 2 from minus infinity to Z 1.

And this is exactly what we are doing in here, we are integrating the pdf from minus infinity to Z 1 and the pdf is the pdf of a Gaussian random variable which has got a mean of 0 and the variance of sigma n square and all other terms will have the same product ok. So, you simply take this term and you multiply this M minus 1 times ok. So, this thing will give you probability of being correct if we have transmitted the hypothesis as 1. And what you can also recognize in here before we move to next slide is all these terms have the same contribution. So, we can also simply write this in this way.

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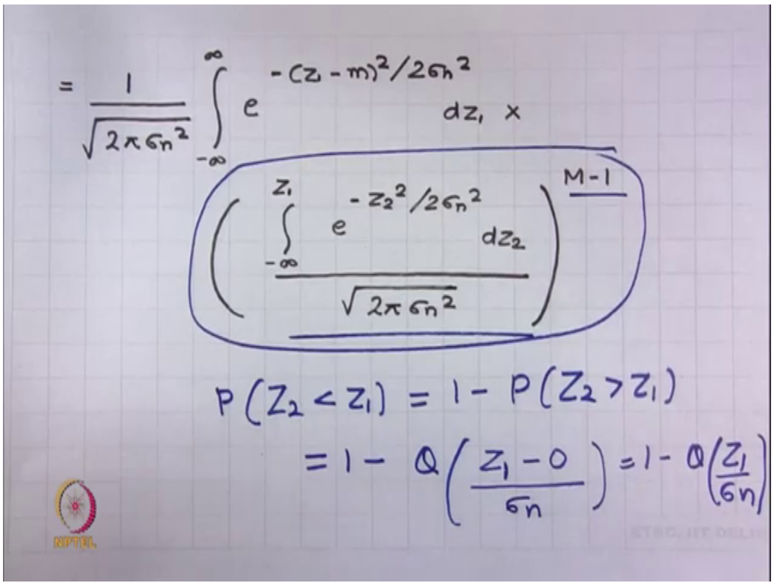

So, each term has this contribution and we have to simply multiply this M minus 1 times that is it, everything else same here. Now what we want to do is we want to express this thing in terms of q function. We know that this is the probability that Z 2 takes in a value less than z 1 and this is 1 minus probability that Z 2 takes in a value greater than Z 1 and we know how to work this out this is Q Z 1, what is the mean of Z 2? Is 0.

What is the variance of Z 2 sigma n square? So, what is the standard deviation is sigma n. So, we know that this thing is simply 1 minus Q of Z 1 divided by sigma n. Now we can have this instead of this in here and we get to this expression ok.

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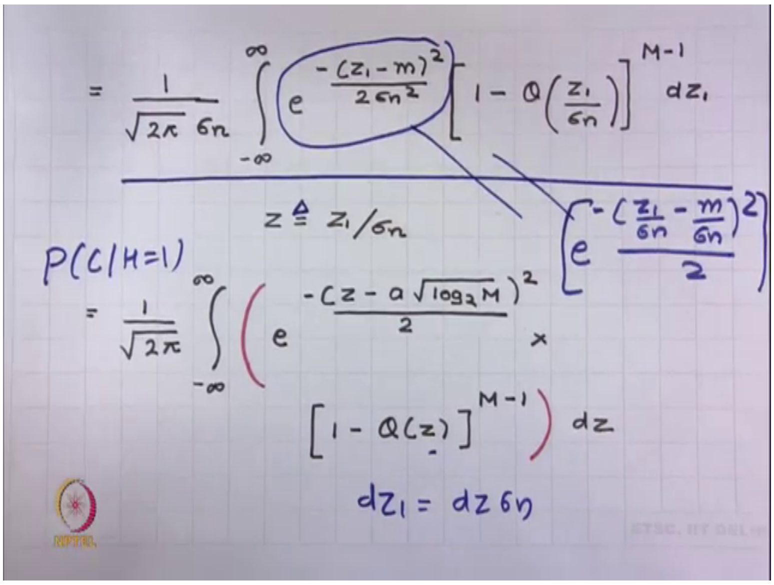

Now we make a trivial change in variables, we say thatZ is simply Z 1 by sigma n ok. So, we have defined Z to be this.

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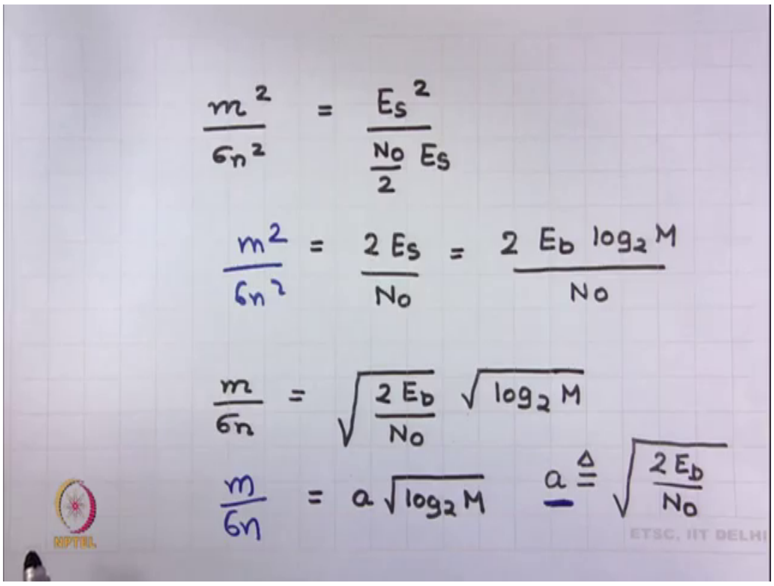

So, before thinking about this let us look at this slide first, I am trying to find out this ratio m square by sigma n square, m square is E s square sigma, n square is N naught by 2 times E s. So, m square by sigma n square is 2 E s by N naught and E s is E b log 2 M.

So, m square by sigma n square is 2 E b log 2 M by N naught; m by sigma n is a square root of this thing which is a square root of 2 E b by N naught times the square root of log

2 M and we define a as a square root of 2 E b by N naught, just to simplify our notations ok. So, we have got that m by sigma n is a times root of log 2 M, where a is square root of 2 E b by N naught. Now, let us look at this again. So, what we are trying to do is we want to write this as z 1 by sigma n minus m by sigma n whole square by 2. So, look at just this term.

Instead of it we want to write it like this, z 1 by sigma n we defined it to be z and m by sigma n we have defined it to be a times root of log 2 M. So, this simply becomes this, Q of z 1 by sigma n is simply Q of z and dZ 1 was dZ 1; dZ 1 is d Z times sigma n and thus this sigma n gets cancelled. So, what we have got is probability of being correct, if we have sent the hypothesis 1 is simply this integration.

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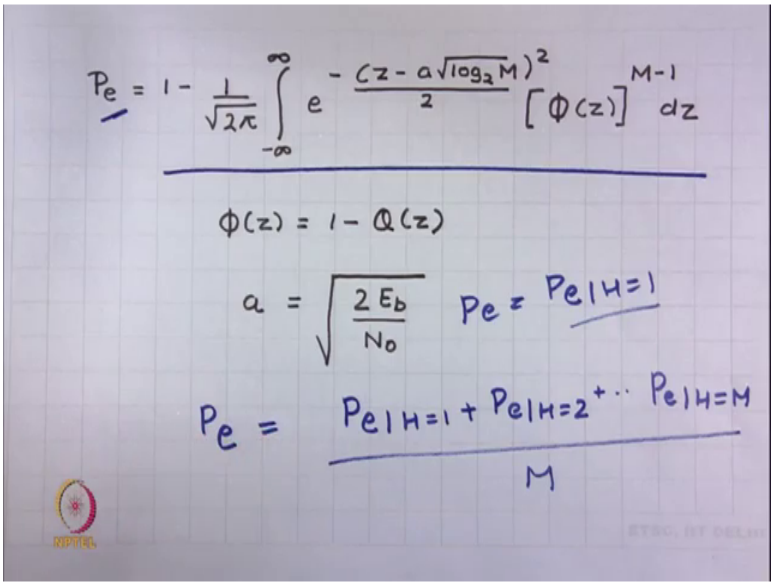

From this we can easily calculate the probability of error which is 1 minus probability of being correct ok and we have already calculated that probability of being correct is this thing.

