Principles of Digital Communication
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Lecture – 02
Signal Spaces
Waveforms & Vector Spaces

So, welcome to the first unit of this course. In this unit, we will be studying about waveforms and signal spaces and today is the first lecture of this unit. So, let us start this is the outline of the lecture.

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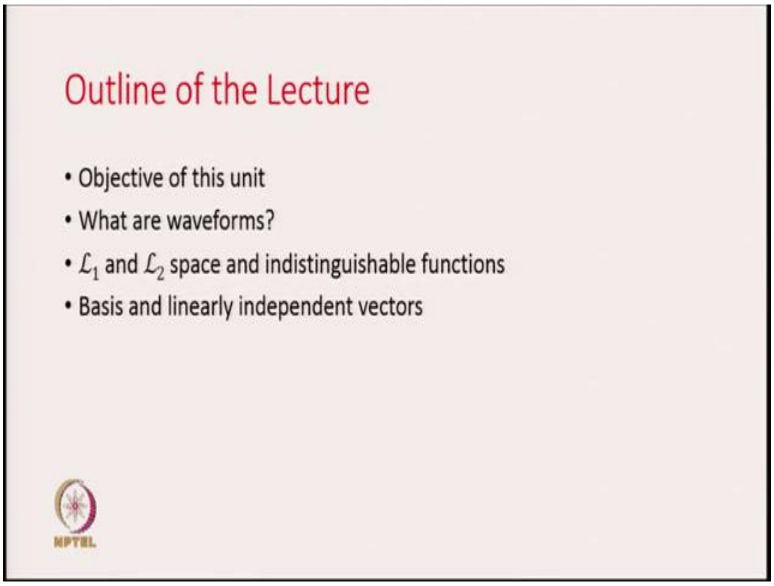

First we will see why it is important to be able to treat waveforms as vectors. We will also answer the question what are waveforms, then we will study what are known as L_1 and L_2 spaces and indistinguishable functions. And finally, we will talk about basis and linearly independent vectors.

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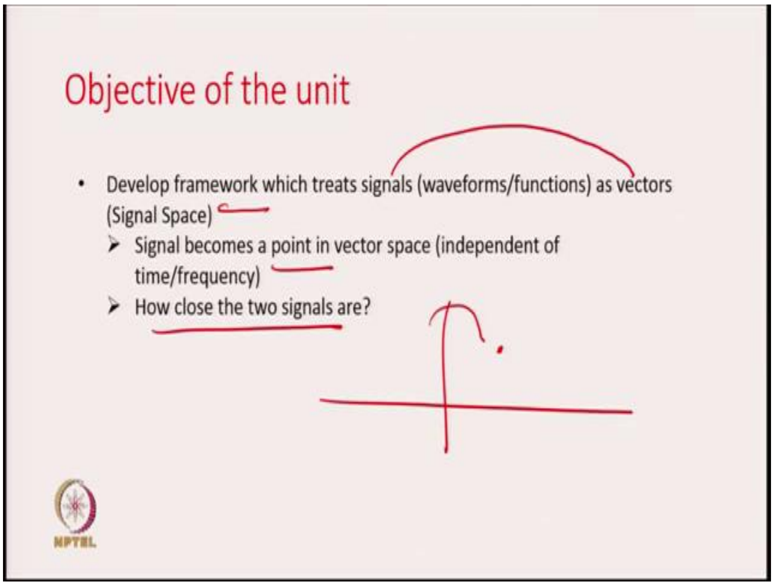

So, let us see the objective of this unit. In this unit, we would be developing framework which would allow us to treat signals as vectors and once you can convert signals into vectors, the signal becomes a point in a vector space. So, you will have a vector space and signal will be just a point in a vector space. Normally you would have seen in the courses like in signals and system that signal is either a function of time or signal is either a function of frequency. But here what we would be doing is we would be able to convert signal into a point in a vector space.

And this will be very useful and helpful and with this, we would be able to define how close the two signals are, what the angles between the two signals are and so on. So, this is a really important unit for this course on “Principles of Digital Communications”. So, let us see it more clearly.

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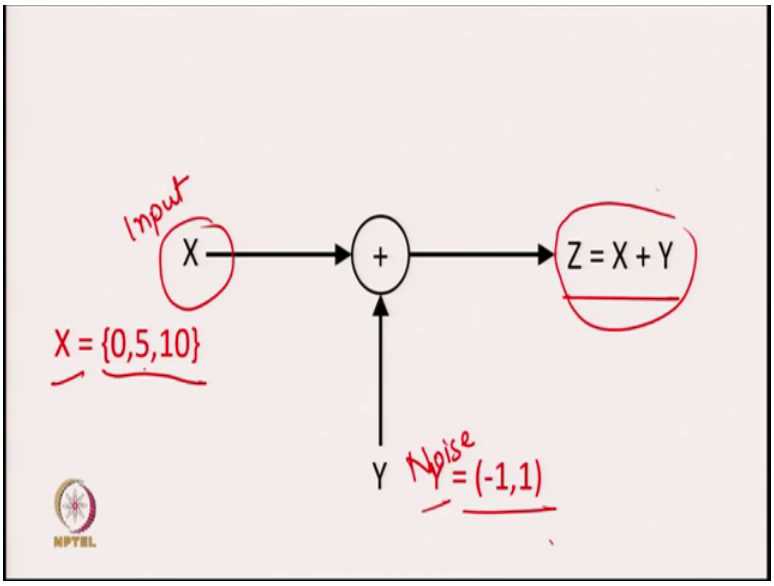

To understand this let us start with a simple receiver. We have already talked about the additive white Gaussian receiver. So, it is a receiver like this, here we have this input. So, this is the input and to this input we have some noise addition. So, noise is additive to the input and this is the output of the receiver X + Y.

Now, let us assume for simplicity that X takes in a value from this set so, either it can transmit 0, 5 or 10. So, this is a finite set. So, we know that we are in the regime of digital communication and let us assume that Y takes in a value between - 1 and + 1. So, noise is limited or the noise amplitudes rather are limited between - 1 and + 1 and what arrives in the receiver is a combination of a transmitted signal and a noise.

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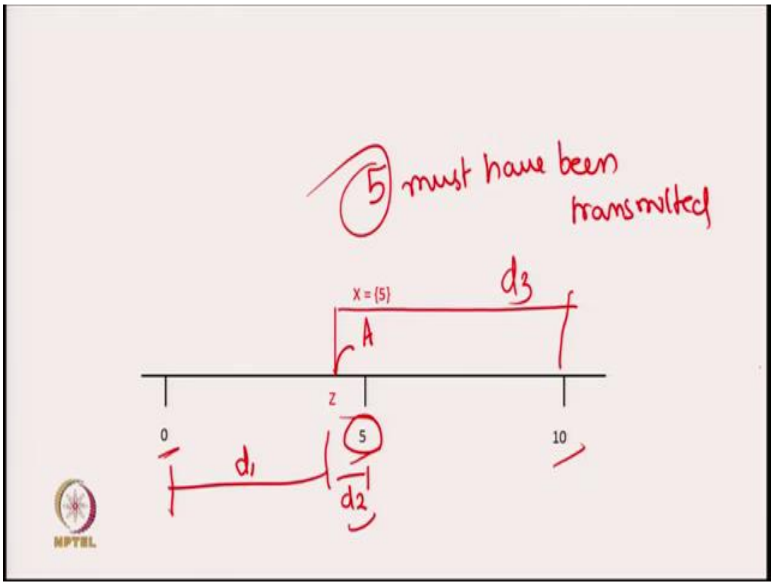

Now, what receiver does is it first maps these possible transmitted values on a real line. So, it maps this transmitted values 0, 5 and 10 and let us assume that the received value pumps up this real line on a certain point let us say a point A. Now what does a receiver do is it calculates the distance of the received value from the possible transmitted signals. So, it calculates distances d_1, d_2, d_3 from all possible transmitted values and then what it does is it finds out the transmitted symbol or signal with minimum distance.

So, in this case for example, d_2 is minimum out of this d_1, d_2, d_3 and thus it selects 5 as the possible transmitted signal. So, receiver decides that 5 must have been transmitted and so things look very simple as now there are no real complications involved in this. Of course, we have simplified this problem a lot, we have converted a complicated problem into a toy problem, but if we can do or if we can convert the signals into numbers this problem of reception and deciding what has been transmitted looks rather a simple problem.

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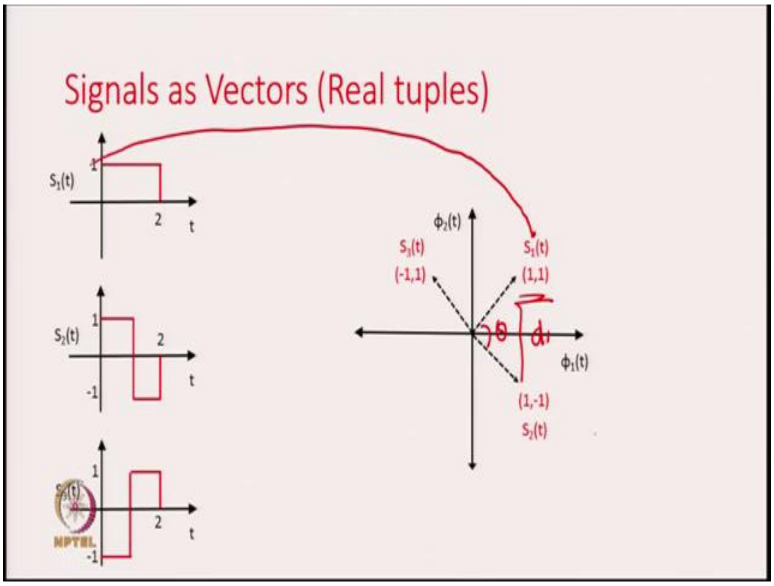

So, in reality a transmitter would transmit signals. So, it would transmit $s_1(t)$, $s_2(t)$and $s_3(t)$. Now if we are able to convert these signals into vectors for example, we have been able with some framework we have not talked how, but let us say there is some framework which allows us to convert the signal into a vector where this signal can be understood by some real numbers, then we can do what we did just previously. So, we can talk about the distance between the two signals, we can talk about the angle between the two signals and so on.

So, whatever you have studied about vectors applies here now so with this framework. So, this framework allows us to talk about the distances between the two signals, angle between the two signals and we will be able to convert signals into numbers and things will become really simple. And so, this is what we are trying to learn in this unit how to do this. So, let us start and first thing that we have to learn is what waveforms are.

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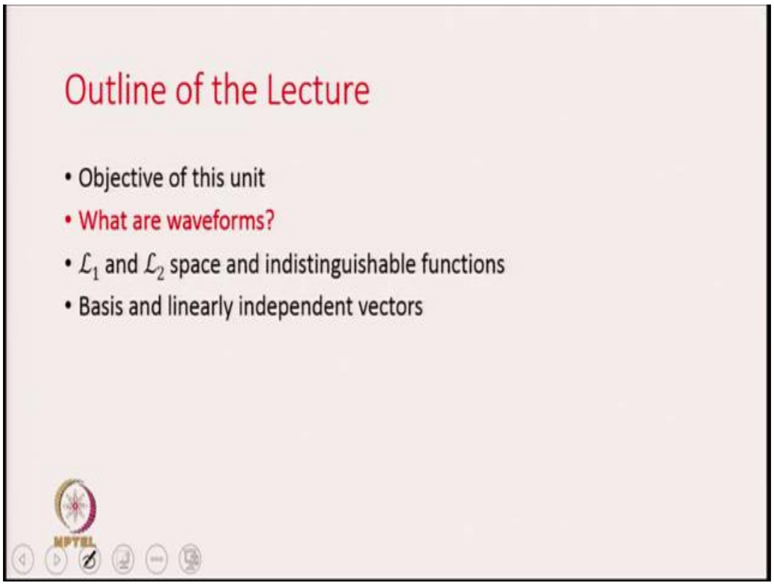

So, we have already seen a hint what are waveforms in the first lecture and we would be talking more about this today.

