Principles of Digital Communication
Prof. Abhishek Dixit
Department of Electrical Engineering
Indian Institute of Technology, Delhi

Lecture – 03
Signal Spaces
Inner Product & Orthogonal Expansion

Welcome to second lecture on unit 1 and in this lecture we will talk about inner product spaces. So, let us first revise what we discussed in the last lecture. In the last lecture we covered what are waveforms. We have looked into what is a signal space, we have set signal space will allow us to treat signals as vectors and that will simplify digital communication a lot because then we can treat signals as numbers. Then we have discussed what is L_1 and L_2 space and we have defined what indistinguishable functions are and finally, we have defined what are basis vectors and linearly dependent vectors.

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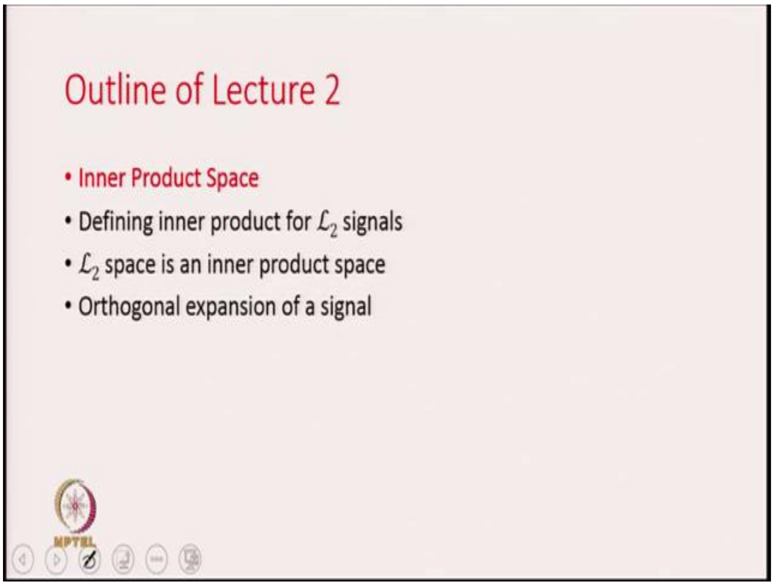

In today's lecture we will be talking about inner product space. Inner product space will allow us to talk about the length of a vectors and the direction between the two vectors. Then we will define the inner product operation for L_2 signals, remember L_2 signals are the finite energy signals. And we are interested in L_2 signals because in digital communication we usually transmit and receive L_2 waveforms. Then we will prove that L_2 space is actually an inner product space and finally, we will discuss how to do orthogonal expansion of a signal which will also be really useful in understanding the transmitter and receiver designs.

So, let us get started with inner product space.

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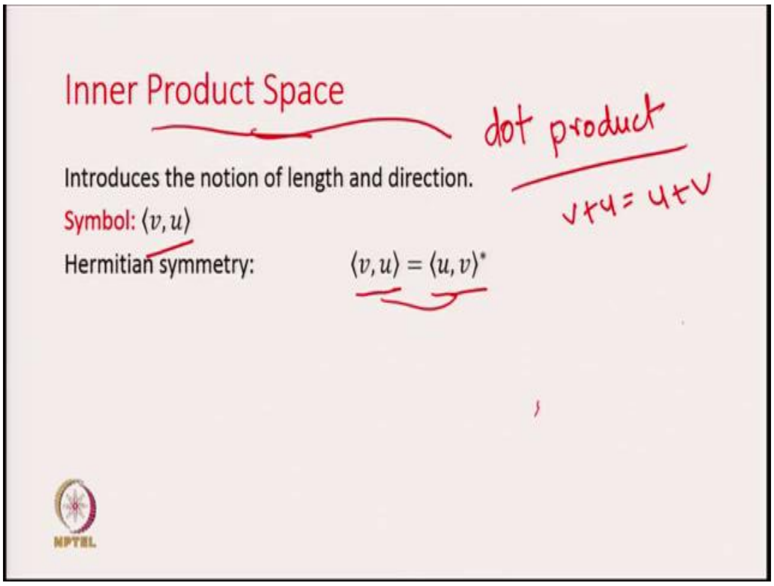

So, remember in the last lecture we talked about the vector space. And in the vector space we said that for a valid vector space, the addition of two vectors should be defined and the scalar multiplication should be defined. And there were certain properties which the vector addition and scalar multiplication must satisfy in order for that operation to be a in a valid vector space. But there was no notion of length and direction of the vectors and in inner product we will be able to talk about the length and direction of vectors. Inner product is same thing as the dot product, but for signals we usually do not say the dot product of signals we talk about the inner product between the two signals.

So, it is the same thing as what you studied in high school about dot product. So, we will use the symbol $< u, v >$ to denote the inner product operation between the vector v and vector u. Now as we have discussed in the last lecture, for a valid vector space certain axioms needed to be satisfied. Similarly for a valid inner product space, certain operations or certain axioms need to be satisfied first let us look what those axioms are. For an operation to be valid in a product operation Hermitian symmetry must be satisfied. What is Hermitian symmetry? It states that inner product of vector v with vector u should be same as inner product of vector u with vector v, but with conjugate.

Hence, the order in which you do inner product matters. Remember when we were talking about the vector addition we said in vector addition the order in which you add vectors does not matter, but when we are computing the inner product, the order in which you take inner product matters especially for the complex signals or vectors.

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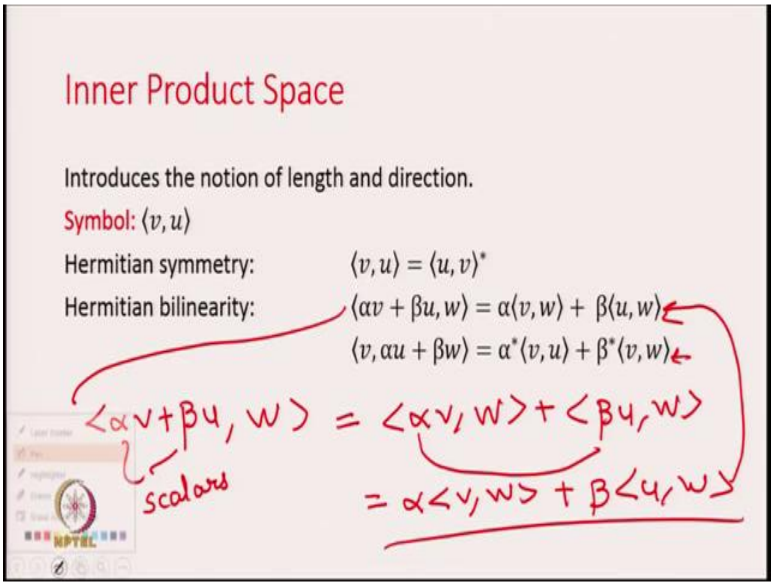

Secondly, the Hermitian bi-linearity must be satisfied. What is Hermitian bi-linearity? So, first condition is for an operation to be a valid inner product operation. If I am taking the inner product of vector $\alpha v + \beta u$ (where α and β are scalars and v and u are vectors) with a vector w, then first of all linearity must be satisfied.

Now remember that we want also to take these scalars outside this inner product operation and if this is scalars are sitting on where the first vector, they can be pulled out without any trouble. So, I can just take the α out, α is sitting with the first vector v similarly I can pull out this β and I get this. And this is this equation now let us see what we mean by this equation.

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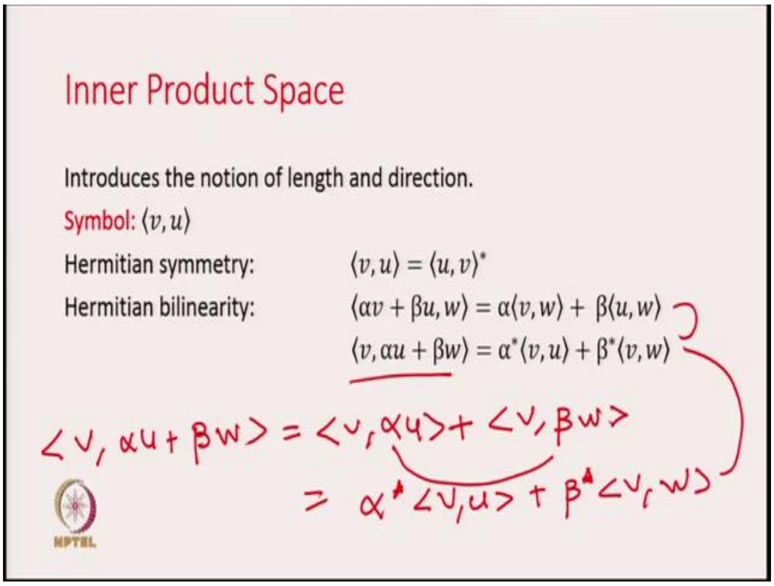

So, what does this state if I want to take inner product of vector v with αu + βw. So, first invoking linearity, I can write this and then now I have again to pull out these scalars, but remember now these scalars are sitting with the second vector, and now if these scalars come out they will come out with conjugate.

So, this is the second equation. So, for a valid inner product operation, these two equations must be satisfied and this is the notion behind Hermitian bi-linearity.

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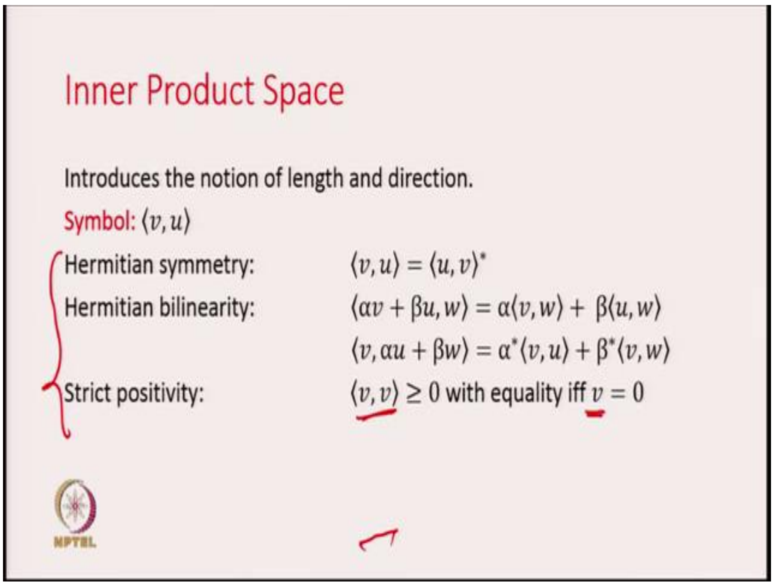

The third property now that needs to be satisfied for a valid inner product operation is, if you take the inner product of vector with itself. This inner product must be a strictly positive this can be equal to zero only if the vector is a zero vector. If vector is not a zero vector, then the inner product of a vector with itself must be strictly positive if the vector is a zero vector than the inner product of vector with itself must be zero.