So, probability of being in error is this thing, what more we have done is we have introduced this phi of z, where phi of z is simply 1 minus Q of z and a in this expression is root of 2 E b by N naught. As if you use this MATLAB and if you carry out this integration, you can get precisely the probability of being in error if hypothesis 1 is transmitted.

But this is also the overall probability of being in error, because probability of being in error is simply the probability of being in error if hypothesis 1 is transmitted plus probability of being in error if hypothesis 2 is transmitted so on so forth plus probability of being in error if hypothesis M is transmitted divided by number of hypothesis. And if all this probability of error are same, then the overall probability of error is also simply the probability of being in error if hypothesis 1 is transmitted ok. So, thus this also gives us the average probability of symbol error ok.

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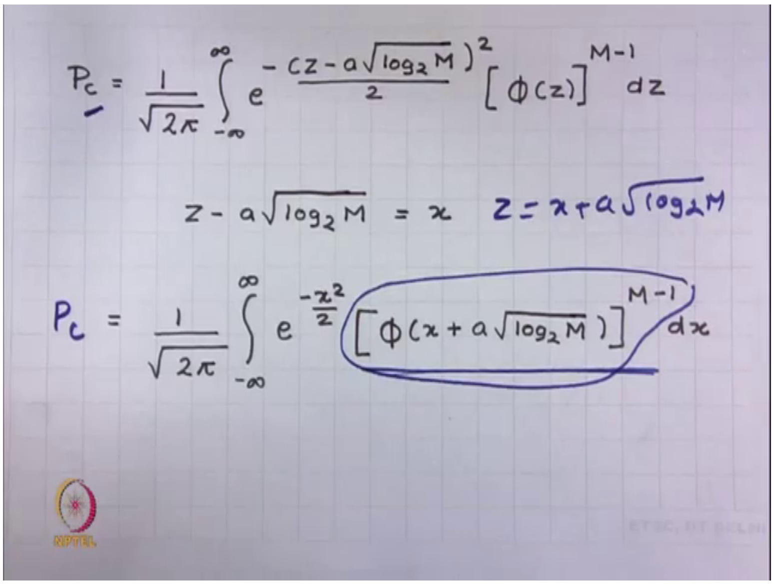

We are not yet done because what we want to do is we want to think about what happens to this probability of error if M tends to infinity that is the question that we like to ask. Also; obviously, this is also an important result because this gives you probability of error of any orthogonal modulation scheme ok. But we are trying to wonder about another question also in here that what is the probability of error if M tends to infinity ok. To think about that let us confine ourselves to probability of being correct and we have said that probability of being correct is this thing. Now we make another change in variable we say that z minus a under root of log to M is x.

So, probability of being correct is simply this thing. So, we are replacing this joint expression simply by x square by 2, and this z thus becomes x plus a root of log to M times M minus 1 dx why we are doing this because we want to handle all terms containing M at the same place ok. So, here it had a term containing M we have shifted this to this side. So, now, only this term has the terms containing M ok.

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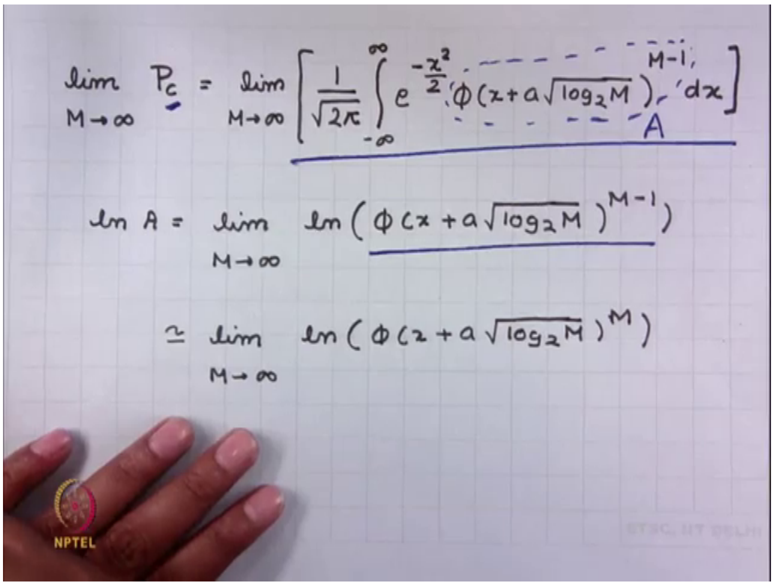

So, the question that we want to ask is what is the probability of being correct if M tends to infinity ok. So, you have to ask this question in limiting sense. So, you want to make M tending to infinity and you have to solve this limit. So, what we want to do is, we want to define this as A when M tends to infinity this quantity as a when M tends to infinity. And so ln A is simply limit when M tends to infinity ln of this quantity.

We will first try to see how does this a behaves when M approaches infinity and then we will plug in this value of A in this expression and then we will see how the probability of being correct behaves with M tending to infinity ok. So, that is the goal of this exercise alright ok. So, if M tends to infinity M minus 1 is same as M. So, from here we can get to this simple and now we can also bring this M to this side and so that limit is simply this where instead of writing M I have written it as 1 by M ok.

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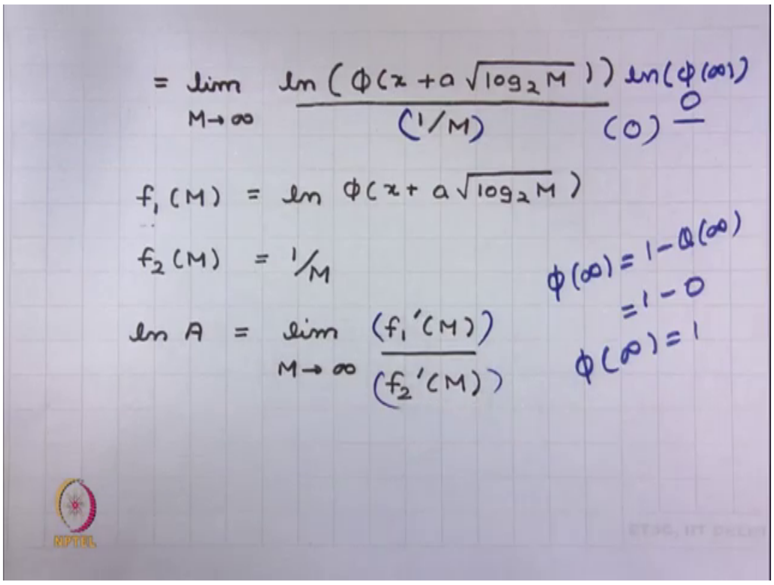

For some strange reasons that will become clear now and let us see what is this limit as M tends to infinity. So, when M tends to infinity this is 0, when M tends to infinity I have ln phi of infinity. What is the ln phi of infinity, what is phi of infinity? Phi of infinity is 1 minus Q of infinity, what is Q of infinity is 0.

So, this is 1 minus 0 phi of infinity is thus 1 and what is l in 1 is 0. So, this limit is of 0 by 0 form and thus to solve this limit I have to use L'Hopital ok. So, first I define two functions. So, I take in this numerator. So, f 1 M is the numerator and f 1 M is the denominator of this expression, then l n A is limit when M tends to infinity derivative of f 1 M divided by derivative of f 2 M and now you take in the derivatives and you hope that you get anything ok.

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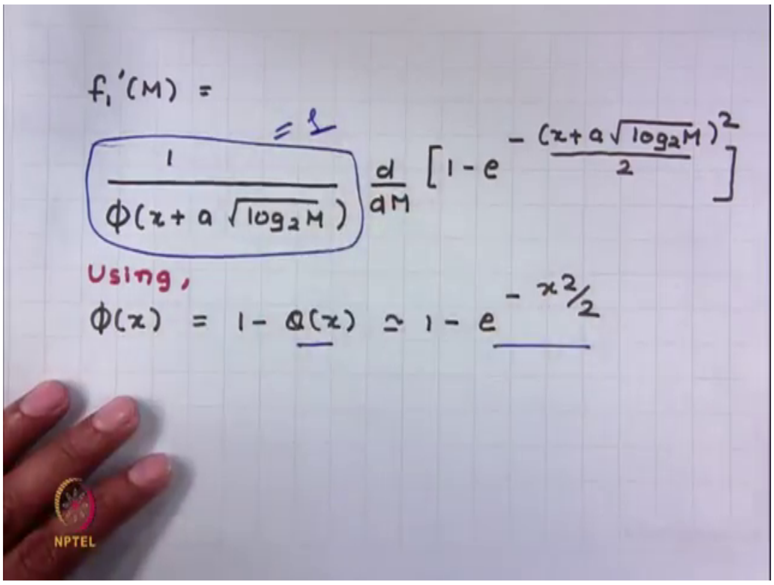

So, then we are differentiating this f 1 M and you do not need lot of skill to differentiate this. So, ln x has a differentiation 1 by x. So, this terms becomes one by this and then you have to differentiate the argument of this thing ok, and to before differentiating the argument of this thing you have to use this identity which will make things easier.