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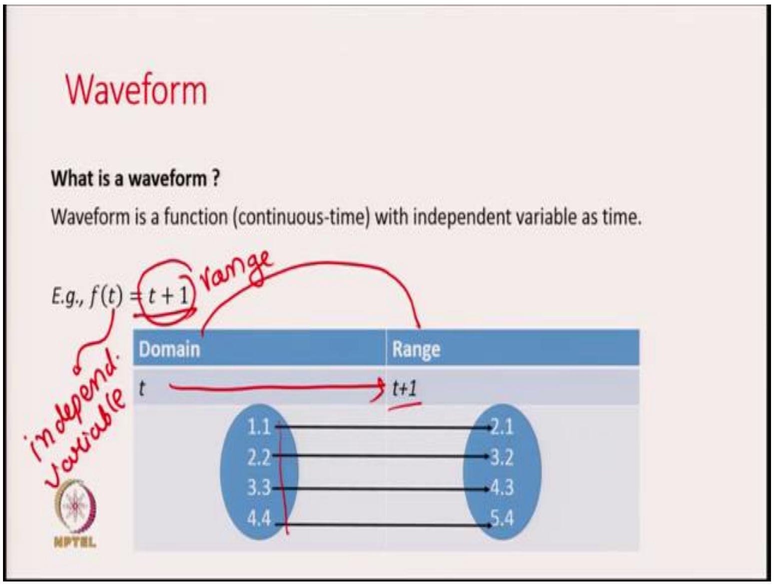

So, waveform as we have already stated is a function and it is function with independent variable as time. It is a continuous time function, it is something that can travel over channel it is something that could be produced by electrical circuits. So, it is a continuous time function with independent variable as time. What is a function? A function provides mapping between domain and range. So, if I take a function as t + 1, then this function provides a mapping between domain range and t is the independent variable. So, t is an independent variable and the functions of interest in this course the independent variable is always time and maybe sometimes frequency, but mostly it is time.

And this is giving us the range of the function and it is providing a mapping between domain and range. So, for example, if domain takes these values then, the range can be simply obtained by just adding one to this. So, this is a very simple example of function; function provides a mapping between domain and range and waveform is a function with independent variable as time. It is a continuous time function.

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Now, the functions can be of two kinds. It can be a real valued function or a complex valued function. Now because we have said that the functions in which we are interested in the independent variable is time. So, it is a function like $f(t)$ independent variable is time. Now time is always a real quantity, time is never a complex quantity it is always a real quantity.

So, these functions would be providing mapping between real space (because time is always real) to a real space; (that means, the value of this function can be a real value) or it can provide mapping between real space to complex space; that means, $f(t)$ can take a complex value. So, we can have two kinds of functions: the functions which provide mapping between real space to real space, or the functions which provide mapping between real space to complex space. The functions which provide mapping between real space to real space are known as real valued functions while the functions which provide mapping between real space to complex space are known as complex valued functions.

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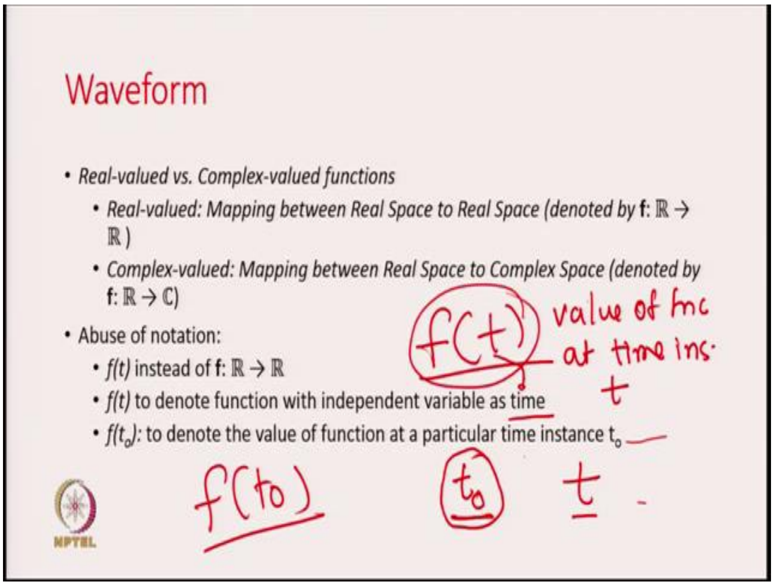

There is abuse of notation that is normally being done in engineering textbooks and we will continue to do that. And this is that when you write $f(t)$, normally this will mean the value of function at time instants t. But when we are writing f of t, it simply means that this is a function where the independent variable as time and it means that time can take value in the real space. So, time can take any value between - ∞ to + ∞. Now to talk about the specific value of this function at let us say at a specific time instance let us say, we are interested in a specific time instants t_0. So, this is how we would be differentiating between a specific instance of time and the time as an independent variable. So, when we are saying t it simply means that time is independent variable, when we are saying t_0 it means that it is a specific value of time.

So, when I am talking about $f(t_0)$ it means that this is the value of a function at a particular time instants t_0, when I am just writing $f(t)$ it simply means that this is a function with independent variable as time. So, this distinction has to be made. This is not precisely how things should have been done, but this is what has been done in most books and we will continue to use this notation.

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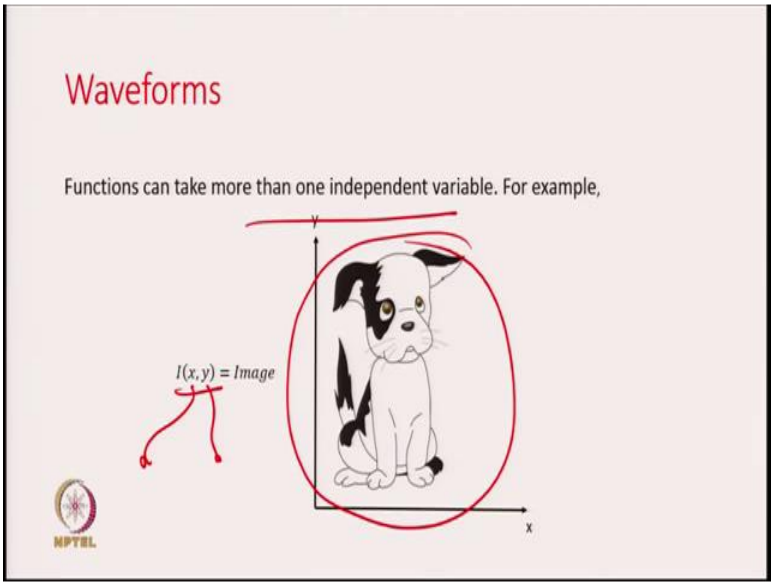

Now, normally the functions can take more than one independent variable. So, we have been looking for functions with one independent variable, but the functions in general can have more than one independent variable.

For example, let us look at this picture of dog. This image is also a function and this function could be constructed by knowing the intensity of this image along x and y coordinates. So, this is a two dimension function; it has two independent variables x and y, but lately in this course we will only be talking about one dimensional functions or one dimensional signals. So, things are pretty easy in this course.

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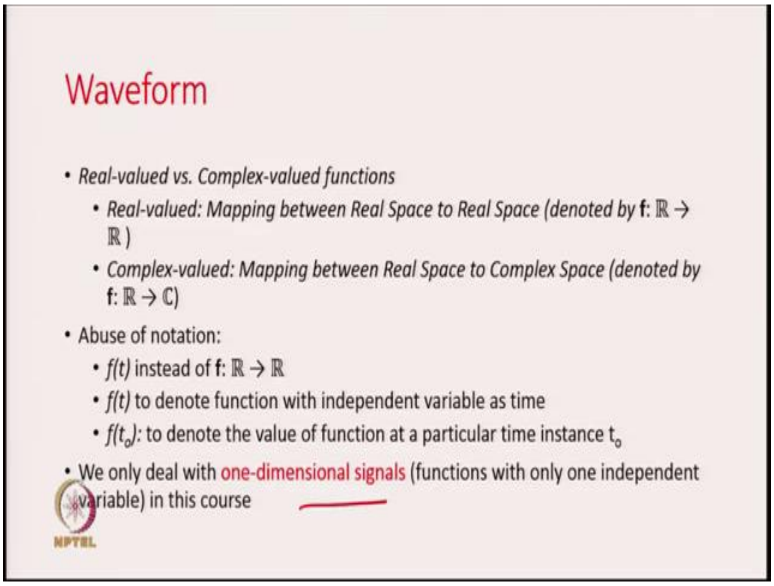

So, this is the last point that we only deal with one dimensional signals of functions in this course. I am using this word function and signal interchangeably because they are so. So, what we have done? We have already understood the objective of this unit which is trying to treat signals as vectors and we have already defined what a waveforms; waveforms of functions with independent variable as time. We only talk about one dimensional signals or functions in this unit. Now, let us see what are L_1 and L_2 spaces and L_1 and L_2 functions or signals and what these indistinguishable functions are.

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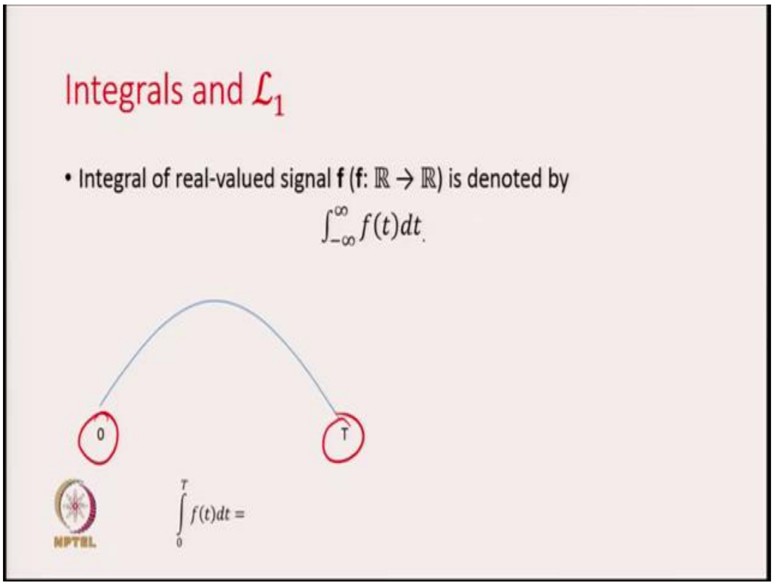

So, let us first define what an integral of a real valued signal is. So, real valued signal provides a mapping between real space into real space and this is how the integration of a function is defined. Everyone must have learned the integration in high school or secondary schools and so on so forth. So, how do we do integration of a function?

Now, the one way to think about doing the integration of a function and the way we have been taught mostly is we take a function, like this is a function which spans from 0 to t seconds. And when you calculate the integration, basically you are interested in finding the area under the curve and the simple way to do this is to divide this function into chunks.

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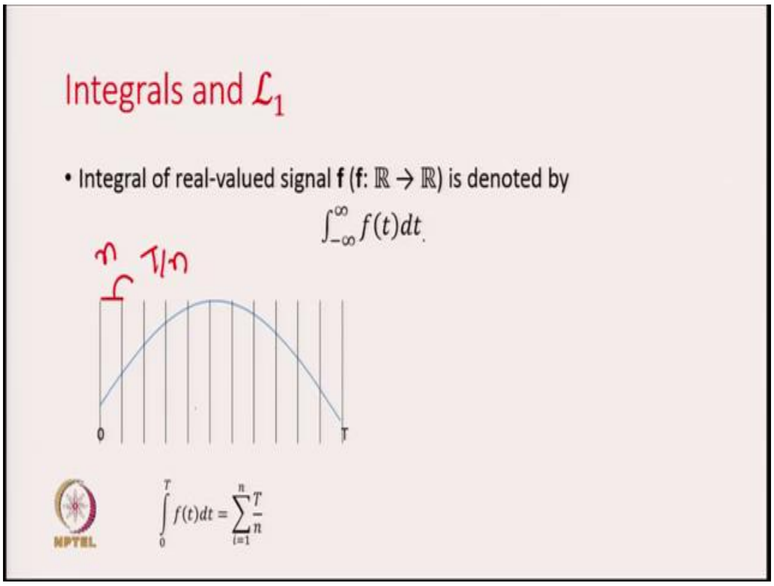

Let us say we have divided it into n chunks. So, the width of each chunk would be T/n. Now to calculate the area under this curve, what you can do is you can break this function into chunks with width of T/n and you find out the value of this function in each chunk.