So, this is the meaning behind strict positive positivity. Now these are the three properties that needs to be satisfied for an operation to be a valid inner product operation.

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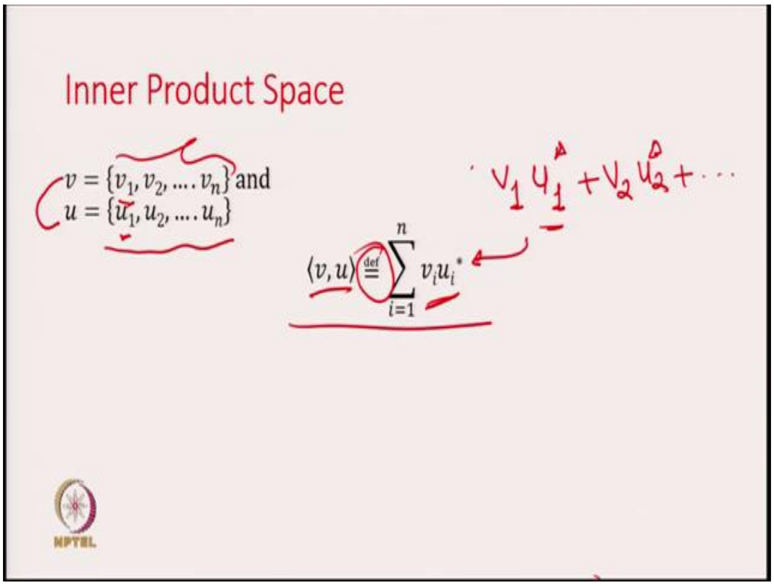

Let us now take one example and let us now see or assume that we have two vectors. Let us say that we have a vector v and we have a vector u and this vector v is consisting of n complex numbers, it can be n real numbers or n complex number does not matter and vector u also consists of n complex numbers. Now we have seen in the last lecture that we can think about vector as n tuples, where the vector can be thought as consisting of n complex or real numbers. So, the same ideas we can think about the vectors as n complex or n real numbers.

Now if I have to take the inner product between the two vectors, I can define. So, this is by definition I can define the inner product operation as this. So, it simply means that I need to pick up an element from vector v. So, let us say I have taken up the first element from vector v, I need to multiply with the corresponding element from vector u that is u_1 corresponding element, but this number comes with a* and then I need to sum this up. So, I need to collect similar products and I should go on and on. So, this is what this equation represent.

So, I need to multiply v_i with u_i* and then I have to sum this up for all values of i. So, this is by definition the inner product operation between two vectors; we have vectors I am thinking as consisting of n complex or real numbers. Of course, if these elements are real numbers then conjugate does not have any influence, but if they are complex then of course, taking conjugate makes difference.

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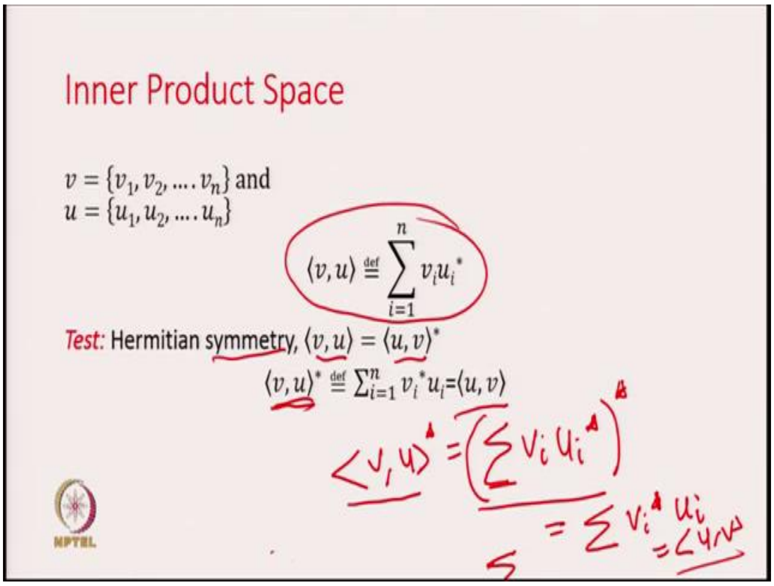

Now if this operation has to be a valid inner product operation this operation has to satisfy the three properties that we have listed out. So, Hermitian symmetry, bi-linearity and strict positivity must have been satisfy. So, let us test let us test whether this definition satisfy Hermitian symmetry. For this I have taken the inner product of vector v with u that this thing should be same as inner product of vector u with v, but with the conjugate. So, now, let us see. So, I am interested in what is this inner product of v with u* and so, v with u* it simply means that. So, inner product of vector v with u is this thing and conjugate means that I have to take conjugate here and conjugate of summation is nothing, but the summation of conjugates.

So, this thing is similar to v_i* and u_i and this is nothing, but by definition this is the inner product of u with v. So, this is what we are saying the inner product of vector v with u* should be same thing as inner product of u with v to satisfy Hermitian symmetry and this is what we have got. So, this definition looks to us of course, we have to test whether the other properties are satisfied and I leave that to you think about proving that other two properties are also satisfied, if this definition is taken to define the inner product between the two vectors please work it out yourself.

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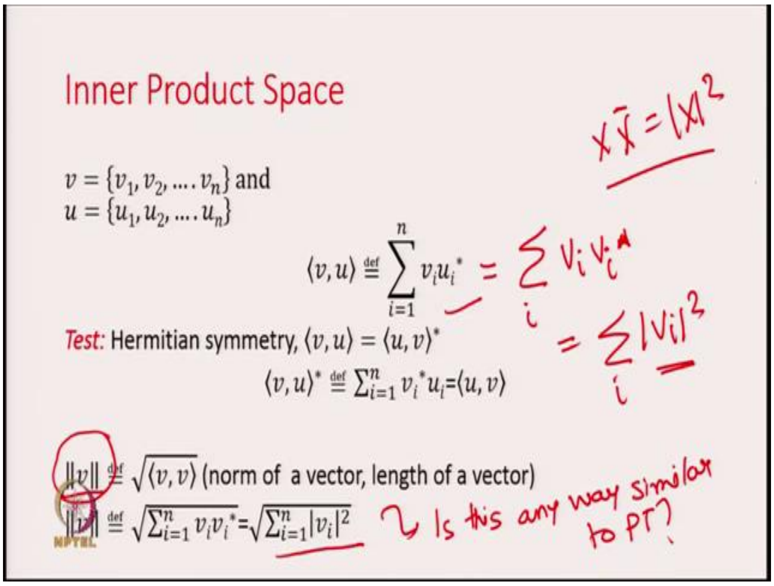

Now, let us define some other important quantities first let us define what the norm of a vector is. So, norm of a vector physically means that we are talking about the length of a vector length of a vector is an important quantity and what is the length of a vector? Length of a vector is also known as a norm of a vector and what is that? It is nothing, but square root of inner product of vector with itself. So, you have to take the inner product of vector with itself, you have to take the square root of that thing and that will correspond to the length of a vector. So, this is an important idea how we should define the norm of a vector.

Let us take an example for if this definition of inner product is adopted, what is that norm of a vector? So, I have to take the inner product of vector with itself; that means, now u_i should be replaced by v_i* because inner product is with the vector itself and what is $v_i v_i$*? It is nothing, but $|v_i|^2$. So, you know that xx* is nothing, but $|x|^2$ if x is a complex number you must have seen this. So, this turned out to be this. So, this is how the norm of a vector v will look. So, this quantity will correspond to length of a vector. I also leave you with something interesting try to think is this anyway similar to Pythagoras theorem; try to think about this and try to get some intuitions.

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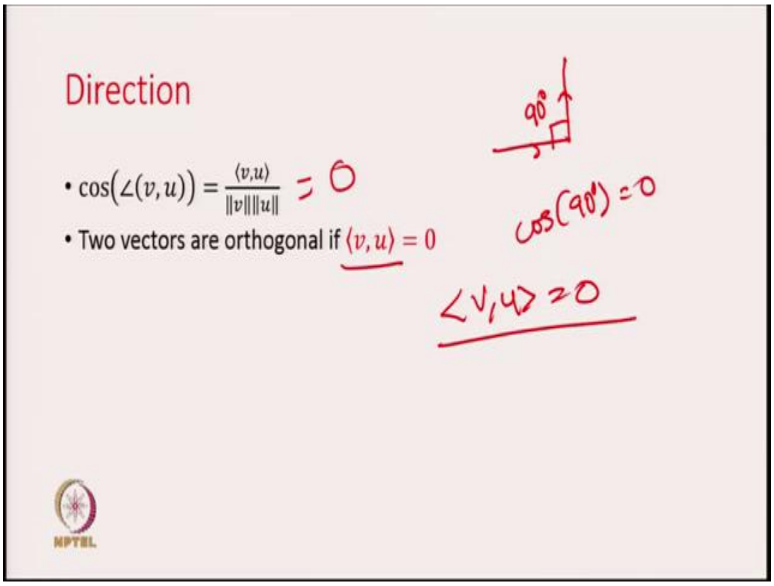

Let us move on. Now it's the turn to define the angle between the two vectors and this is one identity, which I assume that you must have done if I am interested in the cos of angle between the two vectors this can be simply obtained by taking the inner product of with vector with another vector dividing by the length of two vectors. So, this is how the cos of the angle between the two vectors can be thought about. Now if I know this, I can define what is known as orthogonal vectors what are orthogonal vectors? Orthogonal vectors are same thing as perpendicular vectors you must have studied about perpendicularity in high school. So, orthogonality is the same thing as perpendicularity, but in engineering we rename things we invent new names for the things that you have studied in school and make it look fancy. So, don't get confused, orthogonality is the same thing as perpendicularity.

So, let us talk about the condition when two vectors are orthogonal vectors or perpendicular vectors. If two vectors are perpendicular, we know that the angle between the two vectors must be 90º and hence cos 90º is zero and hence this must be zero. Now this can be zero in two conditions either the inner product of vector v with u must be zero or the length of vectors must be infinite, but in this course we are not interested in the vectors infinite length we confine our discussion to the vectors which have finite length. In the case when we are talking about the orthogonality, that can only be achieved if the inner product between the two vectors is zero.

So, this is the test that you need to make to predict whether the vectors are orthogonal vectors or not. So, if you are interested in finding whether the vectors are orthogonal you take the inner product between the two vectors. And if it turns out to be zero; that means, the vectors are orthogonal vectors. We can also prove what is known as Cauchy Schwarz inequality.