That phi of x is 1 minus Q of x and Q of x is e to the power minus x square by 2 if x is pretty large we have used this property several times. So, phi of x is simply 1 minus e to the power minus x square by 2. So, phi of this thing is simply 1 minus e to the power minus x plus a root of log 2 M square divided by 2 and you have to differentiate this term with respect to M and this thing will give you derivative of f 1 M. Now let us first deal with this term when M tends to infinity this term is 1. So, we do not have to worry about this ok. So, we can replace this simply by one and get rid of it and now what remains is the derivative of this quantity.

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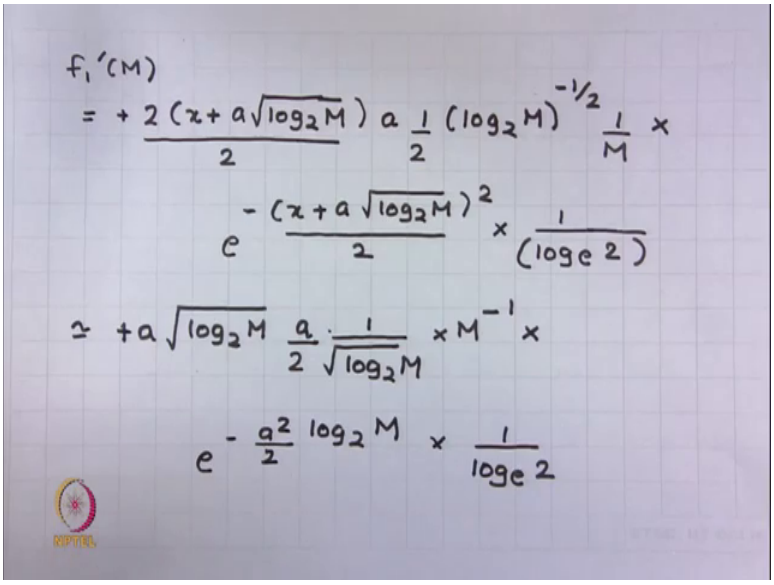

And I will not explain the derivative of this because anyone can do this. You can look at this slide here I have carried this derivation.

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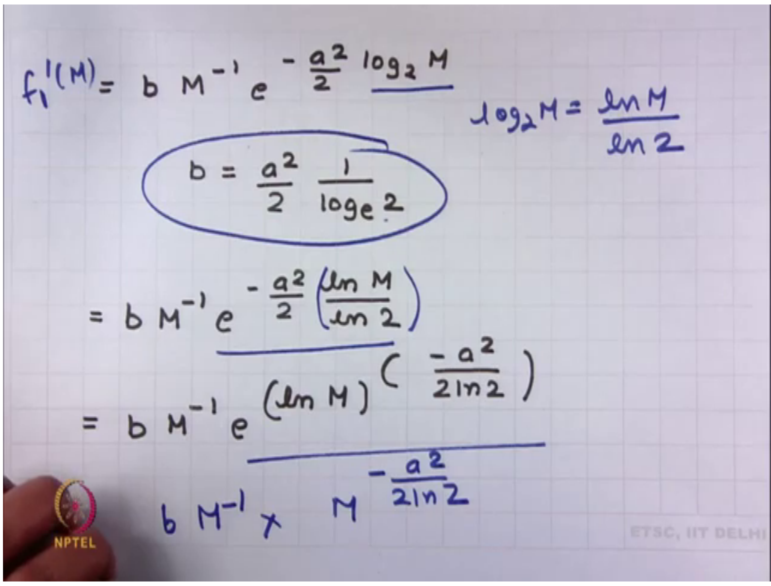

Buts what you can get in the end is that this is simply of this form b into M to the power minus 1 times e to the power minus a square by 2 times log 2 M where b contains all coefficients that are not function of M we are trying to evaluate this limit as M tends to infinity. So, whatever terms which are not containing M are really not relevant what you might notice at this point here though is that b is some positive constant ok.

So, f 1 dash of M is simply this, now what I can do is I can write this log 2 M as ln M by l n 2. So, instead of this I can write it like this trivial and instead of thinking about this I can think about this quantity, because I know exactly what it is.

This is simply M to the power minus a square by 2 ln 2. So, f 1 dash M becomes b times M minus 1 times M to the power minus a square by 2 l n 2. So, this is what we have here.

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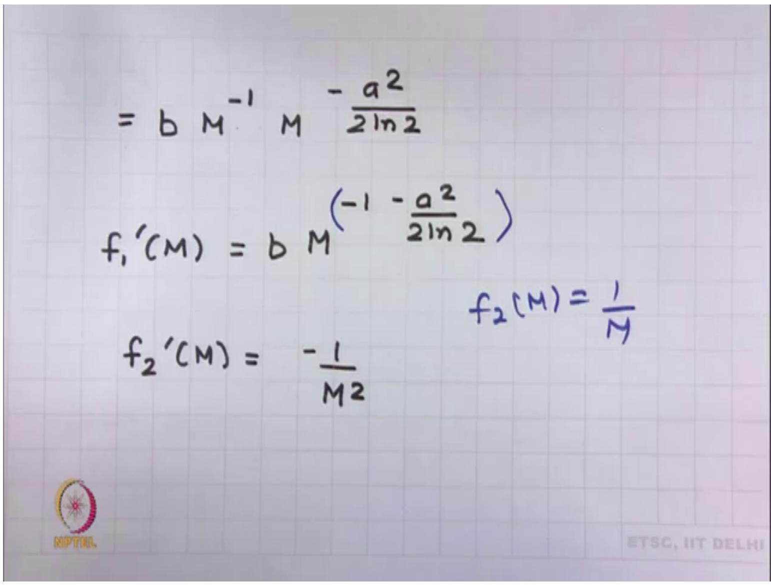

And thus I can write f 1 dash M as also b times M to the power minus 1 minus a square by 2 l n 2. What was f 2 M f 2 M was really simple 1 by M. So, its derivative is simple minus 1 by M square.

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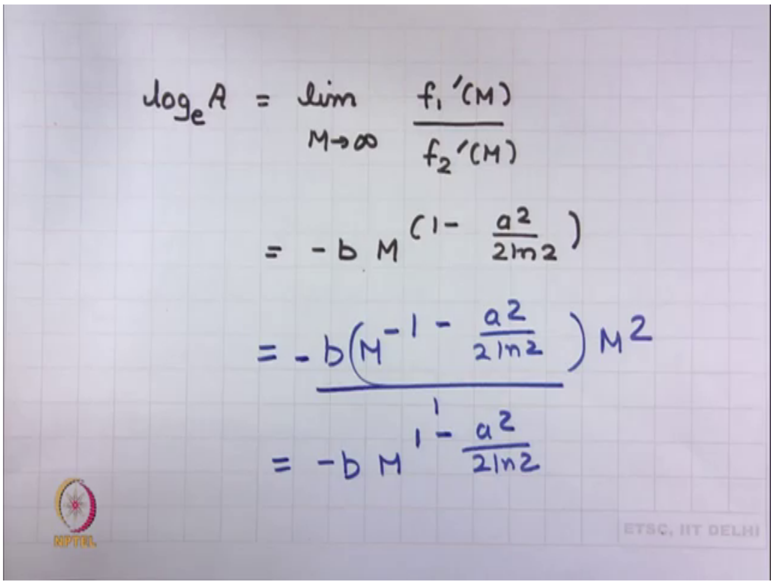

Now what is this ln a? ln a is limit when M tends to infinity f 1 dash M divided by f 2 dash M and this if you work out. So, we have seen that this is b M to the power minus 1 minus a square by 2 l n 2 and that was minus 1 by M square. So, we can have M square here so this is simply minus p M to the power 1 minus a square by 2 ln 2 this thing.