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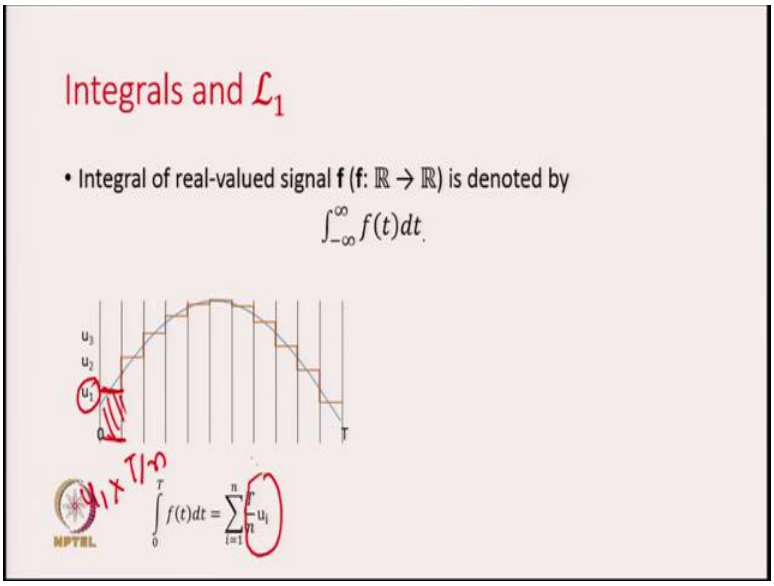

For example, in this chunk I can approximate the value of the function as u_1 of course, if this width to be pretty small. So, I can approximate the value of this function in this chunk as u_1 and then the area in this chunk would be $u_1 \times T/n$ because the width is T/n. And similarly, I can find the area under this function by summing up the areas corresponding to each chunk and this is what we are doing. So, we have to find the area of one chunk and then we have to sum up the areas of all chunks to evaluate the integration of this function. This approach of finding integration is known as Riemann integration. You can also interpret integration in other ways.

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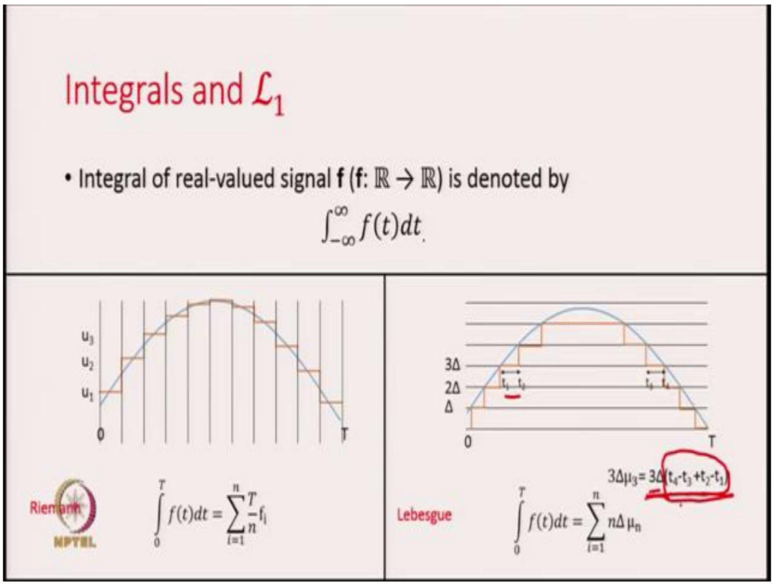

For example, you can think about integration using Lebesgue approach. And what is that approach? Instead of discretizing the time axis so, this here we discretized the function along time axis. What you can also do is you can discretize the function along y axis; instead of x axis you can discretize it along y axis. So, now, what you can do is let us just focus on one part, let us say this function takes in the value three delta. And let us now in order to find the area of this part how can you find the area of this part would be 3Δ times the time interval for which this function takes in this value of 3Δ. For example, in this case these time instance is $t_4 - t_3$ and then there is a second part the width would correspond to $t_2 - t_1$. So, this is the total time for which this function takes in a value of 3Δ. So, the area corresponding to this could be obtained by multiplying 3Δ. So, this is also known as the measure of the function. So, the idea is similar instead of discretized it along time axis, you have discretized it along the amplitude axis and then you are finding the area for each chunk and then you summing this up. So, this is known as the Lebesgue approach.

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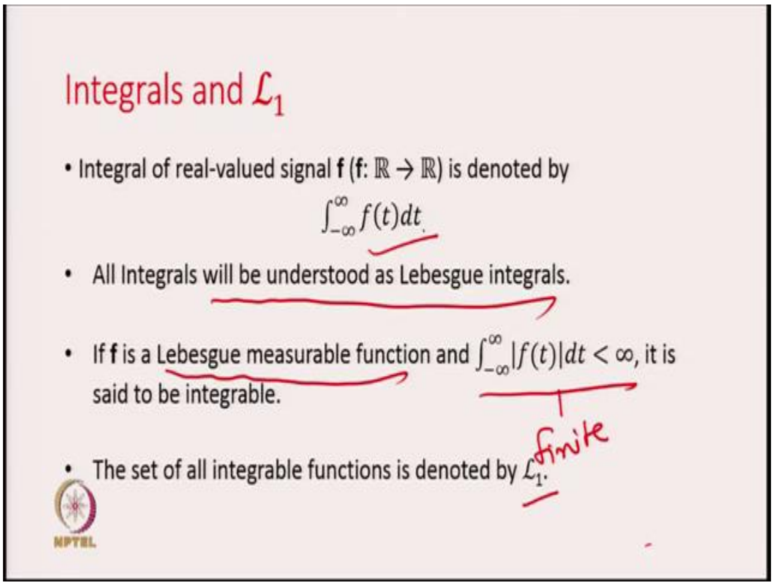

Now, whether you take the Riemann approach or Lebesgue in approach for most functions you would have the same output, but for certain functions the Lebesgue integration would give a different result than Riemann integration.

So, all integrals in this course are understood as Lebesgue integrals and this is because when we are stating theorems we want to state them precisely and some theorems are only applicable if you consider the integration as Lebesgue integration. So, we will like to think about the integrals as Lebesgue integrals; just for preciseness. Now we can also define whether a function as integrable function, if it follows the two condition. The first condition is function must be a Lebesgue measurable function and the second condition is that the integration of mod of that function should be finite.

If these two conditions are satisfied, then we say that the function is integrable function. So, two conditions first Lebesgue measure, it should be a Lebesgue measurable function and integration of mod of that function should be a finite thing. If the function is integrable function, we say that the function belong to L_1 space. So, the set of all integral functions is denoted by L_1 space.

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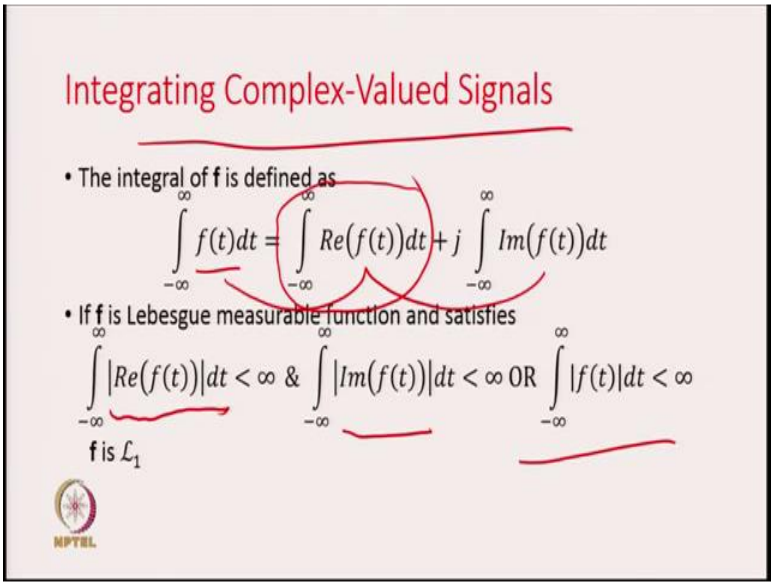

Now, if you want to define this thing for a complex valued signals; so far we have been just talking about real valued signals. So, what happens if we want to talk about these things for complex valued signals, the same thing happens just because its complex you can break this function into two parts; real part and imaginary part and now you apply the condition for integral ability for each part individually. So, you have to say that these things have to be Lebesgue measurable functions and the integration of $|Re\{f(t)\}|$ should be finite and integration of $|Im\{f(t)\}|$ should be finite or integration of $|f(t)|$ should be finite. We have not yet talked about what are Lebesgue measurable function, we have just said that the function has to be a Lebesgue measurable function.

Now, trying to understand what are Lebesgue measurable functions is really complicated and we have to restore to measure theory to understand what are Lebesgue measurable functions. But luckily most of the signals and function that we will be dealing in this course will be Lebesgue measurable functions. In fact, it is very difficult to find an example of a function which is not Lebesgue measurable function; all functions are virtually Lebesgue measurable function.

So, we will take it for granted that the functions that we will be dealing with will be Lebesgue measurable function; so that is not so strict condition. More important condition would be trying to find out whether the integration of mod of a function is finite or is not finite.

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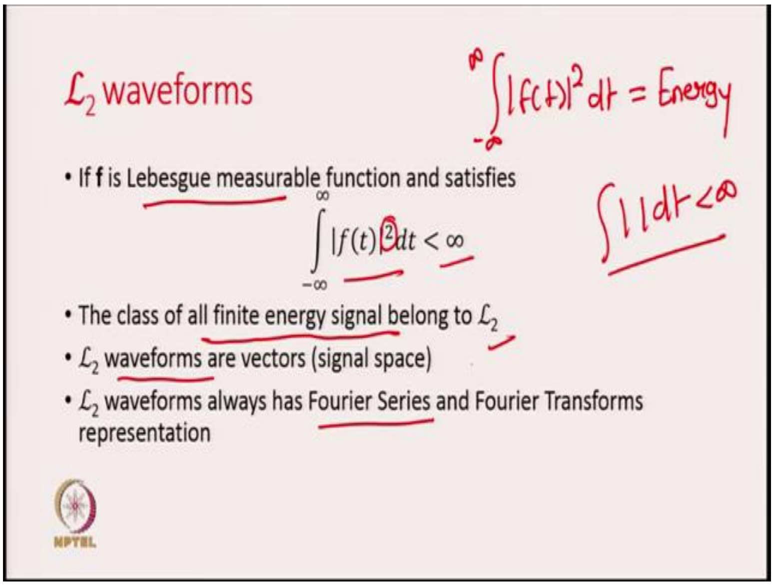

So, now let us study about L_2 waveforms. So, we have already talked about L_1 waveforms and there was that function has to be Lebesgue measurable function and integration of mod of that function should be finite. So, what is that thing in L_2 waveforms? So, for L_2 waveforms the function has to be Lebesgue measurable function and integration of mod square of that function should be finite.

So, when talking about L_1, we were interested that this quantity should be finite integration of mod of function should be finite; here we have a square. So, integration of mod square of function should be finite. So, if you have done a course in signals and system, you know that this quantity is telling me the energy of the function. So, this quantity corresponds to the energy of the function. So, one way in which we can define L_2 waveforms is that class of all finite energy signals because, if this quantity is finite; that means, we are saying that energy of the signal should be finite.

So, the class of all finite energy signals belong to L_2 space. Now can we have signals which have infinite energy? Answer is no; if you could have signals with infinite energy then the energy crisis of the word could have been solved. So, we do not have any signal with infinite energy. So, practically all signals that we deal with have finite energy only, but mathematically we would be interested in certain functions which have infinite energy and does we have to be really careful when we are dealing with those signals.