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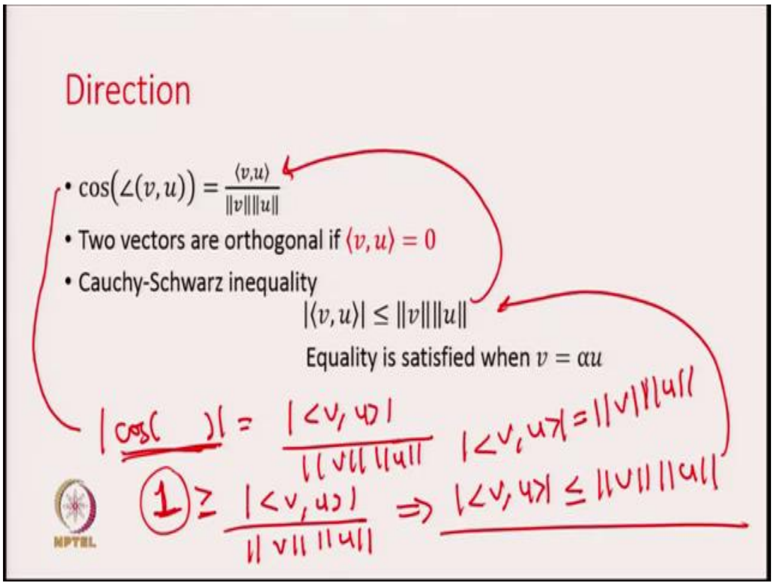

This inequality is very useful and this inequality we will use a lot of time in this course. This inequality follows directly from this relationship this is nothing complicated here.

So, now we know that first let us do we take the mod on both side. So, let us take the mod on both side and now I know that cos takes a value between - 1 and 1 the mod of cos must be less than or equal to 1. So, this quantity is less than or equal to 1 and hence 1 must be greater than or equal to this quantity. So, if this is true this implies that $|< v,u >| \leq ||v|| \times ||u||$ and this is the Cauchy Schwarz inequality. When will this equality be satisfied? This equality will be satisfied when this quantity is same as 1 and when will this be same as 1 when the angle between the two vectors would be zero. So, that cos of zero is 1 so; that means, this equality would exist this equality will exist only when this is 1, this quantity is 1, this will be 1 when angle between the two vectors is zero when angle between two vectors is zero we say that the vectors are collinear vectors, they have the same direction so, that the angle between the vectors zero. So, this equality will be satisfied when v is some scalar times the u vector.

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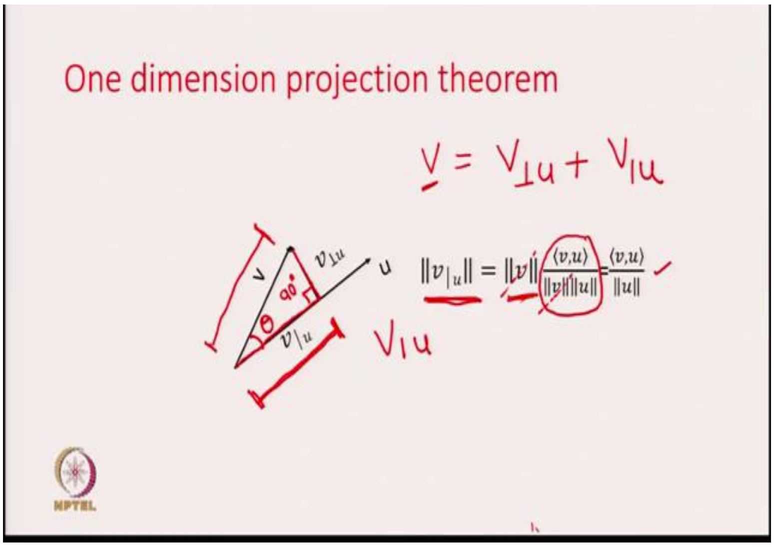

So, let us look at the one dimension projection theorem and here what we are doing is, we are trying to break this vector v. So, we want to break a vector v into two components one is perpendicular component to vector u ($v_{\perp u}$) and the other one is the parallel component to vector u ($v_{|u}$). So, this is like a right angle triangle. So, let us see if we want to find the length of $v_{|u}$. So, this quantity represents the norm or length of $v_{|u}$ which can be obtained by first having the length of v vector. So, this is the length of v vector multiplied by $\cos\theta$, where θ is the angle between vector v and vector u. So, length of $v_{|u}$ is given by this, now we can cancel $||v||$ with $||v||$ and we get this quantity.

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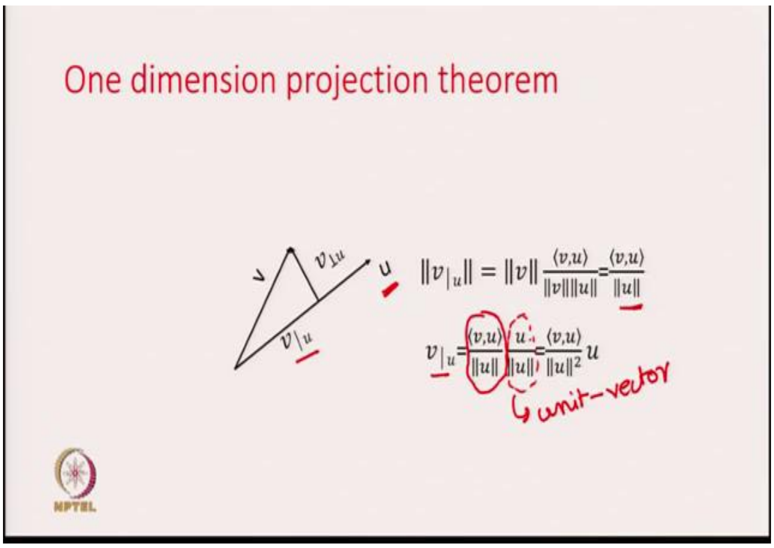

Now, finding the $v_{|u}$, we have to first take the magnitude or length of $v_{|u}$ which as I have said is this quantity, and then I have to multiply with the direction of $v_{|u}$. Now if we look $v_{|u}$ has the same direction as vector u and thus I have to multiply it with vector u. But I have to multiply it in such a way that this does not alter the length of $v_{|u}$ which we have already obtained and the way to do this is by multiplying it with a unit vector. So, this is a unit vector the one easy way in which you can obtain a unit vector is just have the vector and divide this by the length of the vector. So, this is the unit vector.

Now, from this we get $v_{|u}$; it's a very important result and thus I write it again.

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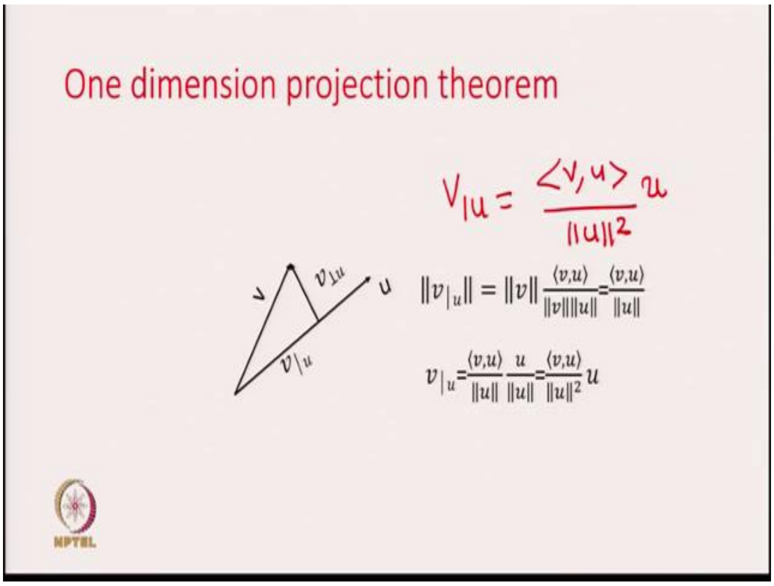

$$v_{|u} = \frac{< v, u >}{\left||u|\right|^2} u$$

Now if you have obtained $v_{|u}$, it will be straight forward to find $v_{\perp u}$ which will be given by the difference v - $v_{|u}$.

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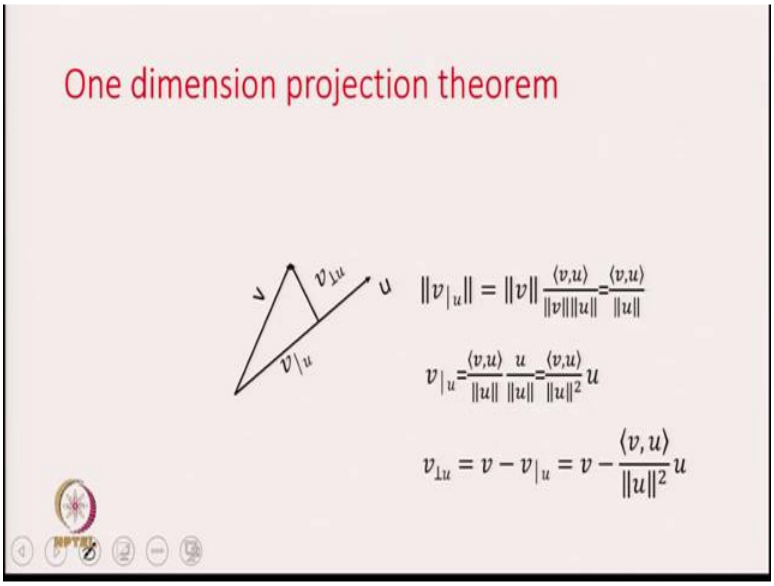

Let us try to look at how we can define the inner product for L_2 signals.

So, let us get started while assuming that I have a vector v and I am trying to approximate this vector v in terms of another vector u and c is some scalar.

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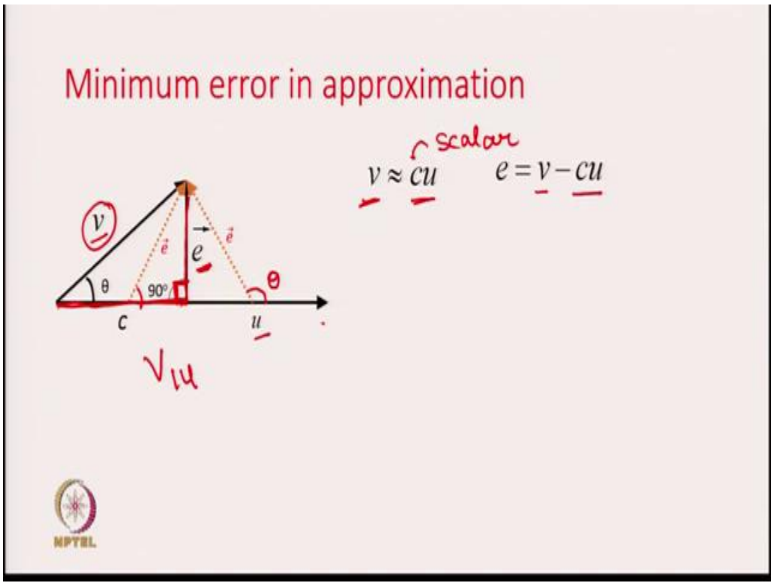

The angle between vector v and vector u is θ. Now because vector v is not same as vector cu, there is an error in approximation and this error is given by the difference between vector v and cu. For example, in this case I have shown that this error vector is perpendicular to vector u error vector in general can take any angle with vector u here for simplicity I have assumed that error vector is making an angle of 90º with vector u.