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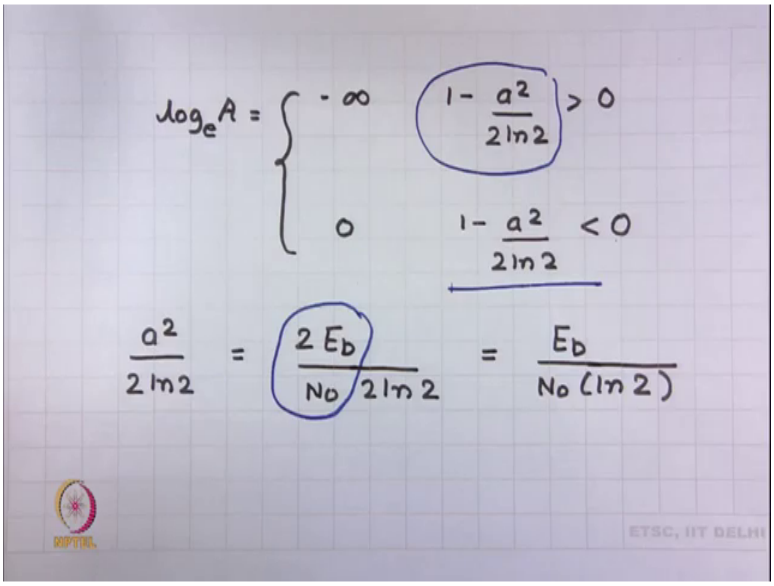

Now, you can see that if 1 minus a square by 2 ln 2 is a positive quantity then what you have is infinity raised to power a positive quantity which is infinity, this thing is infinity and because of this minus it will be minus infinity ok. So, if this 1 minus a square by 2 ln

2 is greater than 0 then ln a is simply minus infinity and if this thing is less than 0, then you have infinity to the power less than 0; that means, you have infinity raised to something negative and this is simply 0. So, in this condition ln a is 0. What is this a square by 2 l n 2? a square you if you remember we have defined it to be as 2 E b by N naught. So, a square by 2 l n 2 is simply E b No by l n 2.

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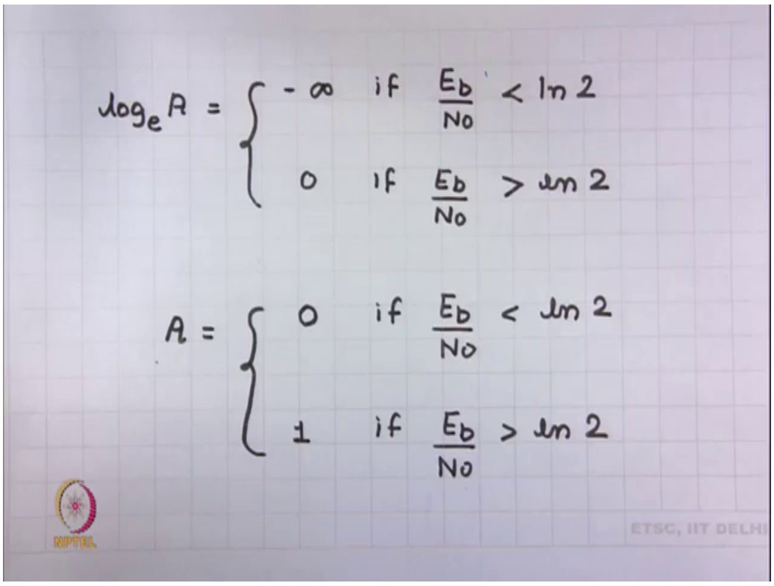

So, instead of writing this I can write it like this ln e is minus infinity if E b No is less than l n 2 ln a is 0 if E b No is greater than ln 2. From here I can say that a is 0 if ab no is less than ln 2 and a is 1 if E b No is greater than ln 2 ok.

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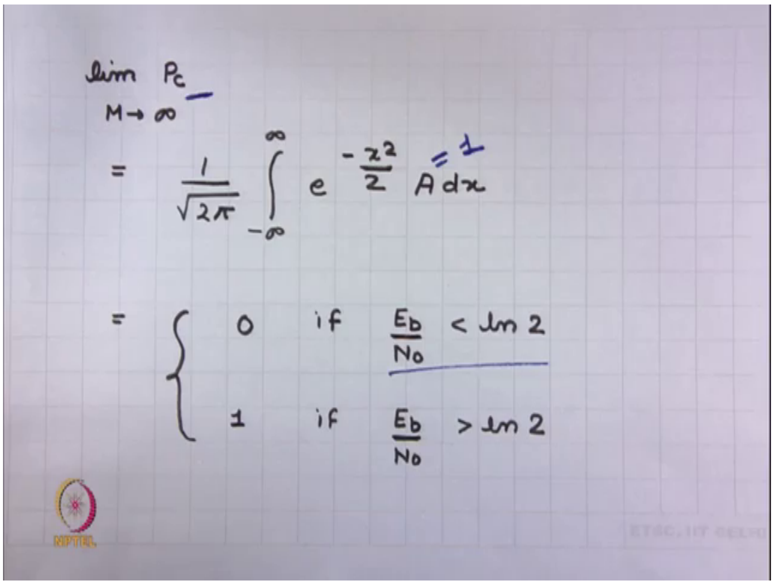

Now let us put this a back into this probability of being correct. So, probability of being correct was this if I E b No is less than l n 2 then a is 0 and probability of being correct is also 0 if E b No was greater than ln 2 then a was 1 and then it is an integration of pdf from minus infinity to plus infinity which is also 1. So, what we have got is if E b No is less than ln 2, then the probability of being correct is 0 and if E b No is greater than l n 2 than the probability of being correct is 1 ok.

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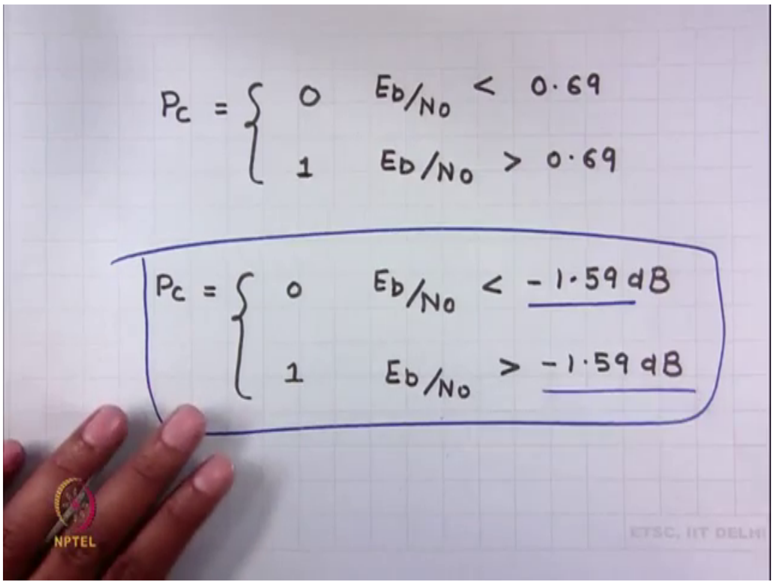

We can write it like this as well that probability of being correct is 0 if E b No is less than 0.69 the probability of being correct is 1 if E b No is greater than 0.69 ln 2 is 0.69 and you can also express this in dB which is usually done most of the time.

So, we have got to this expression that probability of being correct is 0 if E b No is less than minus 1.59 dB and probability of being correct is 1 if E b No is greater than minus 1.59 dB. And this proved what we were talking about that if you have M tending to infinity; that means, if you have infinite bandwidth available. So, that you can have M-ary orthogonal modulation scheme where M tends to infinity, then the E b No requirements drop down to minus 1.59 dB and this is an important result.