Now, one important advantage of these L_2 waveforms is that L_2 waveforms are vectors. You can treat these L_2 waveforms as vectors, you can use the ideas of signal spaces you can do simple things. Secondly, L_2 waveforms always have Fourier transform and Fourier series. So, you must have studied these tools in signals and system codes that you can go from time domain to frequency domain. You can go back and forth using these tools and all this theory is applicable if the underlying signal is an L_2 signal or is an L_2 waveform. So, L_2 waveforms comes with lot of advantages, means you can use Fourier series over a transform you, can use the ideas of signal spaces and all practical signals anyway are L_2 waveforms. So, most of this makes sense.

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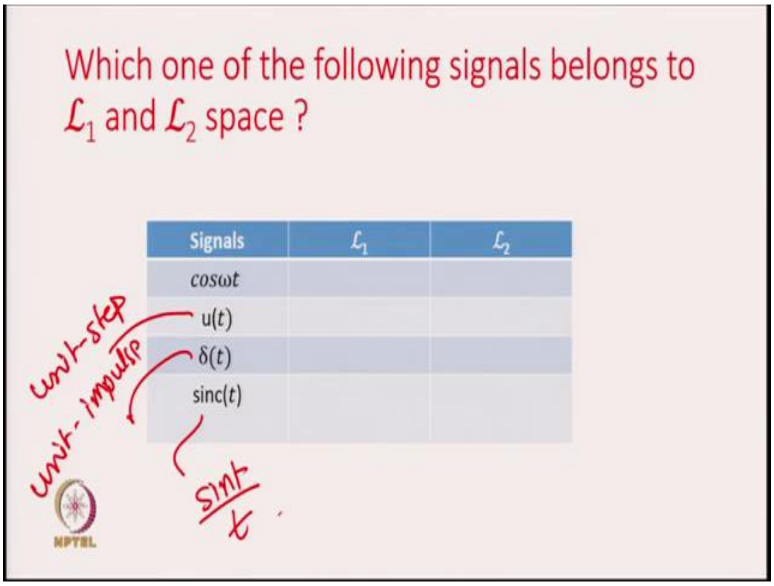

Let me test you with one question we have four signals here. So, we have $\cos\omega t$, we have $u(t)$ which means unit step, we have $\delta(t)$ which means unit impulse, I have $\text{sinc}(t)$ which means $\frac{\sin t}{t}$ and you have to find out whether these signals are L_1 signals or L_2 signals. Now let us look them one by one.

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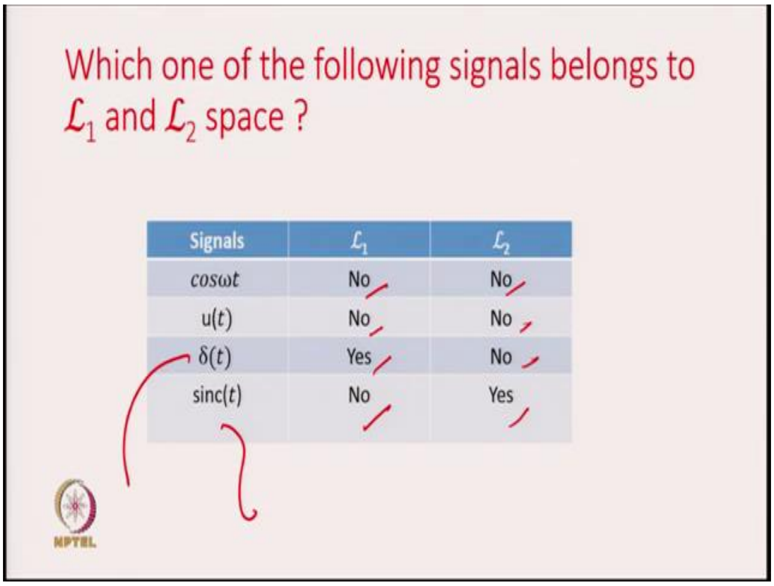

So, let us start with $\cos\omega t$. So, $\cos\omega t$ you can see is neither L_1 nor L_2; unit step is neither L_1 nor L_2, unit impulse is L_1, but it is not L_2, $\text{sinc}(t)$ is not L_1, but it is L_2. So, the first thing that you should appreciate is the function can be an L_1 function, but it need not be L_2.

For example, $\delta(t)$ is an L_1 function, but it is not an L_2 function the function can be L_2 function, but it need not be L_1. For example, $\text{sinc}(t)$ is L_2, but is not L_1. So, there are functions which are L_1, but not L_2 there are functions which are L_2, but not on L_1. There are functions which are both L_1 and L_2; there are functions which are neither L_1 nor L_2. So, from one you cannot make easy conclusions about the other. So, let us see how can we think about whether the functions are L_1 or L_2. To think about this let us start with $\cos\omega t$.

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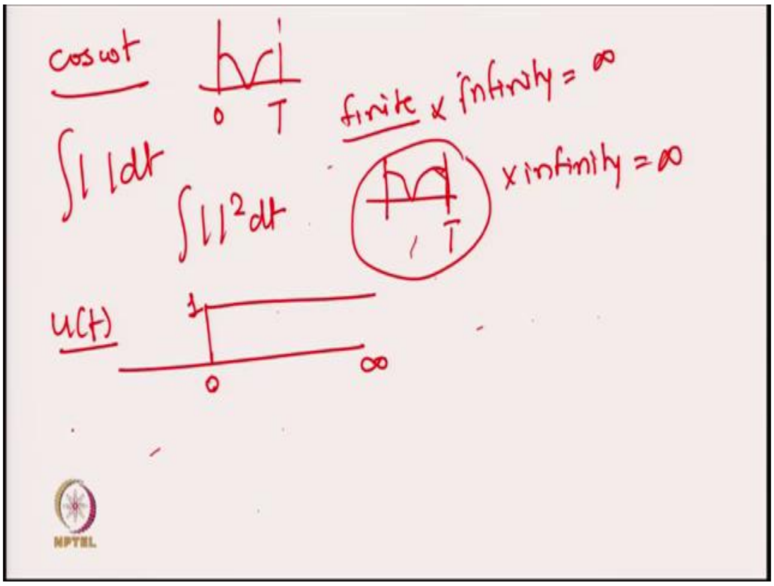

So, in this we would be making very intuitive proof, we would not be going through rigorous mathematical proof. So, how can we think about whether this $\cos \omega t$ is L_1? Let us first start with L_1. So, when we are talking about L_1, we are interested in integration of mod of that function. So, for $\cos \omega t$ if I think about one period, this integration of mod of that function would be a finite quantity whatever is that it is not interesting and this $\cos \omega t$ runs from - ∞ to + ∞. That means, you are multiplying a finite quantity with ∞ and so this would be giving us ∞.

So, this is not an L_1 function. Similarly for L_2 you need to check whether this is finite or infinite and again the same logic holes when you do this in one period, you will end up with a finite quantity. Again because $\cos \omega t$ runs from - ∞ to + ∞ you have to multiply a finite quantity with ∞ and you will get ∞.

So, $\cos \omega t$ is neither L_1 nor L_2, what about unit step? Unit step also you can think unit step goes from 0 to ∞ takes in a value 1 and because it exists for infinite duration and it exists for infinite duration with a value of 1, this is also not going to be L_1 neither L_2. What about $\delta(t)$?

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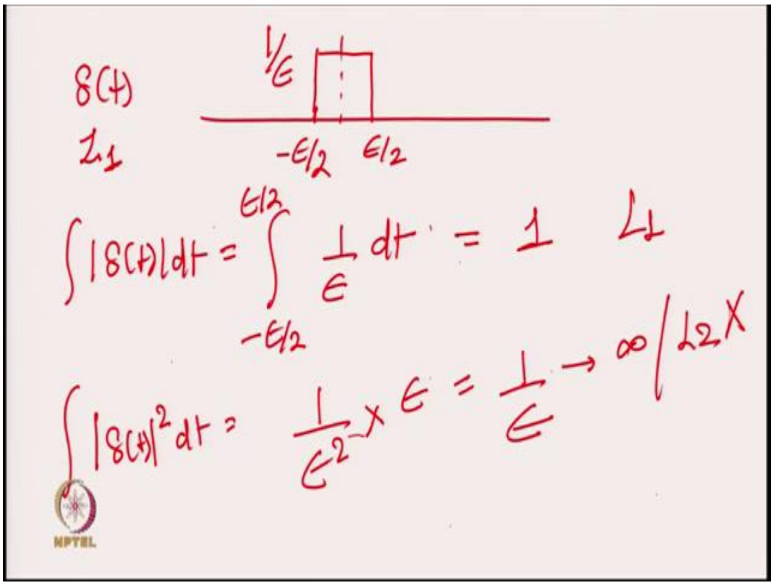

So, if we are talking about $\delta(t)$ this is interesting, let us check whether it is L_1. Let us first define $\delta(t)$. Let me assume that $\delta(t)$ takes in a value $1/\epsilon$ between $\epsilon/2$ and - $\epsilon/2$ where ϵ is very small number. So, if I am interested in integration of $|\delta(t)dt|$; it runs from – $\epsilon/2$ to + $\epsilon/2$ and you get 1. If I do this for $|\delta(t)|^2 dt$, I get $1/\epsilon^2 \times \epsilon$ and this turns out to be $1/\epsilon$ because ϵ is very small quantity, this will approach ∞. So, clearly this is L_1, but this is not L_2. Let us now check for sinc(t).

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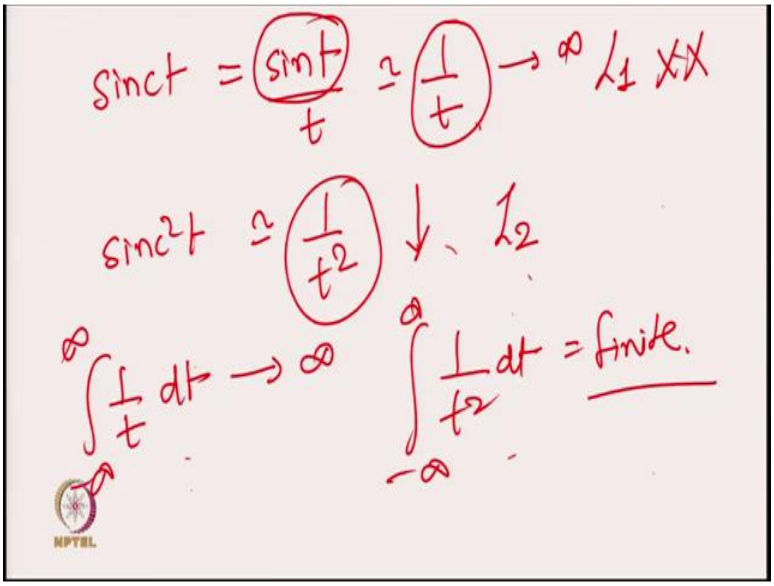

Now, proof for sinc(t) would be most complicated out of three of them, but we can use one idea that sinc(t) because it is represented as $\frac{\sin t}{t}$ and sin(t) is an oscillatory function it goes from - 1 to + 1. So, this sinc(t) for large t's can be approximated as $1/t$.

So, sinc(t) decays as $1/t$ whereas, sinc$^2(t)$ would decay as $1/t^2$. Now because this sinc$^2(t)$ decays faster, you can prove that this is an L_2 function and because sinc(t) decays slower with $1/t$; this is not going to be finite. So, this would be infinite and this is this will not be an L_1 function. This will be an L_2 function because this quantity is going to be finite. So, you can simply check whether $(1/t)dt$, you can check this will be close to infinite if it runs from - ∞ to + ∞ whereas, $(1/t^2)dt$ is going to be finite. So, this is not a rigorous proof for this gives you some idea to interpret about this sinc(t)s and this is an important idea which we will use later on as well.