Now if you see carefully if error vector was making any other angle other than 90º with vector u for example, if it was making an obtuse angle with vector u or if it is making an acute angle with vector u whatever angle it makes other than the 90º then the error vector would have been larger. So, the minimum value of error vector would happen, when error vector makes an angle of 90º with vector u. And when this error vector makes an angle of 90º with vector u, the component of v along u is nothing, but $v_{|u}$ this we have seen in the last slide when we have kind of a right angle triangle, this component of v along u is nothing, but $v_{|u}$.

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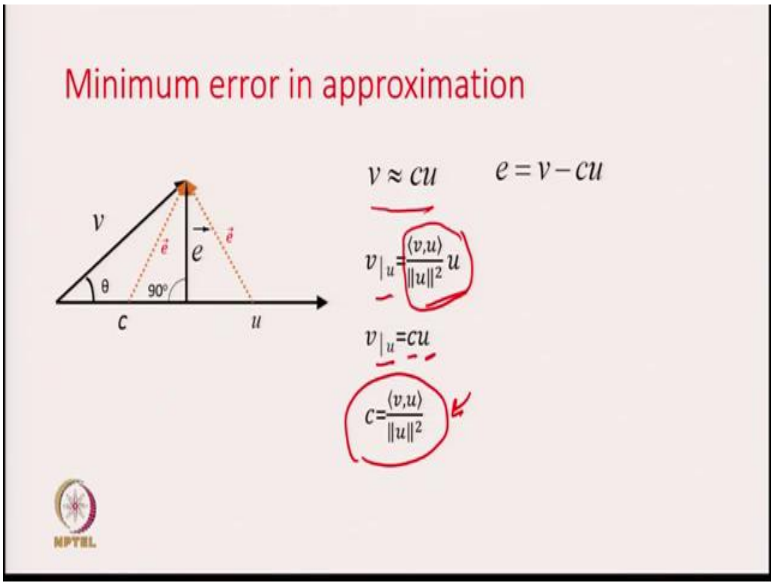

And hence the component of v along u is nothing, but it is $v_{|u}$ and as we have proved in the last slides that this value of $v_{|u}$ is given by this quantity. If I try to think $v_{|u}$ is some constant times u vector, then the value of constant is given by this thing. So, c which is the value of the constant is

$$c = \frac{< v, u >}{||u||^2}$$

So, this value of c will lead to minimum error in approximation when you are trying to approximate a vector in terms of another vector. Following from the same idea let us try to do this exercise in the case of signals. So, we are now doing it in the case of vectors, let us see if we do a similar thing in case of signal what happens?

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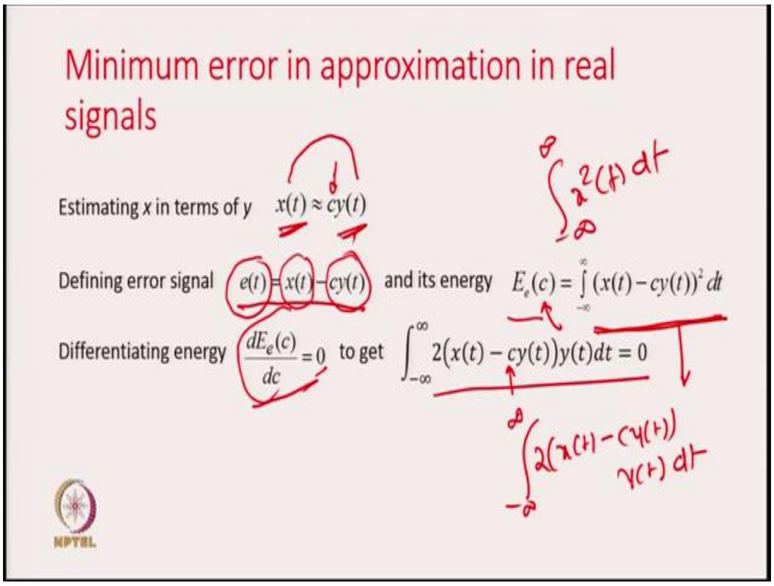

And for this discussion, first I will assume that the signals involved are real signals because this analysis is relatively simple in the case of real signals. So, I start by approximating a signal $x(t)$ in terms of a signal $y(t)$, c is some constant and idea would be let us find out what is the value of c for minimum error.

So, because I am approximating $x(t)$ in terms of $y(t)$, again I will have an error signal, error signal will be nothing, but it is the difference between the actual value of the signal and estimated or approximated value of the signal. If I know this is the error signal, I can also talk about the energy in the error signal energy in the error signal will be nothing, but this quantity. You know that energy of the signal is nothing, but this and the signal error signal is this. So, energy in the error signal is this quantity.

Please see that this energy in the error signal depends upon the value of c and what we are interested in the value of c for which the error signal is minimized; that means, the energy in the error signal is minimized. For L_2 waveforms what is most important is, the energy and that is why we are trying to see for what value of c the energy in the error signal is minimized. To do that we can differentiate this quantity with respect to c, and when I make this derivative equals to zero, I can obtain the value of c.

So, I differentiate this with respect to c this quantity what I would get is 2 $[x(t)$ - c $y(t)] \times y(t)$ and I can keep this thing equals to zero and from this I can obtain the value of c.

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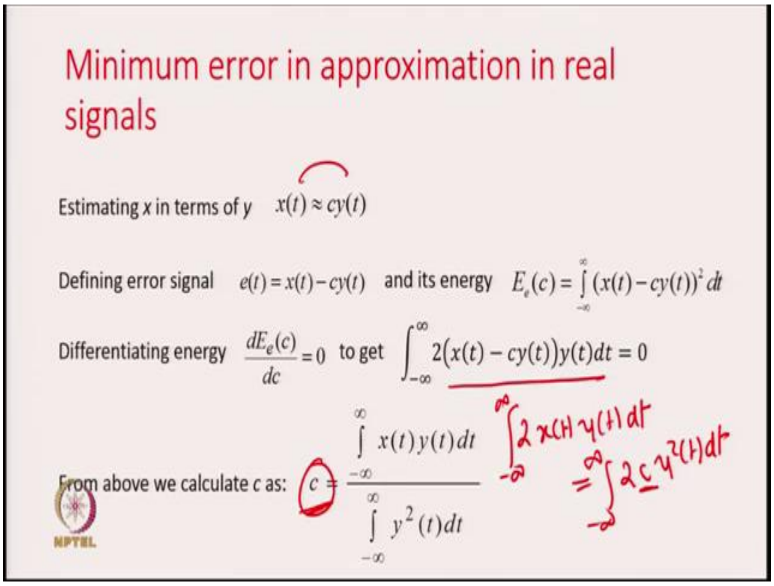

So, let us see what is that value of c? If I want to put this as zero let me do this. So, I have this integration $2x(t)y(t)dt$ is same as $2cy^2(t)dt$. And from this I can take the c out, c is not a function of time it can be pulled out from this I get the value of c as this quantity. So, this is the value of constant that will minimize the error in approximation.

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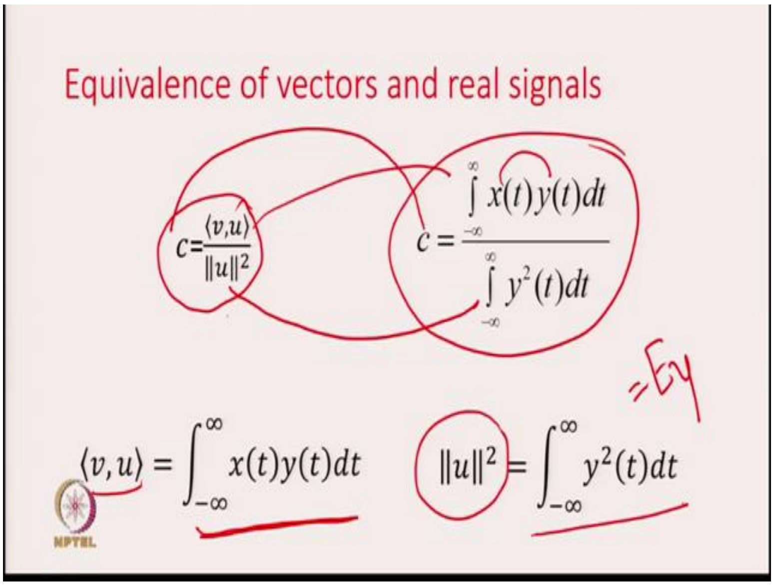

Now, for equivalence between vectors and real signals, we can look in here for vectors we have already obtained that the value of constant that obtains the minimum error in approximation is given by this quantity. So,

$$c = \frac{< v, u >}{||u||^2}$$

For the case of signals, we have got this value of c which minimizes the error in the approximation of two signals, that is when you want to approximate $x(t)$ in terms of $y(t)$, this value of constant minimizes that error in approximation. Now, if we want to define the inner product for signals and we want to treat the signals equivalent to vectors, this will give us some hint on what is the good definition of the inner product. For example, we can choose the definition of inner product for the signals like this, because for vectors this thing is same as this thing. So, this is the definition for the inner product between two signals, remember that the signals involved here are real signals. Similarly I want to equate denominator by denominator, then I get that the norm square of a vector is nothing, but it is this quantity and this quantity is nothing, but it is the energy of a signal. So, energy of a signal is nothing, but it is similar to norm square of a vector and norm is length. So, energy of a signal is nothing, but it is very similar to the length square of a vector.

So, if you want to treat signals as vector, the energy is length square.

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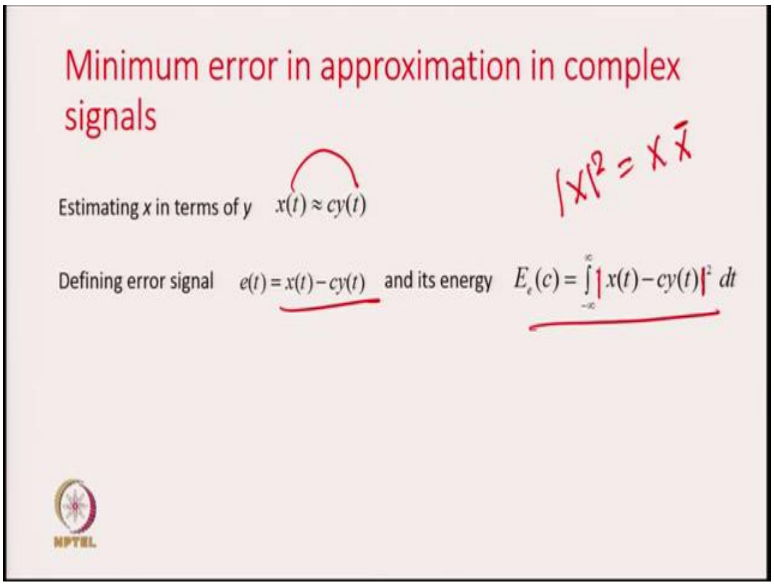

Now, let us do the same exercise for complex signals the idea is similar, we have a complex signal and we want to approximate it in terms of another signal. So, we are approximating $x(t)$ in terms of $y(t)$, where $x(t)$ and $y(t)$ are complex signals. The error signal definition is same as before error signal can be obtain by taking the difference between the two signals, energy in the error signal is also same as before. But because now the quantities involved are complex, we need to take a mod in here and that complicates the thing.