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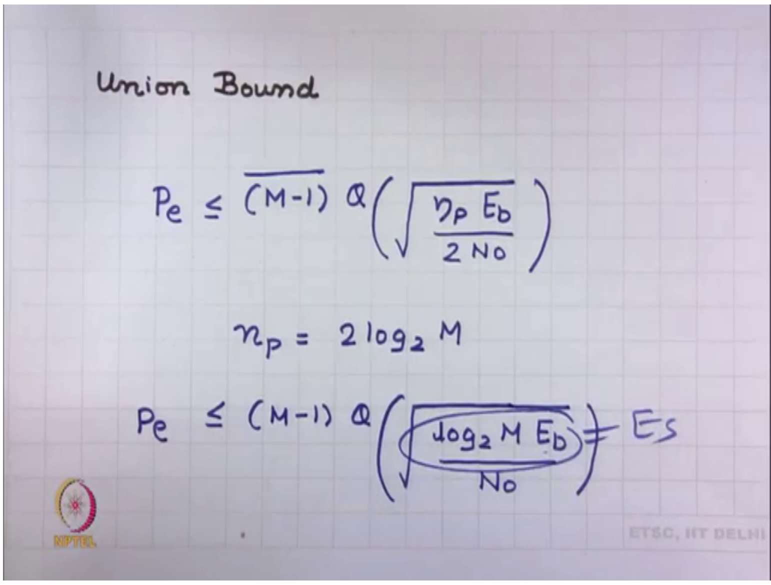

So, do we have to live with this such complicated probability of error and probability of being correct formulas no you also have union bound in this case, union bound is same as intelligent union bound is same as nearest neighbour approximation for orthogonal modulation scheme ok.

This one in the same thing both leads to the same result right, because all signals are equidistant from each other does the number of nearest neighbors is M minus 1 Q of square root of eta P E b No by 2 either be we know is 2 log 2 M alright. So, you can substitute this eta P 2 log 2 M in this and you can write this as this and this is also E s; is not it?

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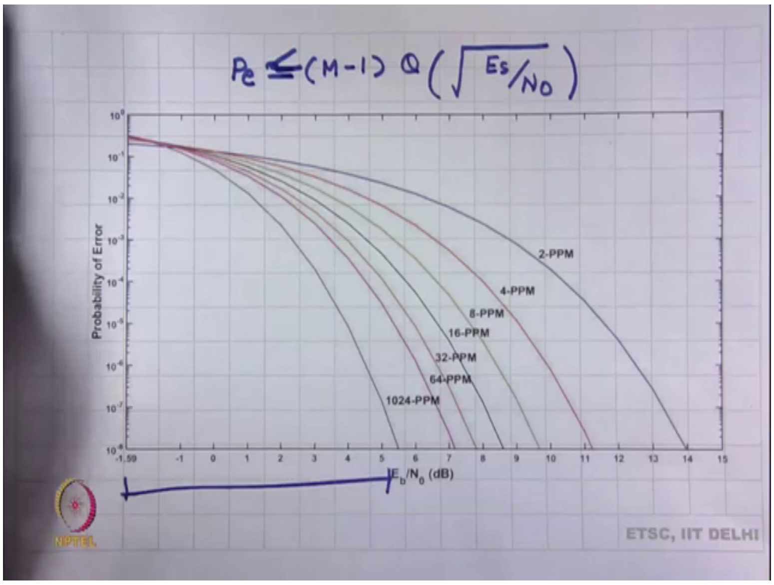

So, what you can say that probability of being correct is less than or equals to M minus 1 times Q of under root of E s by n naught you can use any expression whatever you want.

So, here we show how the probability of error behaves with E b No again it follows a waterfall decrease and you can see that as M increases the E b No requirement decreases for a bit error rate of 10 to the power minus 8, but of course, you are still very far from the Shannon's limit ok. So, M that is required must be pretty large alright and from this curve you can also see that the quick estimates that we had are really precise very close ok.

So, you can use quick estimates to find out whether your simulation models are correct or whether your calculations are correct and or you can use the precise formula that we have used to get to these probability of error curves. So, so far we have looked how to calculate the symbol error rates for various modulation schemes and now it is the time to look at the bit error rates for these modulation schemes.

And going from the symbol error rates to bit error rates is really simple ok. So, that is what we have to do, what happens when you want to identify the bit error rate performance for these modulation schemes ok.

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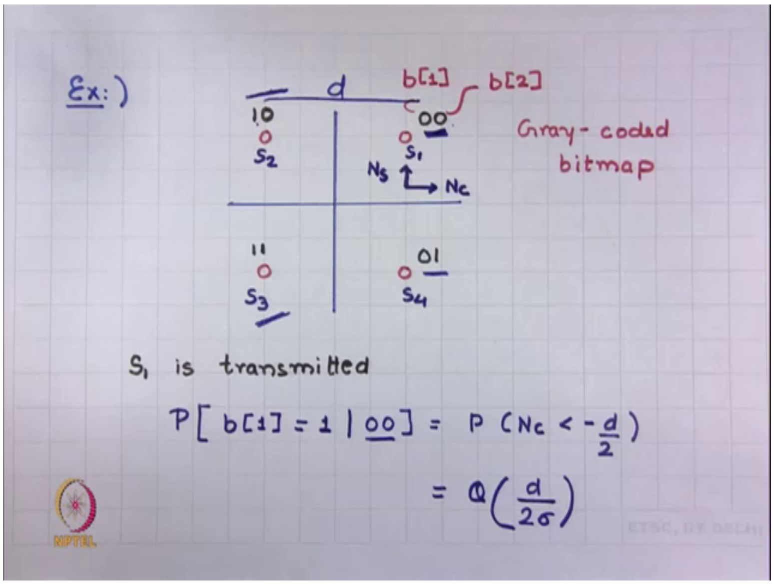

So, let us start with example and as always I would like to start with this 4 qam as the running example. So, here we have 4 symbols s 1, s 2, s 3 and s 4 all these symbols are at a distance of t and they have the equal probability and things like that ok.

Now, what has changed in this diagram if you notice carefully is also we have labeled what are the bits corresponding to these symbols ok. So, symbol s 1 has this bit 0 0 symbol s 2 has the bits 1 0 symbol s 3 has the bits 1 1 symbol s 4 has the bits 0 1. And what you can also notice is that there is only a 1 bit difference between the bits of the neighboring symbols for example, this differ only by 1 bit.

If you compare this with this ok, if you compare this with this there is only 1 bit difference if you compare this with this of course, there is a 2 bit difference. So, what we are saying the nearest neighbors have only 1 bit difference if I am using what is known as gray coding. This is a gray coded bit map which make sure that nearest neighbors differ only by 1 bit and we have to find out what is the bitter rate if I am using this gray coded bit map.

This bit I am saying as b 1 and this path the second bit I am saying as b 2 and again same as before I am assuming that hypothesis that we have transmitted is s 1 and as before we assume that the noise that rise on this s 1 is N c in the in phase direction and N is in the quadrature direction. So, probability that bit 1 is one given that we have transmitted 0 0, will create an error this is the probability that this bit is in error.

And when will this happen? This will happen either if you consider this signal s 2 or you consider this signal s 3. So, these two signals have their first bit as 1. So, this event will happen when N c will become less than minus d by 2. So, this distance is d if N c becomes less than minus d by 2 then the received value will lie to the left of this axis and then either you will have s 2 or s 3 and both these symbols have the first bit as 1 and that will make the first bit to be 1 if you are transmitting signal s 1.

So, this event corresponds to the probability of N c being less than minus d by 2 and this we know by heart is simply q of d by 2 sigma easy.

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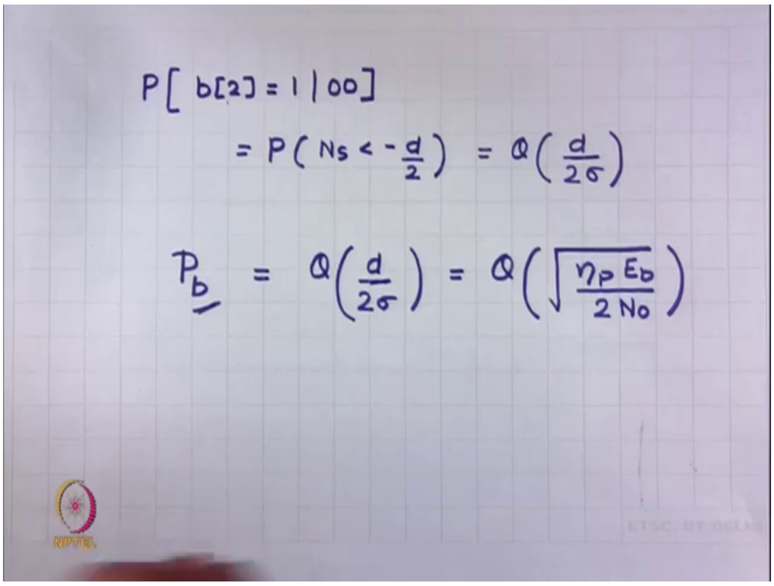

So, now, we have to find what is the probability that bit 2 is 1 given that you have transmitted 0 0 that also corresponds to an error situation and this is probability that N s is less than minus d by 2 because when this happens either you have signal s 4 or s 3 and both signals have the second bit as 1.