So, remember the sinc(t) decays with $1/t$ it is not a fast enough decay. So, if you integrate that function from - ∞ to + ∞, you would have infinite values whereas, when you are considering $1/t^2$ this decays faster. So, even if it spills from - ∞ to + ∞ because it decays faster, the area of $1/t^2$ would be a finite quantity and this is intuitive way to understand why sinc(t) is not L_1, but it is L_2. Rigorous proof of this could be obtained from Fourier series and transforms and I leave this to you to think about how you can prove the same things rigorously mathematically using the ideals of Fourier series in transforms.

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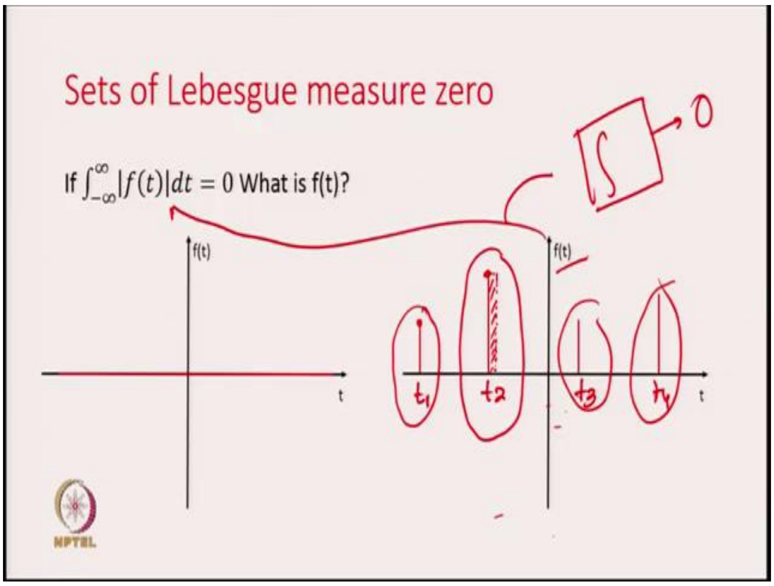

So, let us move on and let us try to find out another important idea and that is thinking about sets of functions with Lebesgue measure of zero. Now let us start this by asking a question. So, if you are given that integration of mod of a function is zero, what are the possible values of this $f(t)$? So, what can be that function? Now a simple function that would satisfy this equation would be an all zero function. So, a function which is all zero would definitely satisfy this. If you take the function, if you take that mod of a function this mod of this function will also be an all zero function and if you pass this through an integrator you are going to get a flat zero.

But are there other functions which satisfy this equation still them being not all zero functions? Yes, there are functions for example, functions like this. So, what is this function? This is a nasty function which complicates things in digital communication and this function takes in some values only at discrete instants of time. So, let us say that it takes in some values for time instances t_1, t_2, t_3 and t_4. Now what happens if you take the mod of a function? The mod of a function would give you the same thing, if I assume this $f(t)$ is all positive and real so taking mod does not make any changes.

And now if I pass this function through an integrator, what happens? Now if you lo, let us just focus on one point. So, if I pass this through an integrator because there is no width integrator calculus the area. So, because there is no width the function exists only at a single point there is no width. So, once you pass this thing through an integrator you get zero. Similarly the contribution to the output of these points will also be zero. So, passing this function through an integrator would give you an all zero signal. So, this is also a valid solution to this equation. So, there are two valid solutions.

So, one is an all zero signal that is good and the second are the set of functions which takes values only at discrete instances of time. So, because at the points there is no width when they are passed through the integrator, you are going to get a zero.

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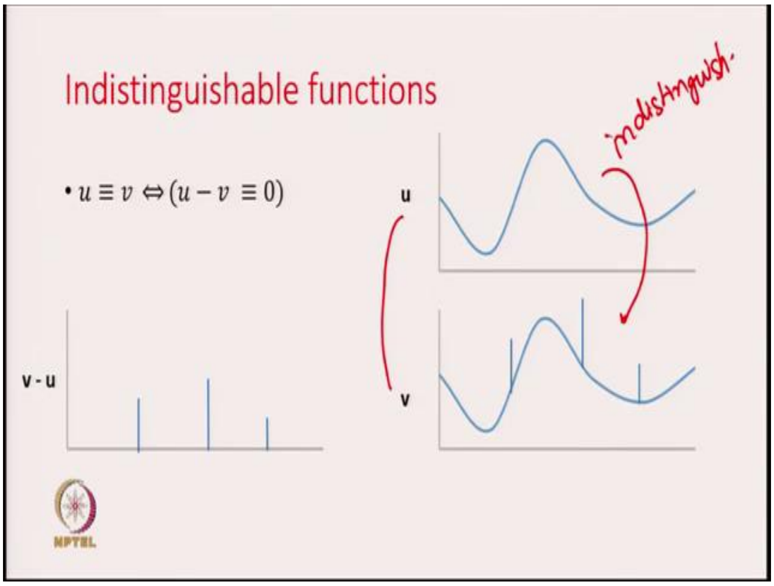

Now, with this idea we can define what is known as what are known as indistinguishable functions. So, the functions two functions are referred to as indistinguishable functions. If you subtract the two functions and you get a function which is like a non-zero function. It need not be an all zero function, it may be like a non-zero function for this. For example, this function is also like an all zero function because once it passes through an integrator, you get an all zero thing.

So, when you are having the systems, the systems are normally built using integrators, multipliers and adders. So, once you pass this function through an integrator anyway are going to get zero. So, whether you feed this to a system or to this to a system, you get the same answer zero. So, this is also like and all zero function. So, indistinguishable for indistinguishable functions, we have to take two functions we have to subtract them and we have to see whether we are getting a function which is like an all zero function because if this is the case, then you can say that two functions are indistinguishable functions. For example, I have taken these two functions. I have taken u and I have taken a function v and let us subtract it what you are going to get is a function like this on the sub subtraction of u with v.

Now, this is like in all zero signal because passing this through an integrator is going to give you a zero. So, these two functions are indistinguishable functions you cannot distinguish between the two of them.

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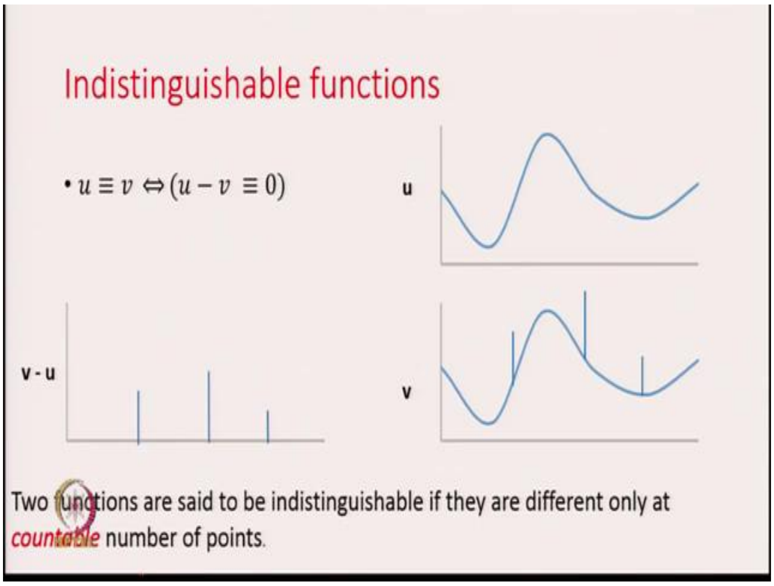

More precisely, we can define indistinguishable functions as the functions which are different only at countable number of points. We have already seen in the first lecture that the countable set is the set which has one to one correspondence with the set of integers. So, we will use this idea to further define what are known as, what is known as equivalence class and so on and so forth. So, we have covered the three parts of this lecture seen the objective waveforms, we have seen what are L_1 and L_2 spaces; we have seen what are these indistinguishable functions.

And now it is our time to look at vectors because this is what we are aiming for. We are trying to establish an equivalence between L_2 signals and vectors. So, let us start by looking into this basis and linearly independent vectors. So, the first thing that we have to see is what these vector spaces are.

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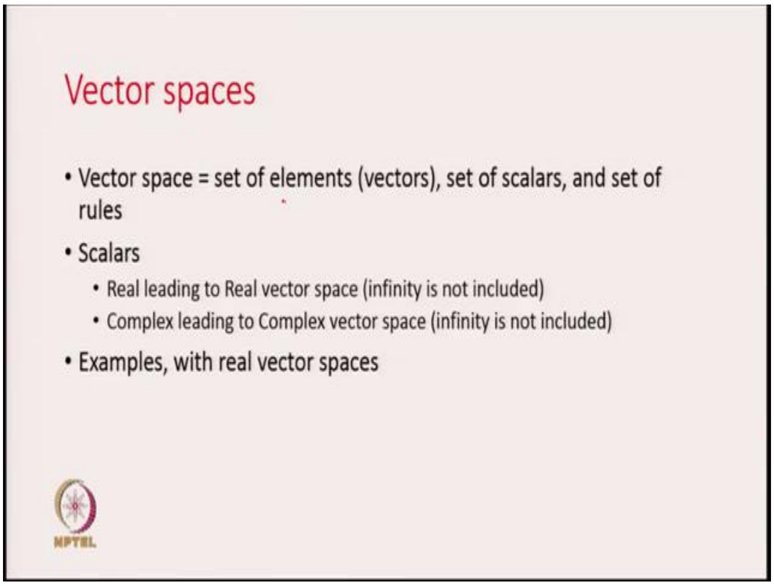

So, vector space is a space which consists of some elements these elements are known as vectors it has some set of scalars and it has some set of rules. So, this is a very basic definition of a vector space something which consists of vectors, it has elements which are vectors it has a scalars and it has some set of rules. So, that is what a vector space is. Now these scalars can be a real numbers, these are scalars can be real and then we are talking about real vector space. These scalars can be complex and then we are talking about complex vector space.

So, we can have vector spaces of the two kinds: real vector space which means that the scalars involved are real. We can have complex vector spaces; that means, the scalars that are involved are complex and of course, in either of these vector spaces ∞ is included. So, throughout this lecture, we will be taking examples of vector spaces which are real vector spaces, but you can easily extrapolate whatever we say for real vector spaces for two complex vector spaces,. So, let us get started let us get started with very simple thing.

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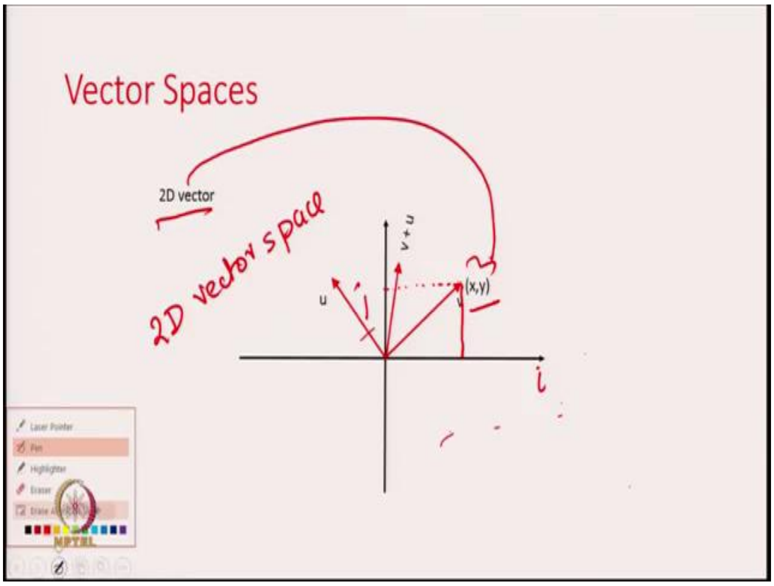

When you are talking about the vectors for example, we have now a_2D vector 2 dimensional vector. This 2D vector can be understood as a vector which has 2 real numbers. So, when you are talking 2D vector, it is equivalent to having 2 real numbers. So, in this case you have 2 real numbers x and y. Remember we are just giving examples from real vector spaces. Now this x as you know would correspond to the value or the projection of this vector along let us say i coordinate and y corresponds to the value or the projection on this vector on this j coordinate. So, this is what you must have done. So, 2D vector remember as we have said is denoted by 2 real numbers.