Now, we can also use an identity that $|x|^2 = xx^*$.

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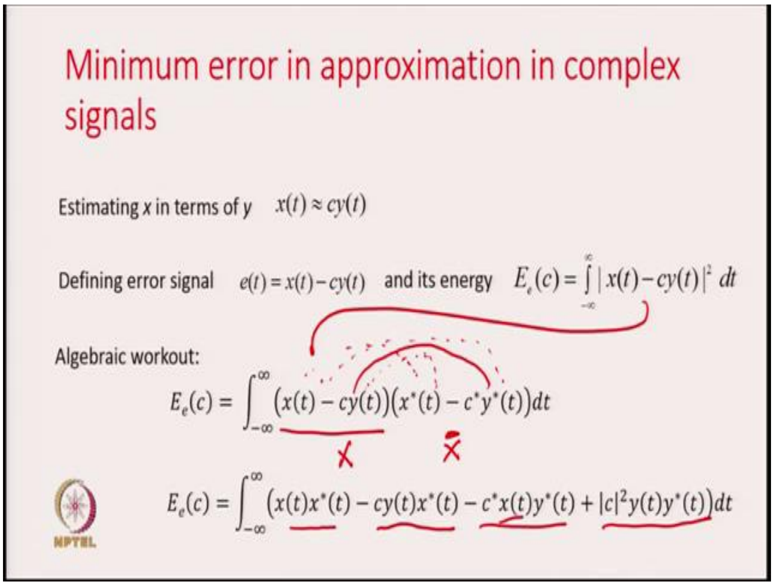

So, from this what we can do is, we can express this thing as this is like x and x* now what we can do is, we can multiply term by term. So, we can multiply $x(t)$ with $x^*(t)$ this is this here, we can multiply c$y(t)$ with $x^*(t)$ this is this now we can multiply this $x(t)$ with this term $x(t)$ with this term and finally, we need to multiply c$y(t)$. So, this with this, which would give us this. So, we will get four terms in the in the multiplication.

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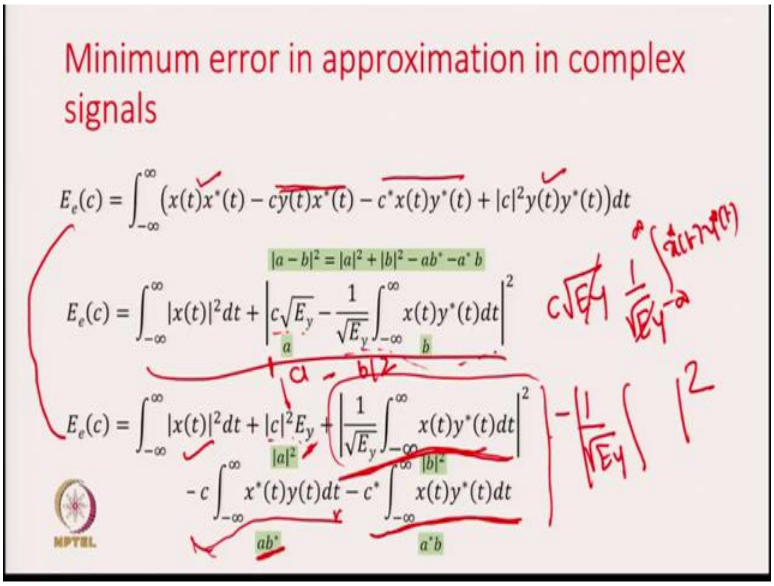

So, this is the expression that we have obtained from the last slide, let us try to rewrite this equation in a different form let us write it like this. So, this is a more convenient way in which I want to express this, why is this more convenient we will see this soon, but let us assume that I want to write this into this form and let us try to see whether this is equivalent to this.

So, now if you see carefully, this is of the form of $|a - b|^2 = |a|^2 - |b|^2 - a\, b^* - a^* b$. So, $a = c\sqrt{E_y}$ and $b = 1/\sqrt{E_y}$ and I have to take the conjugate of this. So, conjugate will come here and here there will be nothing. So, this is b* which is this quantity, similarly I can obtain a* b as this. So, when I am expanding this I get to this form. Let us see if this is equivalent to this form. So, this part is present in this expression this part is also present here.

So, we have mod a square also available here. This part is missing in this expression. So, this is not present is this present ab* is present? Yes. So, this is present here and this part a*b is also present here. So, this is same as this thing, but other than this term. So, we have to subtract this. So, that I make this expression same as this expression. So, let us try to do this. So, subtracting this term we get this expression. So, now, these two expressions are equivalent expressions. So, we will be using this expression rather than this expression.

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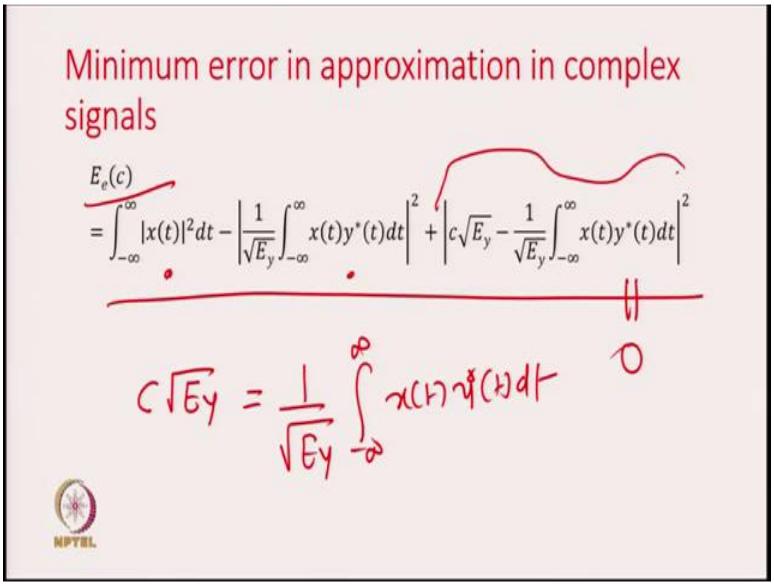

So, let us rewrite this. So, energy in the error signal we have just shown can be expressed like this. Now we have to choose the value of c for which this energy is minimized this term is not a function of c this term is also not a function of c. So, we have to choose a value of c for which this term is minimized. And this term will be minimized when this term goes to zero because this is a positive quantity. So, minimum positive quantity would happen when that quantity is zero. So, we have to choose a value of c for which this term goes to zero. And that value of c can be easily obtained if I substitute this as this. So, from this we can get the value of c which is test value easy.

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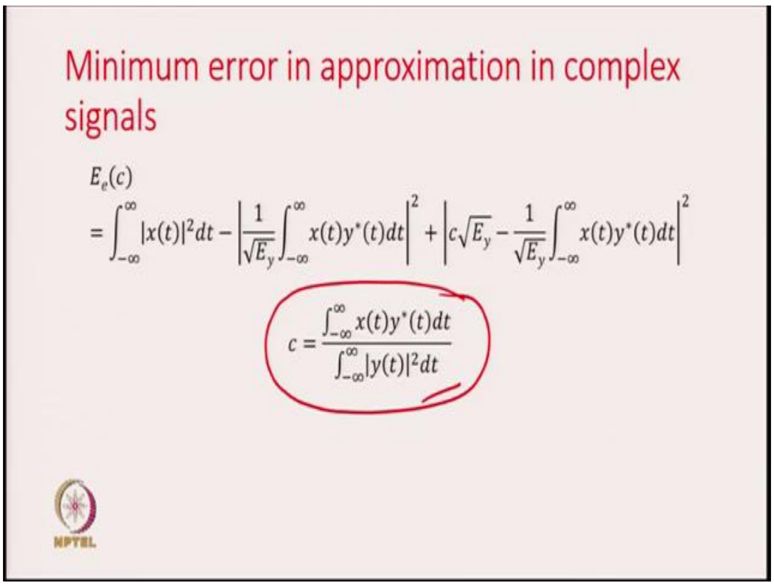

So, for complex signals, this should be the value of c which needs to be chosen if I want to express one complex signal in terms of another complex signal. And let us now do the same thing as we did for real n vectors let us see what the equivalence between vectors is and complex signals.

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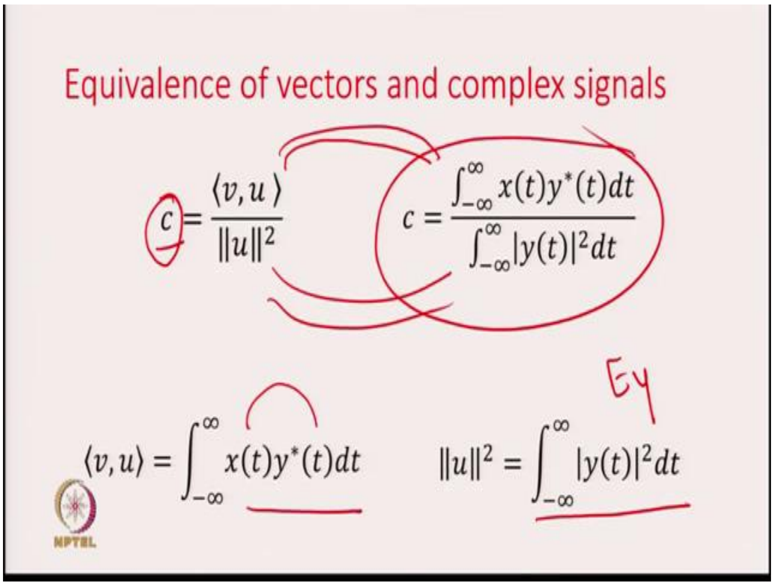

So, in the vectors remember the value of c that minimizes the error in approximation when you want it to approximate a vector in terms of another vector, this was the value of the constant that was giving us the minimum error in approximation, in case of signals this is the value of constant that gives us the minimum error in approximation. And again if I want to equate these two c’s and if I want to think the equivalence between numerators and denominators, I can get the definition of inner product for signals given by this expression, where the inner product of the signals can be obtained by multiplying a signal with the conjugate of another signal and then integrating this up. Again this in the case of the complex signals corresponds to the energy of the signal and as for the real signals, energy of a signal is nothing, but length square or norm square of a vector.

So, these are some ideas which will help you to think about what is the energy of a signal, what is the length of a signal and what is the inner product between the two signals,.

