Principles of Digital Communication
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Lecture - 16
Random Variables & Random Processes: Random Process through an LTI system

Good morning, welcome to the next lecture on Random Processes. So, in the last lecture, we define several classes of random processes; the stationary random process, wide sense stationary random process and we started looking into ergodic random processes.

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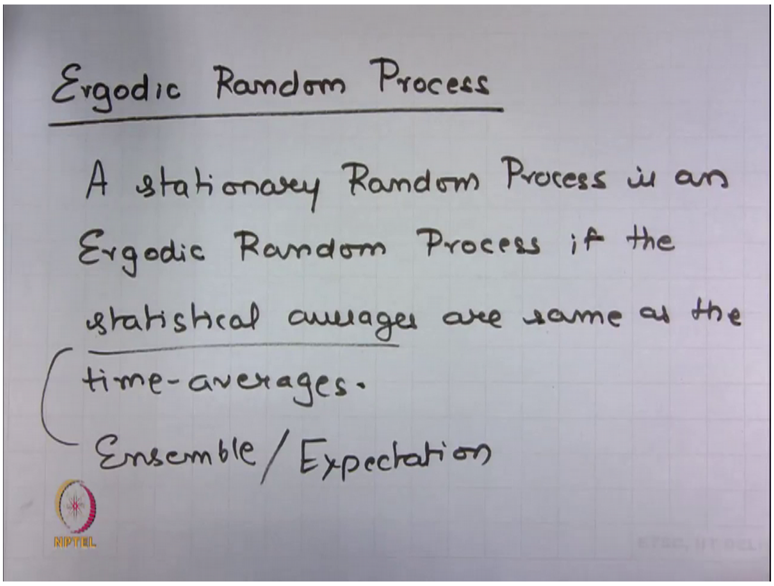

So, we said stationary random process is an ergodic random process if the statistical averages are same as the time averages ok. So, let us now look what are these statistical and time averages. Let me remind this.

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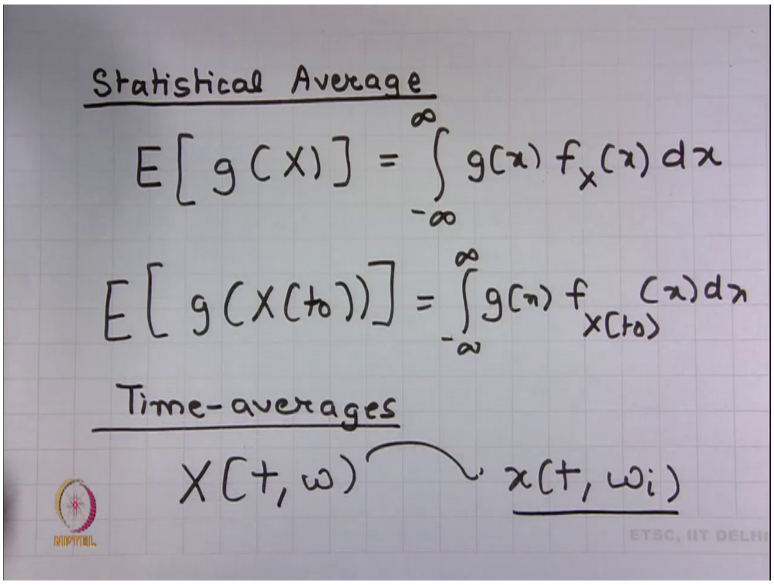

So, if you are talking about statistical average, the statistical average. Now, we have already seen these statistical averages or ensemble averages and we know that this can simply be obtained by taking the expectation of let us say; here we are taking the expectation of a function of a random variable and we know that how to do this. This can simply be obtained by having a function of x multiplying this with the probability density function of x and integrating this thing from minus infinity to plus infinity. And if you want to do the same thing for a random process, it is also easy.

So, we can take or define a similar quantity for a random process X of t but now, we are looking down this random process at a specific time instants t naught. So, we get a random variable X of t naught and like here, we can define this exactly in the same way as we have defined it here. Only that X is replaced by X of t naught because the random variable is X of t naught. What about the time averages? We are used to find these time averages. So, for a time average, we have to take a sample function and we have to integrate that sample function from minus infinity to plus infinity and divide by the total limit of integration ok.

So, for example, if I have a random process; first I need to take a sample function of this random process. So, let us say we are concentrating on this sample function. So, I am having a sample function corresponding to the i'th outcome of the random experiment ok. So, let us see how to find the time average corresponding to this sample function.