So, probability that the second bit is 1 is when and this is less than minus d by 2 alright. So, probably that second bit is 1 is simply probability that N s is less than minus d by 2 and this is also Q of d by 2 sigma the overall probability of bit error is simply Q of d by 2 sigma and which is Q of root of eta P E b No by 2 and remember these bit errors are independent ok.

So, this bit error does not depend upon this bit error or this bit error does not depend upon this bit error and that is why these probability simply add.

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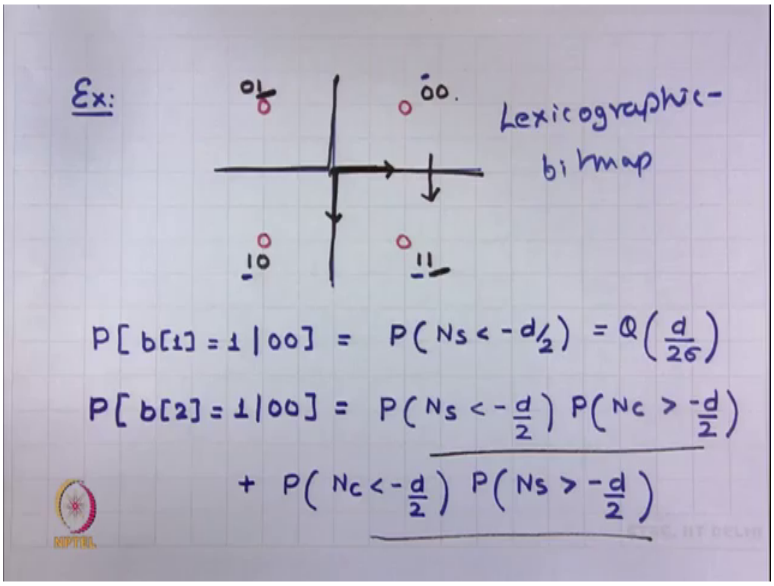

Now, let us reinforce our understanding by looking into different example. So, the underlying modulation scheme remain same; however, we have assigned the bits to the symbols using what is known as lexicographic coding or lexicographic bit map ok. So, this symbol is assigned 0 0, this is 0 1, this is 1 0 and this is 1 1; now you see that these neighbors differ by 2 bits and that is bad.

This we will see. So, again same we have to find what is the probability that bit 1 is one given that you have transmitted 0 0. So, this bit becoming 1. So, either you have one here or 1 here and this will happen when N s is less than minus d by 2 this is simply Q of d by 2 sigma and then we have to find what is the probability that b 2 is one given that you have transmitted 0 0.

And when will this happen? This will happen when either you have this or you have this situation. So, either you received value lie in the fourth quadrant or it lies in the second quadrant. So, this will lie in the 4 th quadrant when N c is greater than minus d by 2. That means, that you lie to the right of this axis and when N s is less than minus d by 2; that means, you lie below this axis.

So, you remain in the fourth quadrant, similarly this event will correspond to the received observation lying in this second quadrant and that will give us the probability that bit 2 is 1 ok. Now we can solve this easily.

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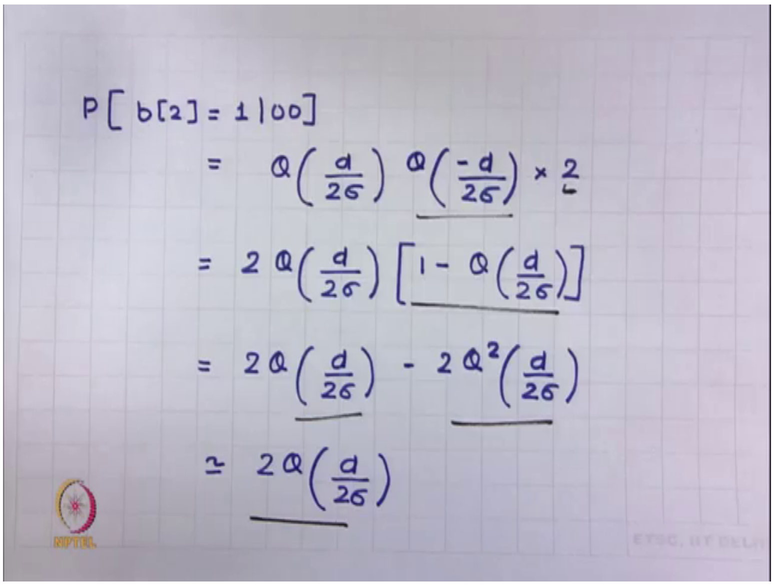

So, first let us appreciate that this probability will be going to be same as this probability just the names of the random variables are different if these random variables have the same mean and the same variances and the same distribution. These two probabilities will be same and thus we have a factor 2 in here, this probability is simply Q of d by 2 sigma this we have seen before and this probability is Q of minus d by 2 sigma and so, you can also write that Q of minus d by 2 sigma as this and you multiply this with this you get this multiply this with this you get Q square function and approximately you can find that probability of bit 2 being 1 is 2 times Q of d by 2 sigma. Of course we are assuming that hypothesis 1 is transmitted.

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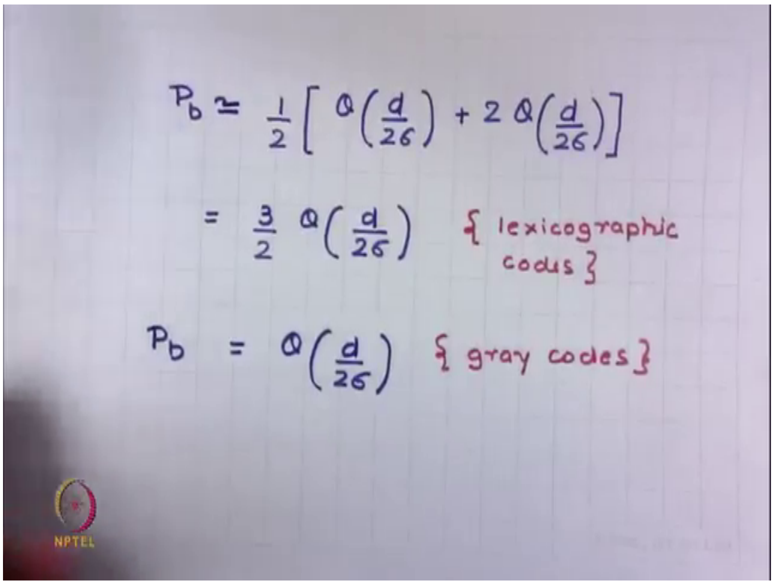

So, what is the overall probability of being in error ok. So, there are 2 bits, number of bits is 2 probability of first bit being in error is this probability of second bit being in error is this the overall probability of error is this plus this divided by 2 which is this. So, what we have got in lexicographic coding, we have caught probability of bit errors 3 by 2 times Q of d by 2 sigma. So, this is factor of 1.5 times larger than the bit error that you have got in gray coding approach. So, though it is not a proof, but it of course, conveys the idea that gray coding should be used while assigning the bits to the symbols ok.