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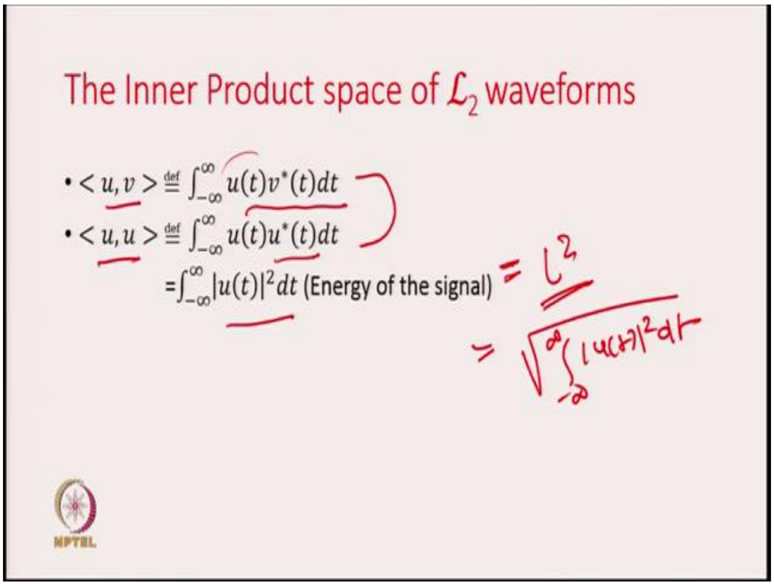

So, finally, we can define the inner product for L_2 waveforms. So, inner product of L_2 waveforms can be define like this, where you take in a signal or a waveform you take in the conjugate of the other signal or waveform and you do the integration from - ∞ to + ∞. I can take the inner product of the signal with itself and nothing happens other than that instead of $v(t)$ you need to have $u(t)$ and this is nothing, but energy of the signal. And this quantity would be saved as the length square. So, if I am interested in length of a signal trying to think about treating the signal as vector, then easily I can take this term square root of this quantity. So, this will tell me what the length of my signal is.

Now, because I now can treat the L_2 waveforms as a valid inner product space, I even though I have not tested whether this definition satisfies all axioms of valid inner product space, but let we will do this, but let us start by assuming that these definitions are valid. These will allow me to do things like I can talk about now the, what is the angle between the two signals, was the length of the signals, was the in the length square of the signal and so, on and so, forth. So, the next question that we need to ask is whether this definition corresponds to a valid inner product space and for that we need to test it through axioms that we have already seen; our symmetry must be satisfied, by linearity must be satisfied and strict positivity must be satisfied..

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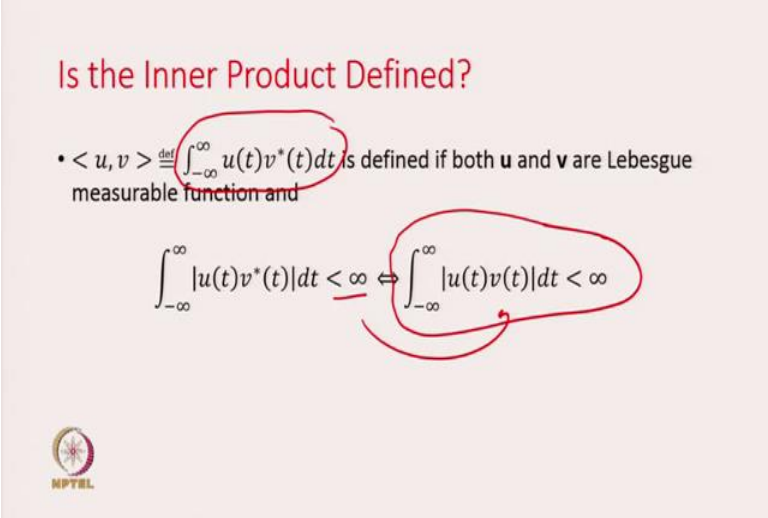

But before all of this one thing that needs to be checked is whether this integration always exists if $u(t)$ and $v(t)$ are valid L_2 signals; that means, the signals with finite energy is this integration always converges otherwise you cannot define the inner product between the two signals. So, that is what we have to check. So, for convergence what we need to do is, we need to check whether the mod of this quantity is finite that is that is when the integral exists we have already seen this in though first lecture. So, for this to happen we can also find out an equivalent expression, that this should be satisfied. So, just replacing the $v*(t)$ with $v(t)$. The mod of a complex quantity or a real quantity is one and the same thing. Now we have to check under which condition this is valid and for doing that the idea is very simple we can use the identity from the complex word.

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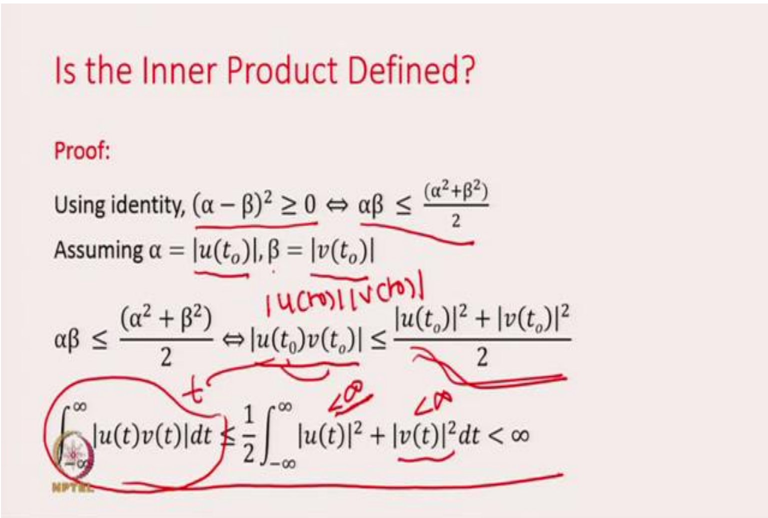

So, you know that $(\alpha - \beta)^2 \geq 0$ and from this we get

$$\alpha\beta \leq \frac{(\alpha+\beta)^2}{2}$$

Now if I assume that $\alpha = |u(t_0)|$ and $\beta = |v(t_0)|$. So, $|u(t_0)v(t_0)|$ is less than or equals to this quantity and $|u(t_0)| \times |v(t_0)|$ is nothing, but it is $|u(t_0)v(t_0)|$. So, from this I get this and then I am integrating this for all values of t_0; that means, t naught is really t, where t spans from - ∞ to + ∞ I can get this expression. Now because my signals involved are L_2 signals I know that this quantity is a finite quantity this quantity is also finite quantity sum of two finite quantities is a finite quantity and hence this will be always a finite quantity. And hence this proves that the inner product for L_2 signals is always defined; that means, if you have a finite energy signal. If you have another signal as a finite energy signal if you want to find the inner product of these two signals is always defined is always finite.

Now, finally, we have to test this definition through the axioms of inner product space.

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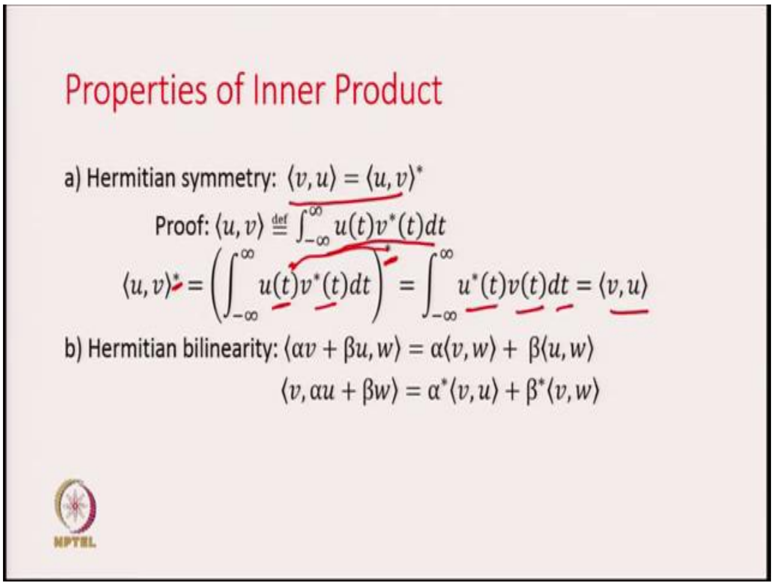

So, first we need to check whether the Hermitian symmetry is satisfied. Now for Hermitian symmetry to be satisfied we know that inner product of v with u should be same thing as inner product of u with v, but with a conjugate. Now by definition we have chosen that the inner product of the two signals is this by definition. So, let us see what happens if I take a conjugate. So, conjugate of integration is nothing, but integration of conjugate. So, this conjugate gets inside this integration, $u*(t)v(t)dt$ and this by definition is inner product of a signal v with signal u and hence Hermitian symmetry is satisfied. Again I will not do the proof of a Hermitian bi-linearity the proof is simple I will request you to do and carry out this proof yourself.

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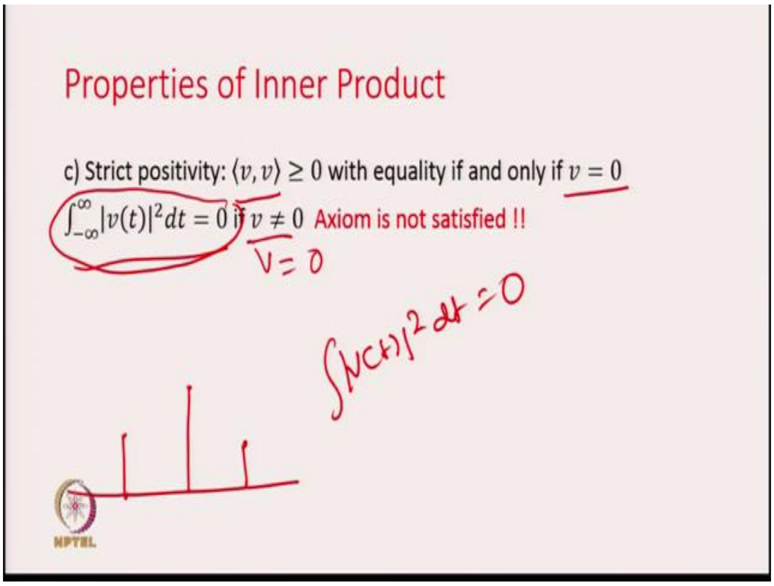

So, this is the third property which we have to see and this property is about a strict positivity. Strict positivity tells me that if I take the inner product of vector with itself this should always be greater or equal to zero. This is zero only if vector involved is a zero vector. For the case of the signals what it would mean is, if you take the inner product of the signal with the signal itself then they should also be a positive quantity. It can only be zero if the signal involved is an all zero signal; that means, this should be zero only if the signal is all zero signal, but you know that this can be zero in the case of L_2 waveforms, even if the signal is not all zero signal. For example, we have seen that if I have a signal like this. So, this signal is taking values only at discrete instances of time, then also for this signal, $\int |v(t)|^2 dt = 0$ because there is no width involved these points and hence when you pass this through an integrator you will get in all zero take. So, for L_2 waveforms this is not satisfied.