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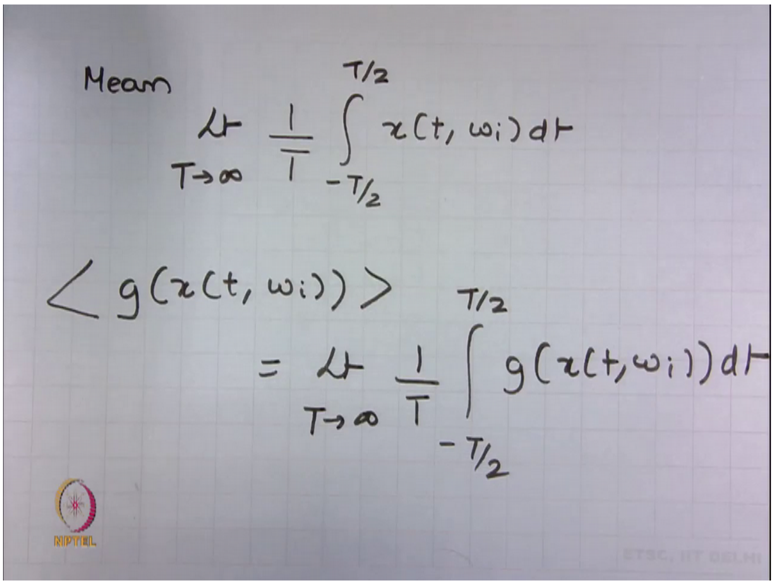

And its time average; let us say we want to take the mean of the sample function you know, the mean can simply be calculated. So, what we are doing? First thing that you realize is that we are integrating it with respect to time and we are dividing it by 1 by T and the limit of this integration goes from minus T by 2 to plus T by 2. So, this is you are taking a sample function from minus infinity to plus infinity and you are taking its time average value.

So, this is the time average right. In general you can think about the time averages as so, let me use this notation to denote that I am taking the time average of this sample function. So, I can take any function of the sample function. If I want to take the time average, this is defined as limit T tends to infinity when by T minus T by 2 T by 2 g of x of T omega i d t. So, what does it mean is in this case we have taken the mean, but you could have taken T time average value of the square of this sample function. So, this means that you can take the time average of any other function of this sample function right.

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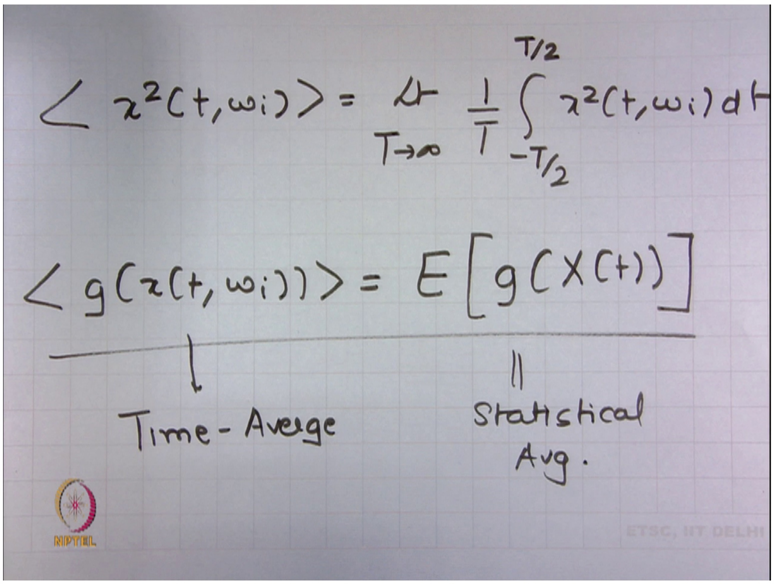

Just to make it more clear suppose I want to take the time average of square of this quantity, I can do this by definition. This is nothing, but limit T tends to infinity 1 by T minus T by 2 to plus T by 2 x square t omega i d t ok.

So, this is taking the time average value of this quantity. So, what is an ergodic process? An ergodic process is a process for which the time averages. So, the time average is nothing, but it is the same as these statistical average. So, the time average this is the time average and this is the statistical average or ensemble average right. So, if time average is the same as the statistical average, we call that process as an ergodic process So, let me see what we mean with this. So, let me write this expression again.