Now the set of all 2D vectors would consist of 2D vector space. So, if you take all such 2D vectors, if you make a set of all 2D vectors then what you end up with is our 2D vector space. Example of a_2D vector space is like the plane of paper or it is computer screen and so on and so forth. This is what you know from before. Now if you have a_2D vector what you can do is you can have another 2D vector.

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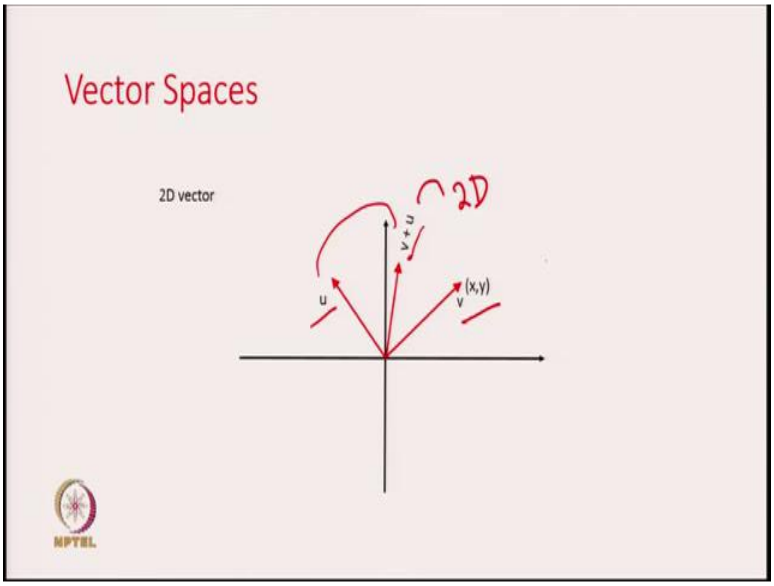

So, if we have a_2D vector here. I can take another 2D vector and I can add these 2 vectors. So, vector addition is defined, it should be possible. Once you add the two 2D vectors what you end up with is another 2D vector. So, addition of a_2D vector with the 2D vector should give you another 2D vector. So, vector addition is possible. What I can also do is I can take a vector and multiply this with a scalar to change the length of my vector. So, this is known as the scaling operation take a vector multiply this with or scalar. So, here I show for scaling operation. So, vector scaling is also allowed.

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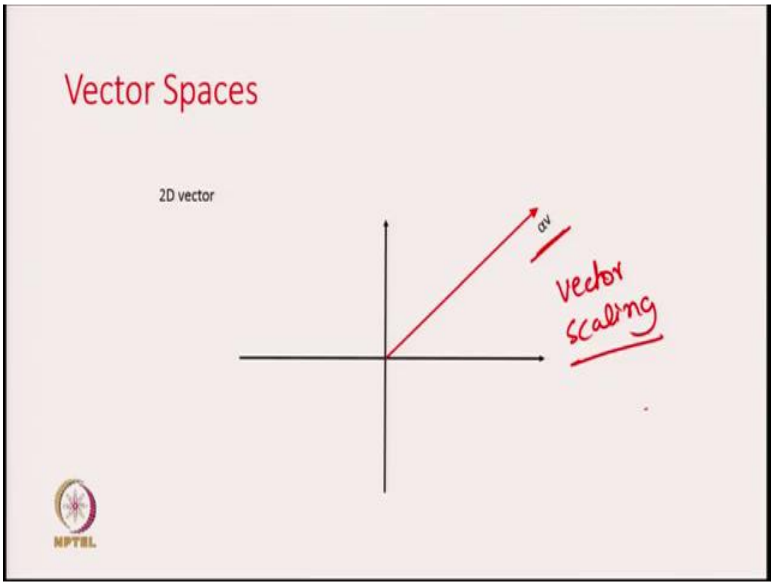

So, vector scaling is allowed and vector addition is also allowed.

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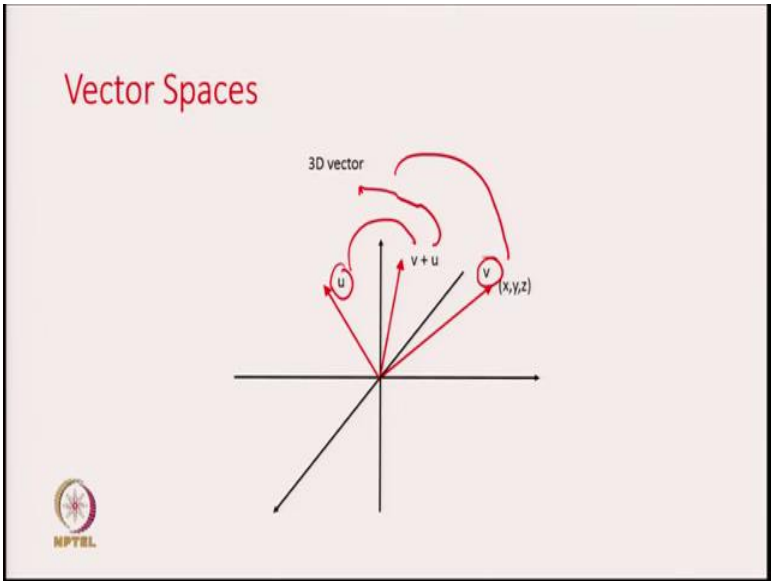

I can talk about the 3D vectors as well the 3D vectors are represented using 3 real numbers. So, instead of 2, now we have 3 real numbers because the dimension is 3D. Similarly as in the case of 2D vectors, the vector addition is allowed; that means, you can take in another vector you can add up these 2 vectors to get a_3D vector. So, I have taken a vector v, I have taken a vector u and I have added these 2 vectors to get another vector v + u which should also be a_3D vector. I can take this vector and I can scale this up by multiplying this with a scalar. So, scaling operation is also defined. So, after scaling operation also the dimension of the vector remains same.

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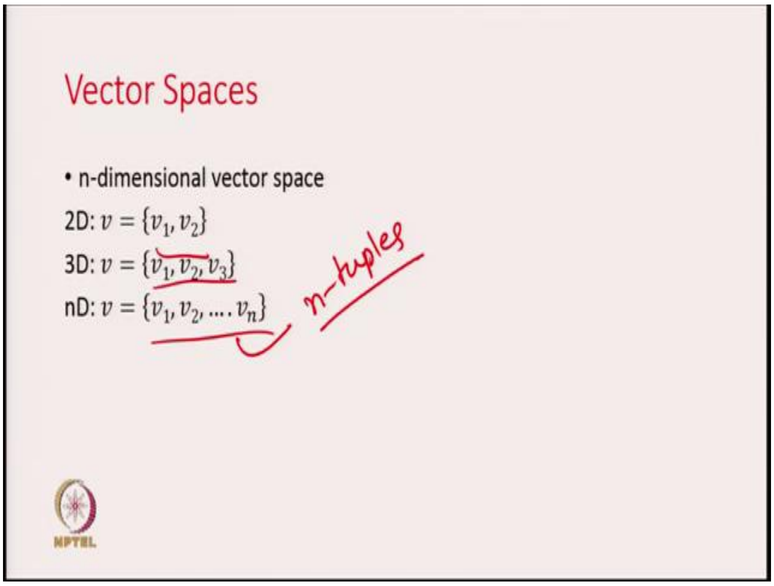

So, I can extend this concept to nD vectors; so, n dimensional vector spaces. So, in 2D, two real numbers, 3D three real numbers, nD it should be represented by n real numbers and we also call this as n tuples. So, you have an array of n real numbers that is what we simply also call as n tuples.

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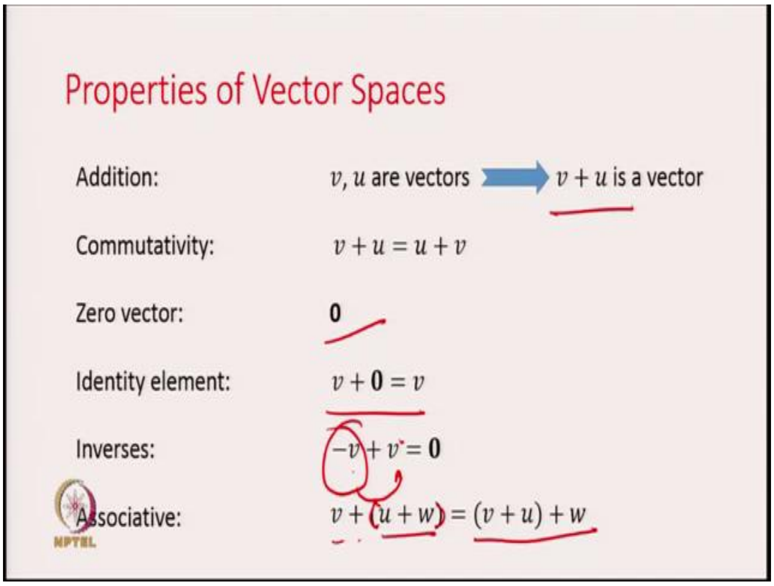

Now, mathematically if you have to prove that something is a vector space, you have to prove that certain axioms or properties are satisfied and now we have we will be going through those properties to see what they are. As we have said if you have two vectors, you can add the two vectors to get another vector.

So, vector addition is defined. So, addition is defined, the order in which you add these two vectors does not matter. So, vector addition is commutative. There all always exist a zero vector which means that if you the vector with a zero vector you get the same vector. So, zero vector exist addition of a vector with a zero vector should give you the same vector. Then an inverse of a vector also exists; that means, you can have an inverse vector which when added to the original vectors this is an inverse of this vector what you get is a zero vector.

So, inverse of a vector must also exist, then vector addition is also associative; you add v + u + w with this bracket involved and this would be same as this thing. So, vector addition is associative. So, all this properties corresponding to vector addition must be satisfied. Now in the case of the vector space, we have already seen that a scalar multiplication must also be satisfied and it should be satisfied by following these properties.

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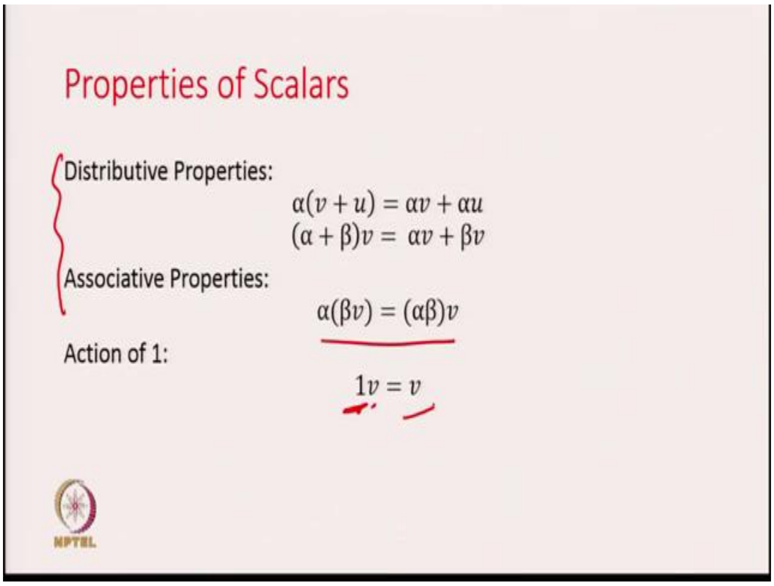

So, distributive properties in the scalar multiplication must be satisfied and associative properties must also be satisfied. I will not be reading out these properties because I feel that these are very simple properties. Moreover we should also have a unit vector. What is the unit vector? Unit vector is a vector by which if you multiply any vector, you get the same vector. So, there must also exist a unit vector. So, in short we can say vector spaces space where vector addition is defined and scalar multiplication is also defined and these are defined in such a way that the properties that we have listed out are satisfied.