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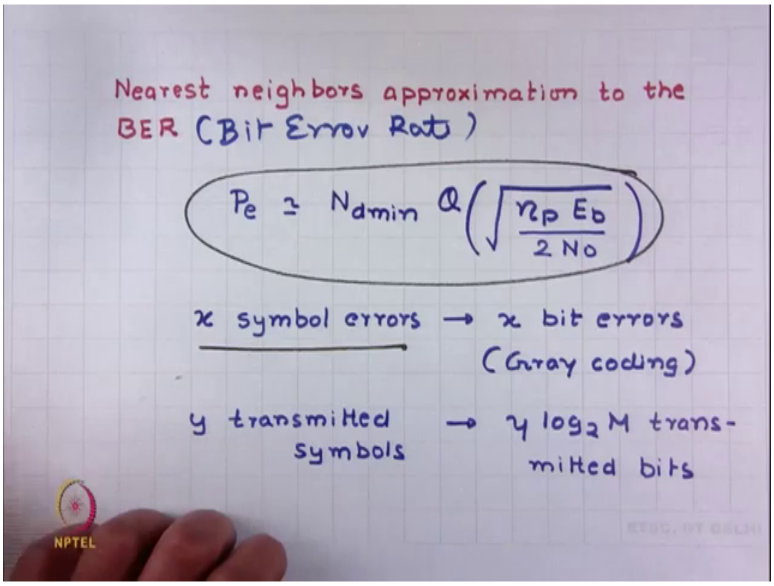

Now, we go and see how to evaluate this bit error rate using this nearest neighbors approximation that we have used for calculating the symbol error rate as well.

And remember this nearest neighbors approximation states that probability of symbol error rate can be approximated by multiplying the number of nearest neighbors with this Q function BER is an important word BER stands for bit error rate ok. So, sometimes we call this as probability of bit errors or simply bit error rate all right BER. Now if we assume that we are using gray coding what we assume also is that these nearest neighbors only differ by 1 bit because we are using gray coding.

Now when we are using this expression is simply means that the error happens mostly when a symbol slips to one of its neighbors ok, symbol is most likely getting confused by its neighbors ok. So, symbol error rate happens when the symbol slips to one of his neighbors decision region.

So, bit errors will also happened when a symbol is slipping to its neighbors decision regions and if the symbol slips to its neighbors decision regions and if we are using the gray coding mechanism; that means, whenever a symbol slips to its neighbors 1 bit error happen ok. So, if this x symbol errors happens and if these errors are mostly by symbol slipping to its neighbors decision regions then x symbol errors corresponds to x bit errors.

Because neighbors only differ by 1 bit, if we are transmitting y symbols; that means, we are transmitting y log 2 M bits because per symbol there is log 2 M bits ok. So, what is the bit error rate? Bit error rate is how many bits in errors divided by total number of bits ok.

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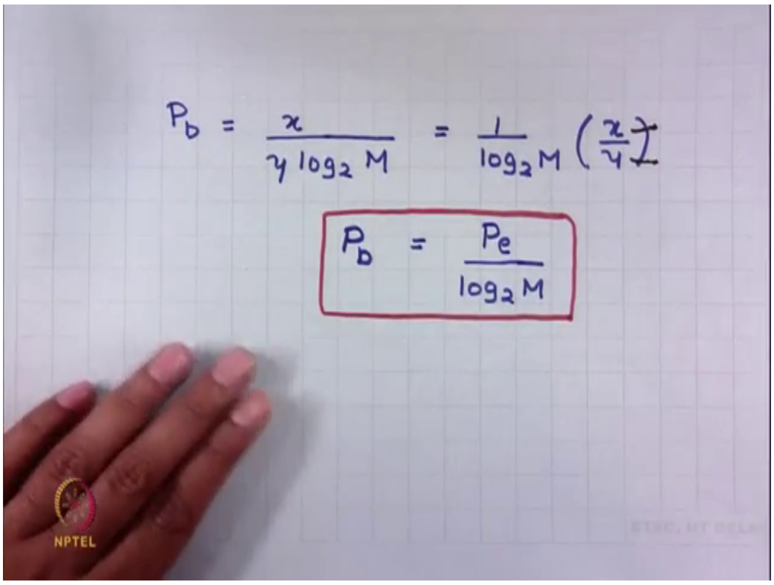

So, x divided by y log 2 M I can write this as 1 by log 2 M times x by y and what is this x by y is the symbol error rate. How many symbols in errors x symbols in errors how many symbols transmitted y symbol transmit this. So, x by y is symbol error rate and P b is bit error rate. So, bit error rate is simply P divided by log 2 M ok. So, we have got a neat relationship between bit error rate and symbol error rate if we are using this nearest neighbours approximation ok.

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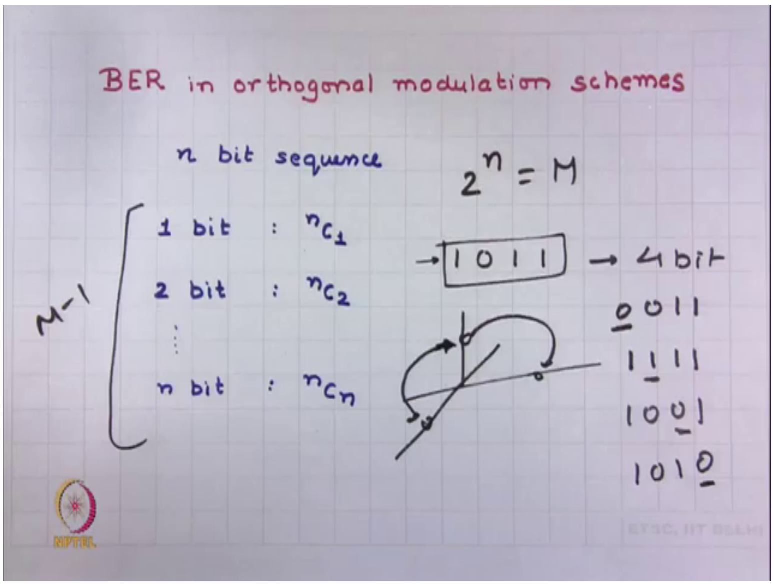

This relation cannot be used when we are trying to estimate the bit error rate in terms of symbol error rate for orthogonal modulation schemes, this relationship works fine for qam, psk pam and so on so forth, but not when you are using orthogonal modulation schemes.

Let us see the idea that is in here. Let us assume that in a symbol I have an n bit sequence; that means, I have 2 to the power n symbols and if I have an n bit sequence let us say I am having 4 plus bit sequence and when I am using these orthogonal modulation schemes remember this picture.

All symbols are at equal distance from the considered symbol thus this symbol might slip to this symbol; this symbol might slip to this symbol and so on so forth. And this in this orthogonal modulation scheme what happens is there is nothing like gray coding, because all symbols are neighbors of are considered symbol and thus you cannot make sure that the neighbors only differ by one bit that is not possible ok.

And that is why you cannot use this idea of nearest neighbors approximation and gray coding that we use to derive this relationship ok. Now so what we have seen is if you have transmitted this symbol this symbol might slip to any other symbol all symbols are equally likely to get confused with and thus I can slip to any of these M minus 1 symbols, any right on an average I would like to slip to each symbol one time.

Now, let us see how many symbols differ by 1 bit if there are n bits in a sequence the number of symbols that differ by 1 bit compared to the considered symbol is N c 1. For example, if I have 4 bits and this is the symbol that I have transmitted if you have to find out how many symbols differ by this symbol in 1 bit then there are 4 symbols. So, you can have 0 0 1 1. So, this symbol differs by this symbol in this bit you can have 1 1 1 1 this symbol differs by this symbol in this bit.

You can have this so this symbol differs by this symbol in this bit you can have this. So, this differs by this in this bit. So, if I have a 4 bit sequence you have 4 symbols which are different from this symbol in 1 bit and that is n c 1. Similarly you can ask the question how many symbols differ by the considered symbol in 2 bits, n c 2; similarly you can ask the question how many symbols differ by n bit compared to the considered sequence n c n. So, from this basic simple idea you can find average number of bits in error in M minus 1 symbols. So, n c i symbols will differ by i bits.