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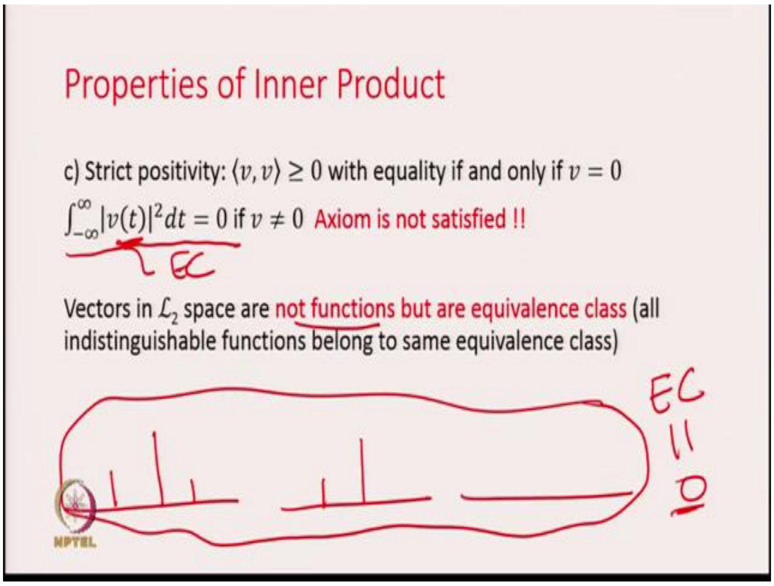

So, how do we get rid of this? We get rid of this by saying that the vectors in L_2 space need not be functions, but their equivalence classes. What I mean by equivalence classes all indistinguishable functions belong to the same equivalence class. For example, if I have a signal like this or if I have a signal like this or if I have an all zero signal, all these signals because they are indistinguishable functions they belong to the same equivalence class. And let us say they belong to the equivalence class of zero. So, if I say that these $v(t)'$s are not functions, but equivalence class then this thing can only be zero if the equivalence classes is zero and hence we get rid of this mathematical subtlety.

So, this is just to convince mathematicians and to satisfy axioms that we say that vectors in L_2 space not functions, but they are equivalence classes, all indistinguishable functions belong to the same equivalence class, but for engineers do we have to worry about this? Answer is no because out of this equivalence class only one function would be practically realizable does whether you call the L_2 vectors as functions or equivalence class, it does not really matter. Now we are moving to the last, but probably the most important idea that how can we do the orthogonal expansion of a signal.

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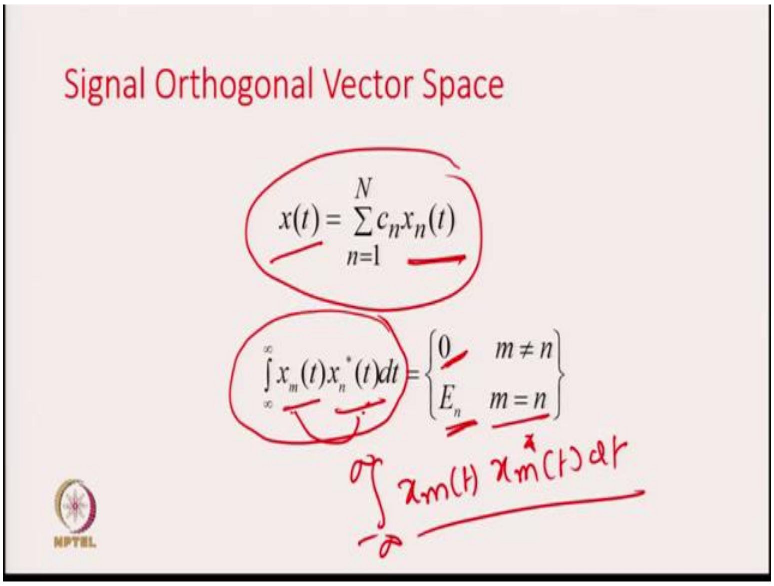

Let us start with vectors. So, let us assume that I have vector x and I can express this vector x in terms of orthogonal components. For example, I can decompose any vector in terms of i, j and k components. So, I can express a vector in terms of orthogonal components x_1, x_2 and x_3. There are three orthogonal components and using these three orthogonal components I can span a 3D space. So, in general what you are used to is assuming that these vectors are unit vectors i, j and k the idea is similar was making it little bit more generate. And we have also seen that the values of these coefficients c_1, c_2 and c_3 are nothing, but can be simply obtained by taking the inner product of vector with one orthogonal component divided by the length square of that vector and similarly I can obtain also c_2 and c_3.

So, what we are saying here is, in the case of vectors you can decompose a vector in terms of orthogonal vectors and using these orthogonal vectors you can span a given space. Remember orthogonal vectors are always linearly independent because they are orthogonal. So, they are linearly independent and if they are giving spanning a given space; that means, they forms a basis set. So, that is important. So, you can think about the basis set, we have already seen why basis sets are important. So, you can obtain a basis set by having some orthogonal vectors in such a way, they span a given space and we call this as basis vectors. Extending the same idea to signals we can say that I can have a signal and I can express the signal as the linear combination of orthogonal signals.

So, x and t denotes a set of orthogonal signals or functions. What are orthogonal functions? To obtain orthogonal functions I have to take a signal I have to take another signal from the same set. So, $x_m(t)$ is different from $x_n(t)$ in general, I have to take the inner product. So, this gives me the inner product operation between $x_m(t)$ and $x_n(t)$and this should be zero if m is not same as n; that means, if the functions involved are different functions then they have to be orthogonal. So, for orthogonality the inner product must be zero and hence if I to choose two signals, then this expression. So, that gives me zero.

However, if the signal is same as this if two signals involved are the same signals, then what this will give me is the energy of the signal. Because then this would be $x_m(t)$ into $x_m^*(t)dt$ and this is nothing, but this would give me energy of the signal. So, this is a test to construct an orthogonal set or orthogonal family. So, this is a big idea constructing a signal in terms of orthogonal functions.

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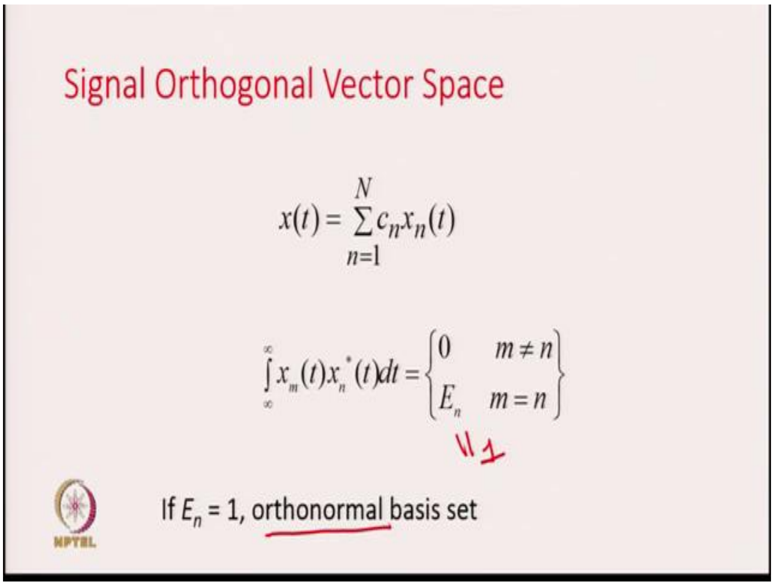

Moreover if this energy is always one; that means, all signals in the set have the energy as 1, this is similar to unit vectors, then we say that set as orthonormal set; that means, orthonormal set is the set where the energy of the signals involved is 1.

So, that is the basic difference between orthogonality and orthonormality. In orthonormality we have added restriction that the signals energy should also be one unity. What can we choose as basis vectors?

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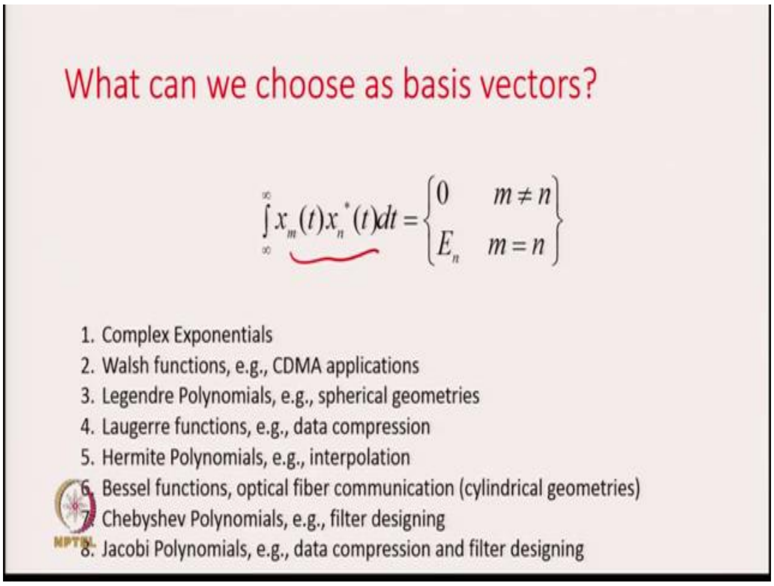

There are lot of examples that we can choose as basis vectors, we can use complex exponentials to construct these orthogonal functions that we do in case of Fourier series. We will learn about this in this course. We have Walsh functions which we use for CDMA applications, we have Legendre polynomials, Laugerre functions, Hermite polynomials, Bessel functions, Chebyshev polynomials, Jacobi polynomials and so, on and so, forth. So, there is wide variety of functions that you can choose to construct your basis factors few examples few important examples of such functions we will see in this course.

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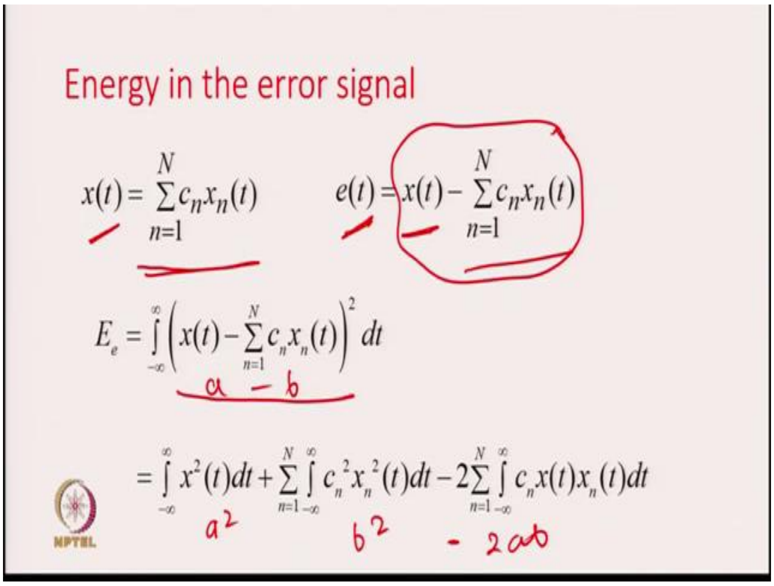

So, let us see what the energy in the error signal is. So, if I have a signal $x(t)$ and I am approximating it as a linear combination of n orthogonal functions, first of all we see what the error signal is. Error signal can be obtained by taking a difference between LHS and RHS. So, this is the error signal we have done and calculated this error signal several times, this is the error signal, energy in the error signal can be obtained simply by squaring this quantity up. So, at this moment we are assuming everything to be real. So, we do not have to worry about magnitude and so on. Now this is of the form $(a-b)^2$. So, we can think about this as a^2+b^2-2ab.