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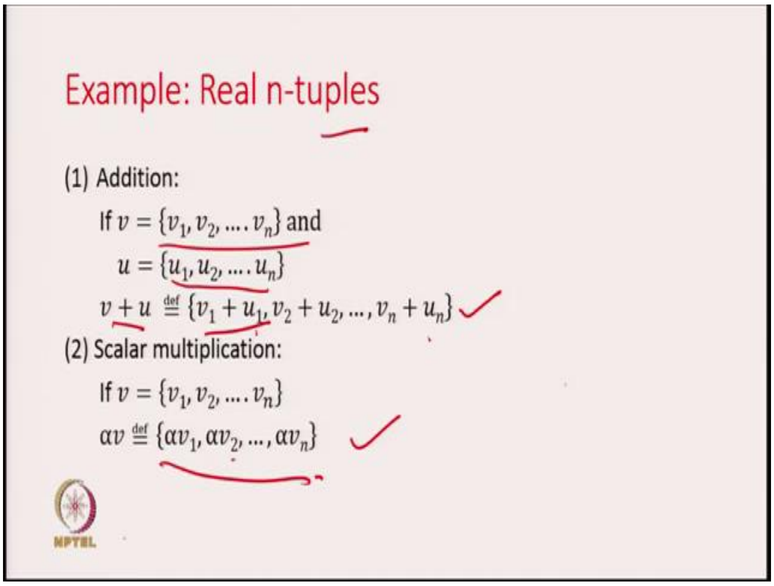

Let us take an example with this n tuples and let us say that I have a vector v which is set of n real numbers, I have a vector u which is set of n real numbers again and I can define the vector addition by taking an element from v and by taking a corresponding element from u and adding these 2 things together.

So, we make an element wise addition as demonstrated here, you have added $v_1 + u_1$ that forms the first element of vector v + u. You can define the scalar multiplication as simply multiple line each element by this scalar. So, there is a scalar multiplication is also easy. Now I leave the proof to you. To prove that if I define the vector addition in this way and a scalar multiplication in this way, then this satisfies all the properties of a vector space. So, this you can prove yourself. Let us move on to an interesting question. Is L_2 space a vector space? Yes.

Now, we have something at our disposal which we can use to prove that L_2 space is a vector space and what are those things? We know which property should be satisfied and let us define first the addition of to L_2 signals and a scalar multiplication of to L_2 signals and let us then check whether these satisfies all axioms corresponding to vector space. Let us get it started.

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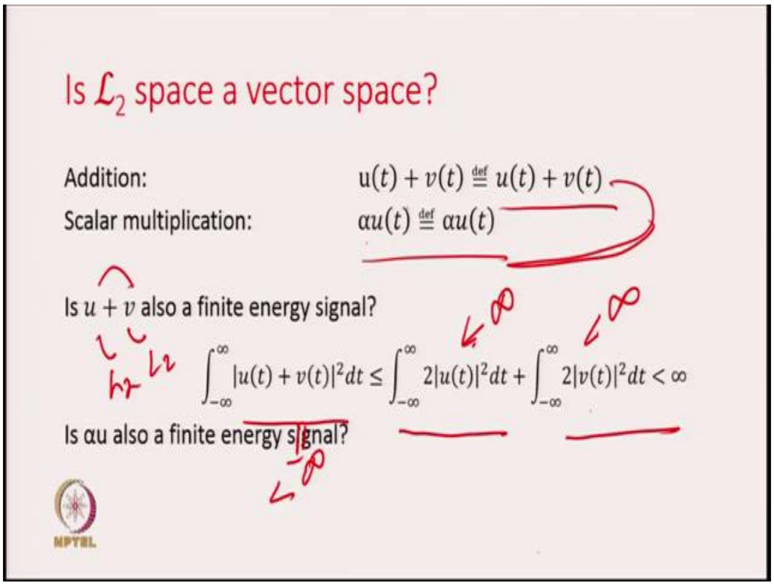

So, addition of two vectors addition of to L_2 signals is easy, it is trivial; you must have done it several times before. You just take the 2 signals and add them. A scalar multiplication of a signal is also simple; you just multiply the signal with any scalar.

Now, if you have these two definitions of how you would want to add to L_2 signals and how you want the scalar multiplication to work on an L_2 signal I leave it to you to prove that this indeed satisfies all properties of vector space and this will prove that L_2 space is actually a vector space. One simple point that you might think about slightly difficult to think about is, if you have u and v if u and v are L_2 signals is it guaranteed that u + v is also a L_2 signal; that means, is u + v is also a finite energy signal, is it guaranteed? Let us check by simple properties from complex numbers that you must have studied which says that this quantity should be less than or equals to this quantity + this quantity this quantity is finite this quantity is also finite because these are L_2 signals; then this quantity should also be finite.

So, addition of 2 finite quantities should give me a finite quantity; this is intuitive. If you if you are adding up a bunch of finite energy signals, then the energy of the signal should remain finite otherwise we could have created energy easily. Similarly if we can prove for a scalar multiplication as well if you have a finite energy signal you multiply this with a scalar which is a finite, then the resultant signal should also be a finite energy. So, L_2 space is a vector space; you add to L_2 signals you get another L_2 signal.

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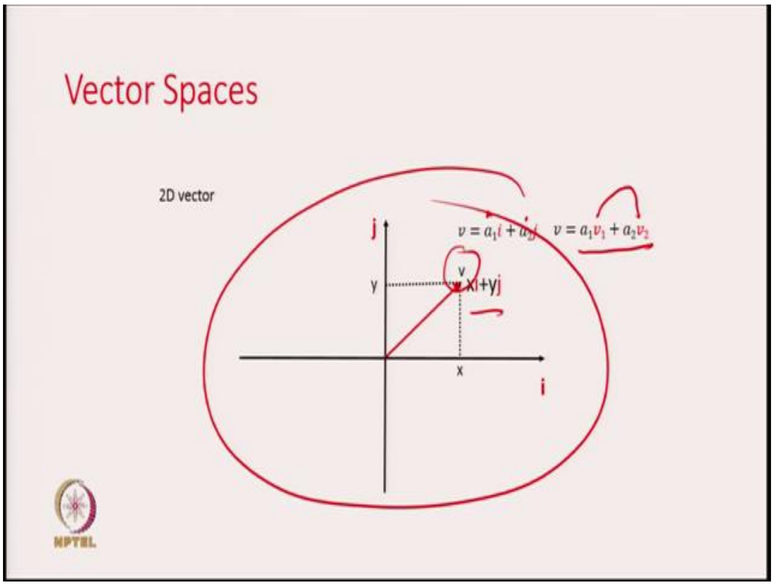

Now, let us talk about this vector spaces little more and let us again fall back to our example of this 2D vector. So, I have a_2D vector again I have represented this like this $x_i + y_j$ and you must have seen this representation before what we are saying is i and j are perpendicular vectors and so on so forth as usual as beck convention. Now to generalize this I can write this is in this term where instead of x, I have a_1 and instead of y, I have a_2 and by changing in these values of a_1 and a_2, I would be able to span the complete 2D space alright. So, complete space could be spanned if you change these values of a_1 and a_2. I can also generalize this further I can write this vector as a linear combination of 2 vectors rather than i and j, I can have a vector v_1 and v_2. I am doing the same thing, but I am just generalizing it making things more general.

So, this vector 2D vector is now a linear combination of 2 vectors v_1 and v_2 of course, this v_1 and v_2 has to be properly chosen and so on and so forth, but let us say we have made an intelligent choice and we have properly selected these two vectors. Now this 2D vector could be represented as a linear combination of these two properly chosen vectors. Similarly I can extend this concept to 3D case in 3D case; I can have a_3D vector which I can obtain by having a linear combination of three other vectors.

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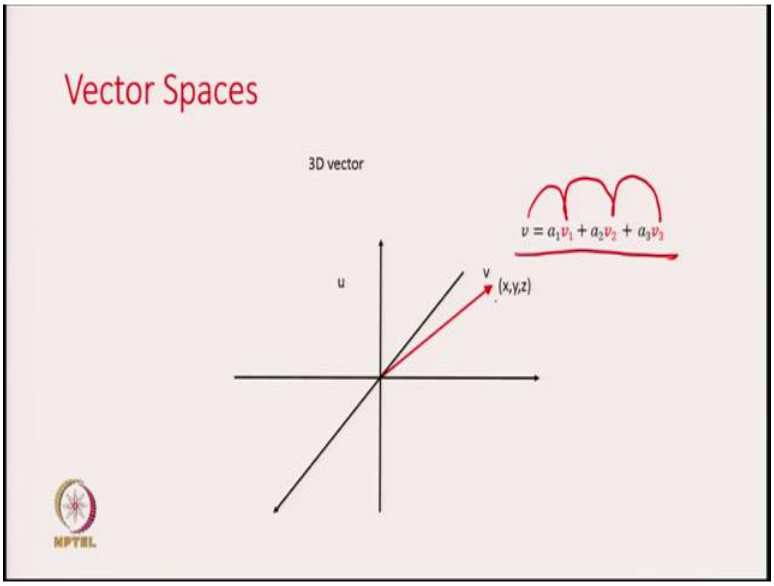

So, for example, I can have a vector v, I can obtain this vector v by a linear combination of v_1, v_2 and v_3. Again I want to generalize this idea, I am always interested in generalizing ideas to nD case.

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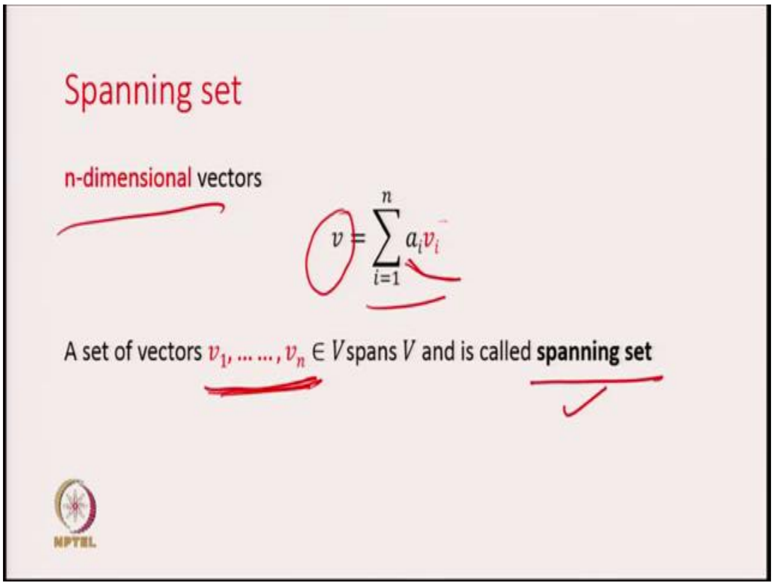

So, an n dimensional vector could be obtained by making linear combination of n vectors. So, I can have an n dimensional vector, these vectors set of vectors v_1, v_2 … v_n which allows us to span a space is known as a spanning set. Why are they allowing to span a space? Because I can change the values of these a_i's and by changing the values of a_i, I can go to any point in that space. So, this is the meaning of spanning the space. So, these set of vectors which allow us to spend a vector space is known as a spanning set this is important. So, what we have learnt so far in vector spaces? I can have a vector, I can obtain a vector by a linear combination of some other vectors and normally if I want to span an n dimensional vector space, I would need n properly chosen vectors intuitively.