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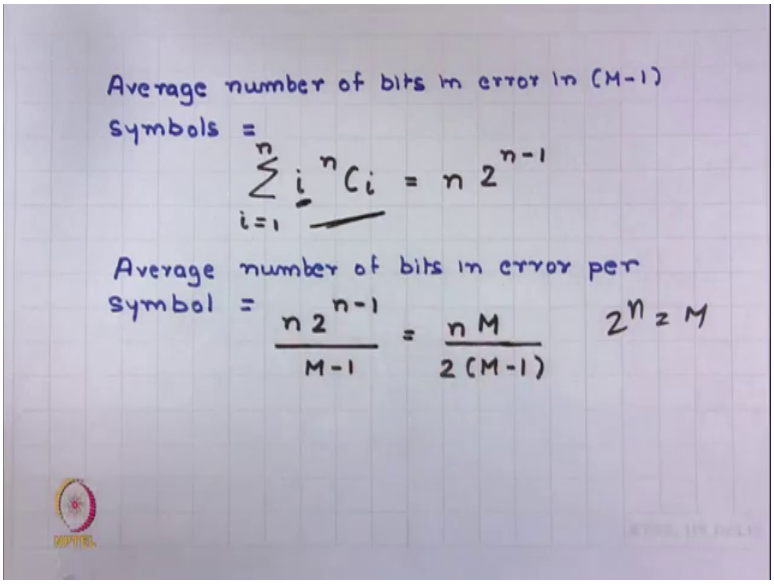

And you can find out the average number of bits in error by running this I from 1 to n and this series is well known this is simply n times 2 to the power n minus 1.

If you know this you can easily calculate the average number of bits in error per symbol by simply dividing this thing by M minus 1 and remember 2 to the power n is m. So, this is simply n M divided by 2 divided by M minus 1 ok. So, this is average number of bits in error per symbol.

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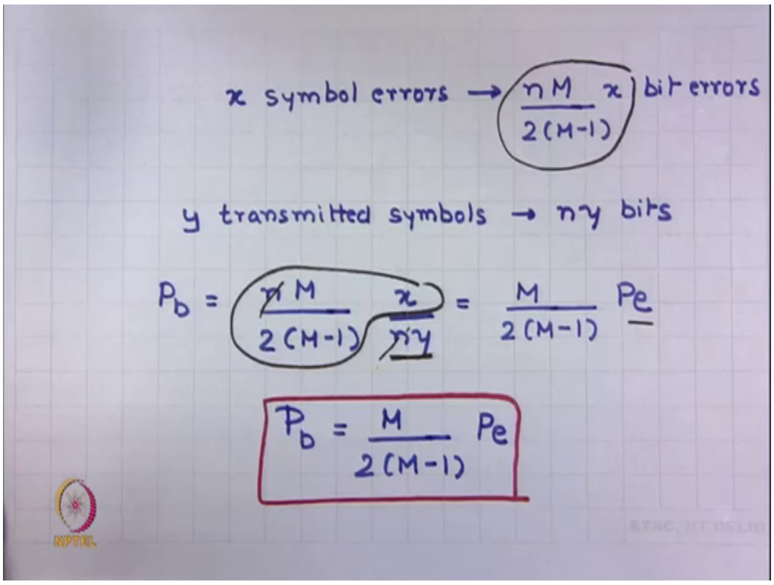

So, if there are x symbol errors we will have these many bit errors and if we transmit y symbols we are transmitting n times y bits. So, similarly bit error rate is this thing divided by this thing. So, this is the number of bits in errors divided by total number of bits transmitted and again we can rearrange this. So, this n cancels with this n. So, we have M 2 M minus 1 and we have x by y which is symbol error rate.

So, we have got relationship between bit error rate and symbol error rate for orthogonal modulation schemes as well. So, this is the relationship that you should use alright.

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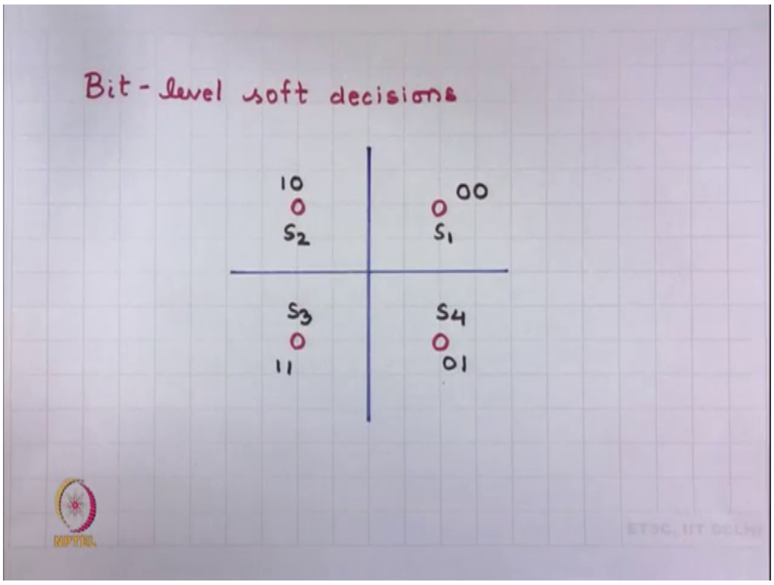

So, last topic for today is the spit level soft decisions. So, we have talked about this soft decisions in one of the previous lectures as well, in soft decisions what we want to do is instead of making a hard decision hard decision is that the detector decides whereas, 1 or 0.

In soft decisions what you like to do is you like to preserve these likelihoods and you tell the detector these likelihoods and when you convey to this detector these likelihoods detectors may use this likelihood to improve their performance ok, and we have seen one example of that before. So, detector can also preserve likelihoods at bit levels and we will see one example of this. So, we are using the gray coding again to assign bits to the symbols and we want to find out the likelihoods at the bit level.

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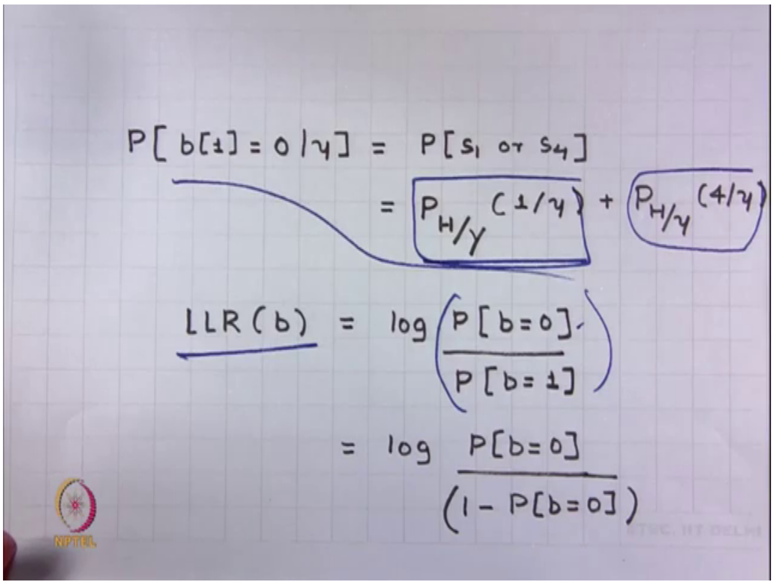

So, detector will like to find what is the probability that bit 1 is 0 given an observation whatever that observation is and if you look at this mapping bit 1 is 0 here and here. So, this probability as same as the probability of signal being s 1 and s 4.

So, this probability is probability of signal being s 1 or s 4 and what is the probability of signal being s 1 this is the a posteriori probability of signal being s 1. So, we have to calculate the a posteriori probability of signal being s 1, we have calculated these a posteriori probability several times ok. Plus a posteriori probability of signal being s 4 ok, the sum of these two a posterior probabilities you will get the probability of bit 1 being 0 and from this we can get the log likelihood ratio.

Log likelihood ratio is simply log of probability of bit 1 to be 0 divided by probability of bit 1 to be 1 and if you know that probability of bit 1 to be 0 is this the probability of bit to be 1 is 1 minus probability of bit being 0 ok. So, once you have calculated this you can easily calculate the log likelihood ratios ok. So, what we are saying is when you have to calculate the probability of bit being 1 or 0 you simply sum up the a posteriori probabilities of the desired signals and from that sum of a posteriori probabilities you can also calculate easily the log likelihood ratios for bit being 0 and 1 ok.

So, with this we have finished posteriori about coherent detection. So, so far we have being considering coherent detection we have not considered the case when you have frequency and phase half says we have assumed that the receiver knew exactly posteriori about the frequency and phase of the carrier. In next lecture we will see what happens when receiver has not complete knowledge about the frequency and phase of the carrier. So, we will learn about detection for non-coherent communication in the next lecture

Thank you.