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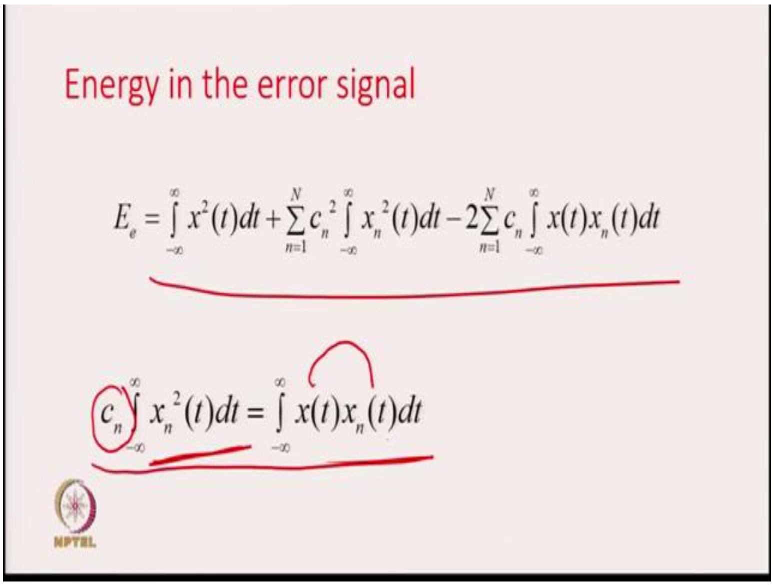

Now, if you look at this expression. So, this is what we have derived in the last slide and if you see this. So, of we already know that the value of the a nth coefficient can simply be obtain by taking the inner product of the signal with nth orthogonal function divided by the energy of the nth orthogonal function.

So, now we know that this is the energy of $x_n(t)$.

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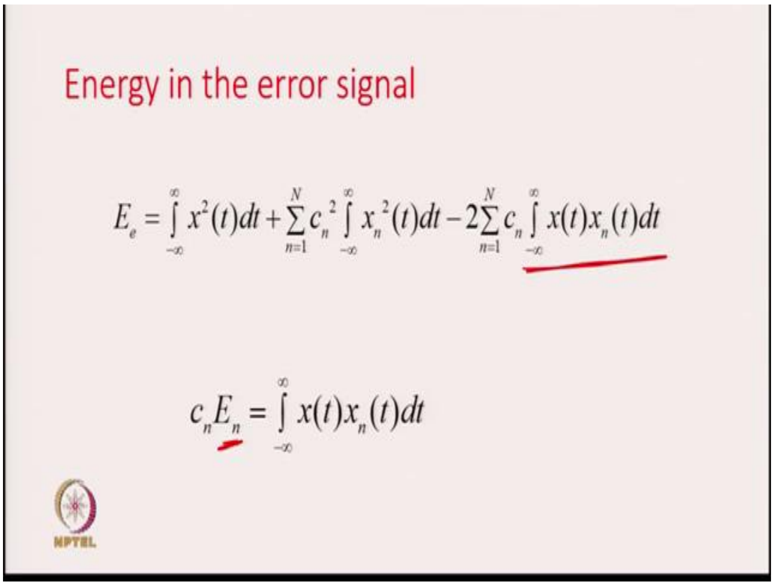

So, I can replace this quantity with $c_n E_n$.

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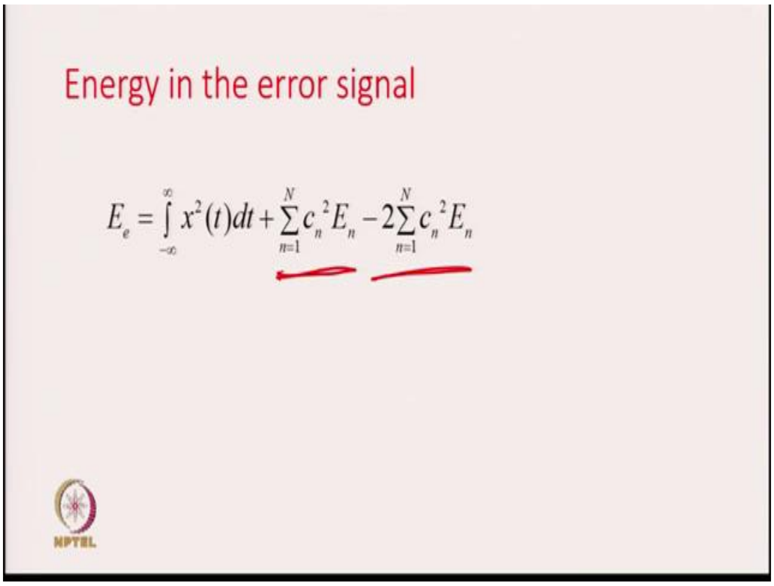

Doing that we get $c_n^2 E_n$. Similarly I can see that this is also E_n. So, I can replace this with E_n and I get this, now I see that this and this can be subtracted to get this.

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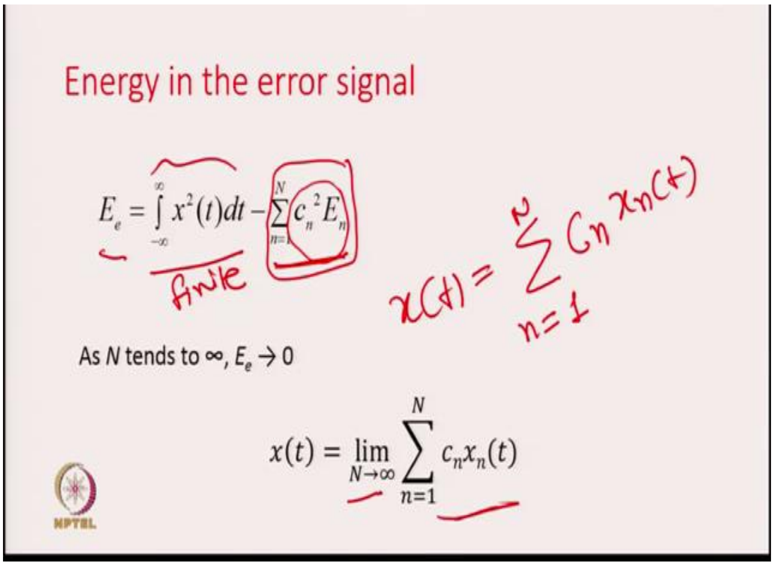

Now, if we look at this expression if this is a finite quantity, this will be a finite quantity if the signal is an L_2 signal or L_2 waveform because L_2 waveform has finite energy and this is the energy. So, this quantity is finite. Now this quantity increases as n increases because this is a positive quantity. So, this quantity increases as n increases and finally, you hope that as the number of terms increases the energy in the error decreases. So, if I make n tends to ∞, energy in the error will go to zero. And hence, I can write $x(t)$ as $\sum C_n \, x_n(t)$, at least if n is pretty large then I hope that the energy in the error will go to zero and this is all about the convergence. So, I can write $x(t)$ in this form at least if n is pretty large, say ∞.

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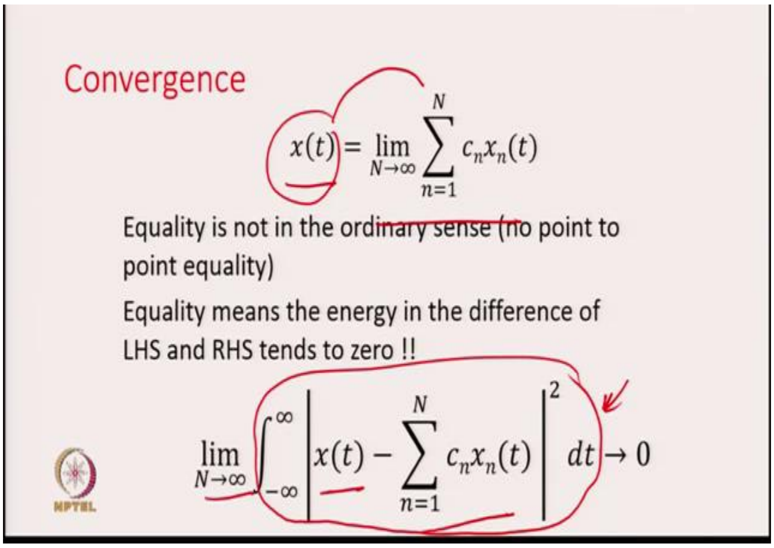

So, let us see the convergence when you are writing this expression you have to understand that the LHS is not same as RHS in point wise sense; that means, for all values of time the LHS is not same as RHS. So, it is the equality is not in the ordinary sense this equality is not point to point equality. So, LHS might be different from RHS at some time instances this equality is in the sense of that the energy in the difference of LHS and RHS tends to zero; that means, if you subtract LHS from RHS, find the energy in that different signal that energy in the different signal will go to zero at least if the number of terms involved in the summation is pretty large.

So, this is what we are writing. So, this is what we are saying if I take a difference between LHS and RHS, the energy in the different signal, this energy will go to zero if n is pretty large and this convergence is set to as limit in the mean convergence. So, this quantity converges to this quantity in the limit in the mean sense in this sense.

So, this is the completion of lecture 2. In this lecture we have learned about inner product spaces we have learned that by using this idea of inner product if we can define the length of signals we can define the angle between two signals, and this is a very important idea which will simplify things for us because we know that dealing with vectors is easy, dealing with signals will also be subsequently easy. We have also seen and looked at the definition for inner product of L_2 signals we have already convinced you that L_2 space is in inner product space.

Finally we have seen orthogonal expansion of a signal; that means, you can take a signal and you can think about the signal as a linear combination of orthogonal functions. From next lectures we will be looking at specific examples of orthogonal expansion of a signal we look at what is known as Gram Schmidt orthogonalization procedure and we will see there that the signal can be constructed in terms of finite number of orthogonal functions and also there is no error in that approximation. So, we will start with Gram Schmidt orthogonalization procedure in the next class.

Thank you.