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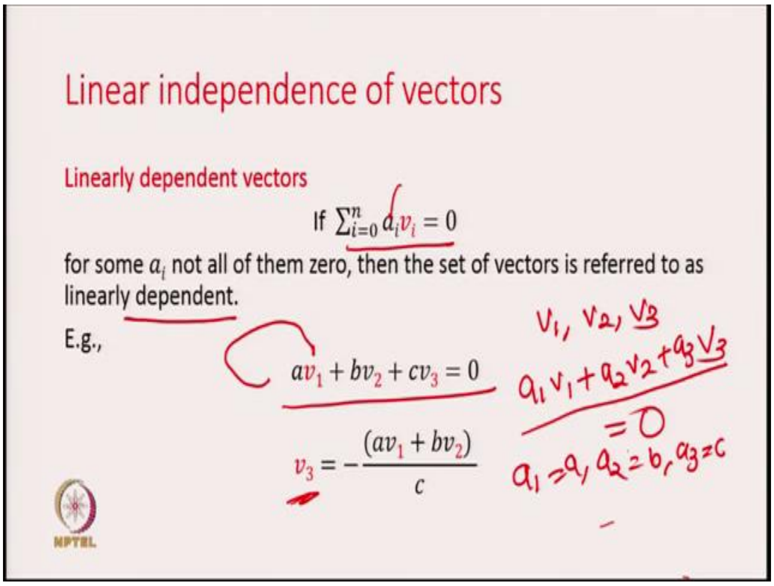

Let us now see the idea of linearly dependent vectors. So, if this linear combination that we have just talked about if this linear combination can be made 0 by choosing some a_i is not all of them 0; if all a_i are 0 then of course, this linear combination would be 0. But what we are saying is if we can have this linear combination turning out to be 0 without having all a_i is to be 0, then we say that this set of vectors is linearly dependent set; the vectors are linearly dependent on each other. Let us see why is this and for simplicity let us just talk about the situation when we have 3 vectors; v_1, v_2 and v_3 and I am interested in this linear combination.

And let us assume that this becomes 0 when the value of a_1 is a, a_2 is b and a_3 is c. So, if I choose the value of a_1 as a, a_2 as b and a_3 as c. I can make the linear combination of $a_1v_1 + a_2v_2 + a_3v_3$ as 0 and we call this situation that these vectors v_1, v_2 and v_3 would be linearly dependent and let us see why is this. So, if this is 0, we can write v_3 in terms of v_1 and v_2 and hence this means that vector v_3 is dependent on vector v_1 and v_2. So, that is the idea behind the linear dependence of vectors.

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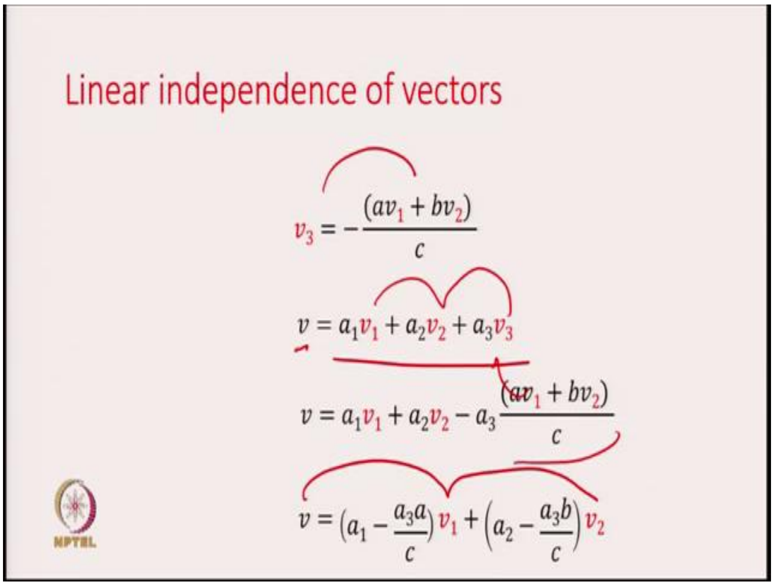

So, if the vectors are linearly dependent we can express one vector in terms of other vectors. Let us see what it does. So, we have already said if the vectors are linearly dependent, then v_3 could be expressed in terms of v_1 and v_2. We already know that vector v can be obtained by a linear combination of v_1, v_2 and v_3. Now plugging in this value v_3 into this expression what we get is v vector could be written simply like this just plugging in the value of v_3 here. Now I can simplify this and get this. So, what does this tell me? So, if factors are linearly dependent, I can express a vector in terms of a reduced set of vectors.

So, I started by expressing a vector in terms of three vectors v_1, v_2 and v_3. So, I was expressing vector v in terms of three vectors. Now because these vectors are linearly dependent, I can substitute one vector in terms of other vectors and I can express this vector in terms of reduced number of vectors. So, I am killing in some I am killing some linear dependence. So, this is the idea of what we call as a basis set.

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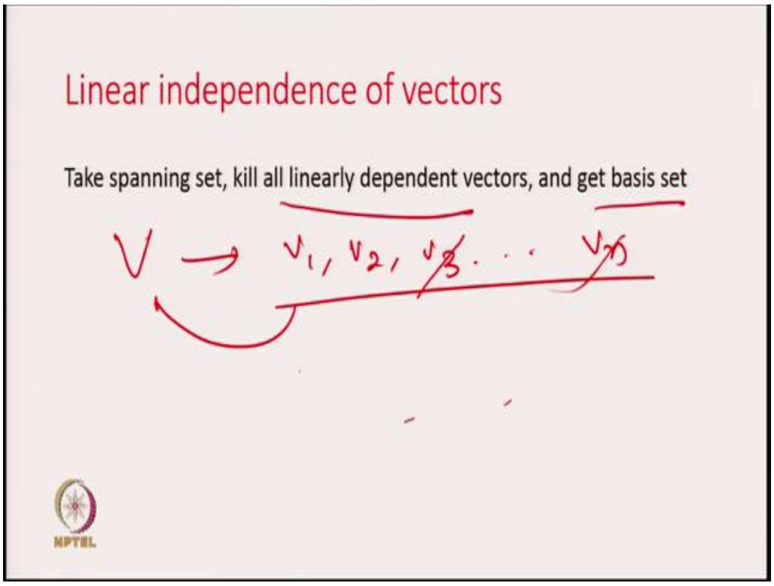

So, you have a spanning set. So, using spanning set, you can you were spanning the given vector space. Now because some vectors were linearly dependent, you can start killing those linearly dependent vectors and once you kill those linearly dependent vectors what you end up with is a reduced set and this reduced set is known as a basis set if there is no linear dependence between the vectors and if this set is still able to span the complete vector space. So, we get a basis set. So, what is the idea?

So, you have a vector space v, you find out the vectors v_1, v_2, v_3 and so on and so forth. Let us say v_n which allows you to spend this vector space, you find that some of the vectors are linearly dependent, you kill those vectors, you get a reduced set, but this reduced set is still able to span the complete space and this reduced set where there is no linear dependence and which is still able to span the complete vector space is known as a basis set. So, let me define because I have not defined what a linearly independent vectors, I have already defined what a linearly dependent vectors and the definition follows straight away there is nothing different.

So, we can define linearly independent vectors as the set of vectors for which the linear combination can only be made zero if you have all scalars being zero. So, you made a linear combination the linear combination only becomes zero if you have all the scalars being zero then those set of vectors are known as linearly independent vectors.

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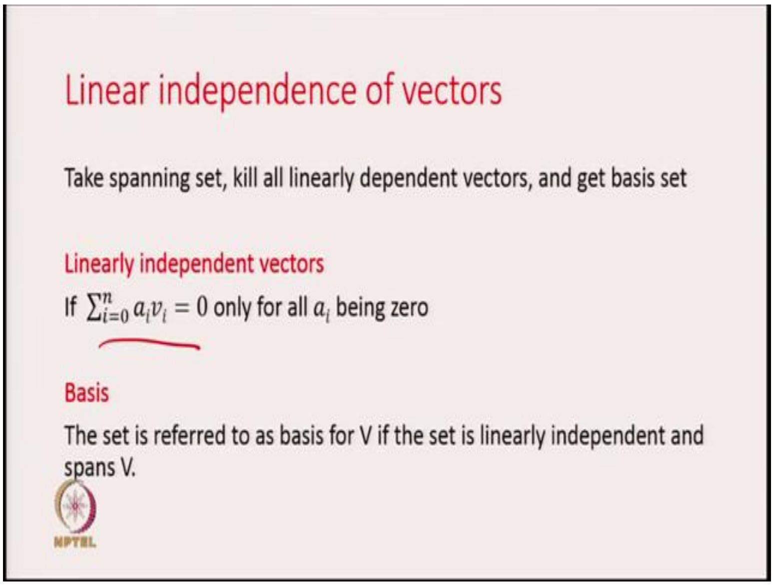

So, I can define the basis vectors again. The basis vectors is a set of a vectors if the set is linearly independent, it spans a vector space v. So, let us now see the dimension of the vector space. So, what is the dimension of the vector space?

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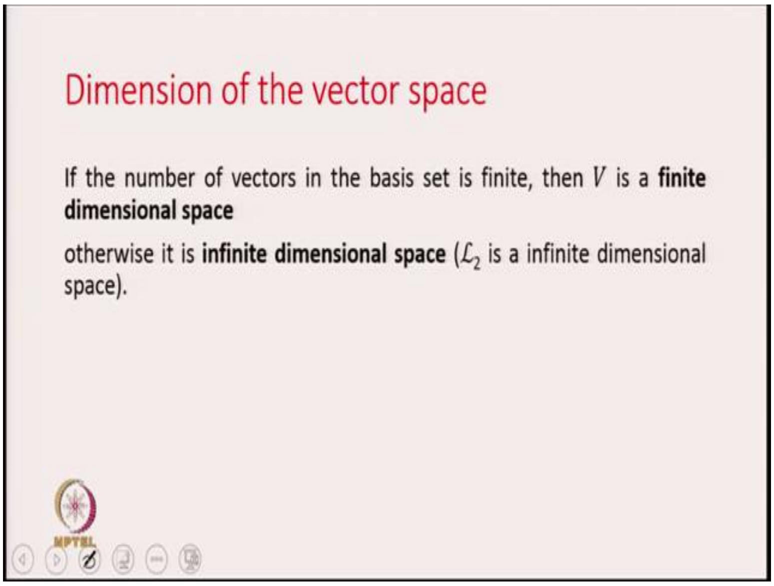

If the number of vectors in a basis that is finite, we call this as a finite dimensional space; otherwise it is known as infinite dimensional space. In fact, the number of vectors in the basis tells me the dimension of this space.

So, if the number of vectors in a basis set is 10, then we say that the vector space of use of dimension 10; if the number of vectors in the basis set is finite of course, is a finite dimensional vector space and the number of vectors in the pacesetters infinite, we call this as infinite dimensional vector space.

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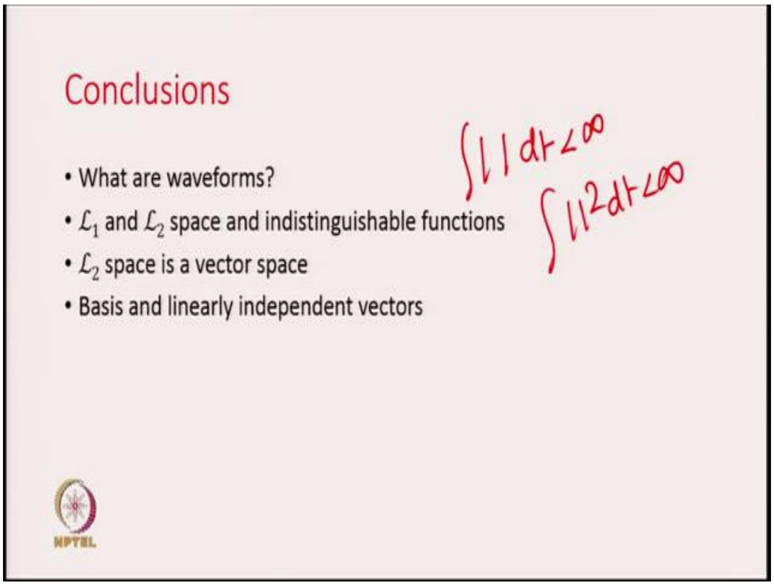

So, we have come to the conclusions of today's lecture. We have seen what a wave forms, wave forms are just continuous function of time. We have seen this, these waveforms are useful in digital communication course because they are used in the transmission of signals. We have seen what are L_1 and L_2 signals and these are defined based on the integration of mod of the function being finite or integration of mod square of the function being finite. We have seen that L_2 space is a vector space. So, whatever we can do in a vector space can be done for the L_2 space as well. We have defined what is a basis and linearly independent vectors.

And in tomorrow's lecture we will be introducing a very useful concept and that is the concept of inner product space by which you would be able to find out that the angle between the 2 signals and so on so forth.