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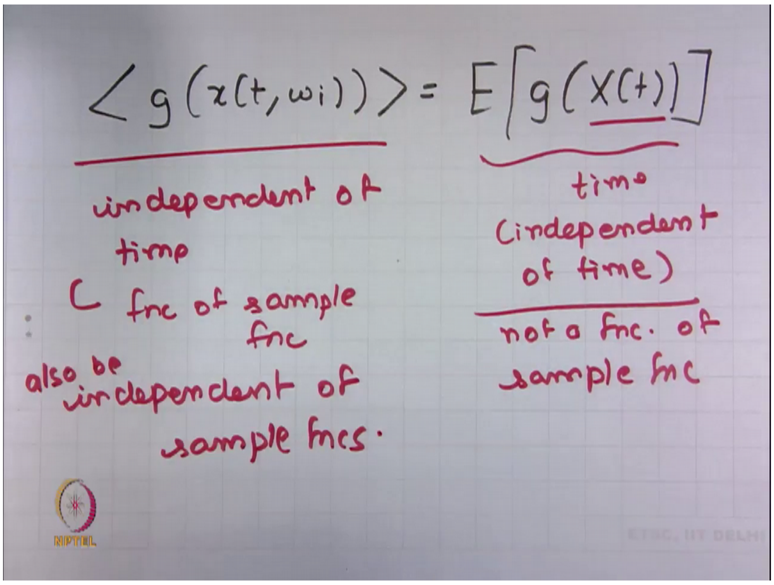

So, let me write that time average is same as statistical average. Now if you look at these 2 things carefully you know that this function is normally a function of time is not it right because this is a random variable which is sampled by looking at this process at a time instants t. So, this is normally a function of time, but because we have considered this to be a stationary process, this is actually independent of time. And hence we require stationarity for process to be ergodic ok. So, this is independent of time because my process involved is a stationary process and does it does not depend upon what is this specific value of this time ok.

Now, this quantity is of course, it is independent of time because its a time average right is independent of time because it is a time average. But this quantity is a function of sample function is not it. And this is not a function of not a function of sample function because its an average across all possible realization right. So, it will not be a function of sample function. So, hence from this we conclude if this has to be same as this; that means, this quantity should also be independent of if the time average is the same as statistical average. That means, it should also be; it should also be independent of sample functions. What can we infer from this is that all sample functions must have the same time averages is not it right.

So, a basic reasoning for this is suppose if you have a sample function which is sufficiently random right. If it is sufficiently random it can mimic the entire random process right. So, a sample function must have sufficient randomness this is what we are drawing from basic intuitions.

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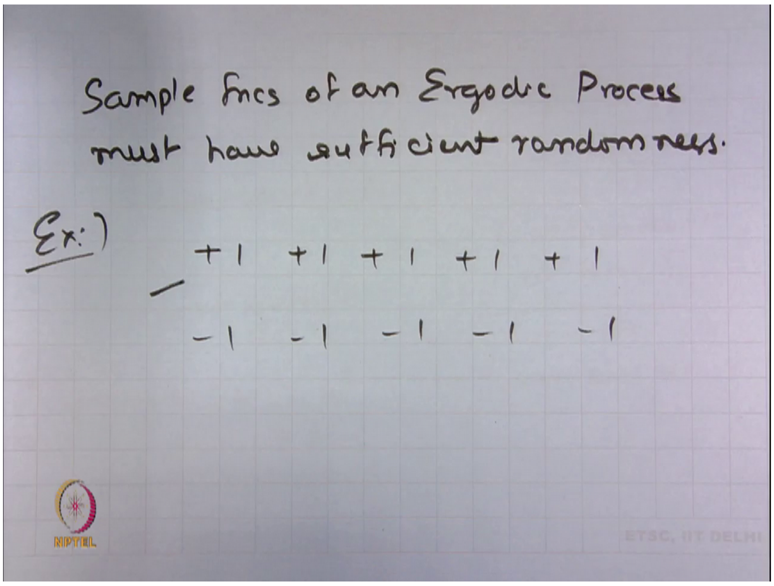

Sample functions of an ergodic process must have sufficient randomness. So, only if they have sufficient randomness their time averages will be same as the statistical averages; that means, each sample function should mimic something like the entire random process. If the sample functions are not random themselves; for example, let us take an example suppose I have a sample function which takes 1 at all times and I have another sample function which takes minus 1 at all timing instances.

Then what happens? This function does not mimic the entire random process because the random process can take a value 1 or minus 1 whereas, this sample function takes only the value one right. This sample function takes only the value minus 1 and as you can see that these sample functions time averages are different right.

So, the sample functions of an ergodic process must have sufficient randomness. So, that their time averages equals statistical averages, is this normally. The case the answer is yes because in digital communication systems we use the scramblers line coder which we would introduce such that these sample functions have sufficient randomness or sufficient transitions for the synchronization circuits to work right.

So, you want your sample functions to have sufficient randomness in them because you want your synchronization circuits to work and to have that randomness in your sample functions, we use the scramblers and line coders. So, we can assume to a fair degree that the sample functions of the waveforms that we deal with in digital communication systems are ergodic ok or they have sufficient randomness and thus the underlying process is an ergodic process.

So, the key point is most practical processes that we will deal with are ergodic process. And what is the advantage of this? Viewpoint is that we can take the average either by looking in the time domain or we can calculate the statistical average whichever is more convenient and we will see that sometimes one is more convenient than the other right. So, it now you have two tools to choose from.

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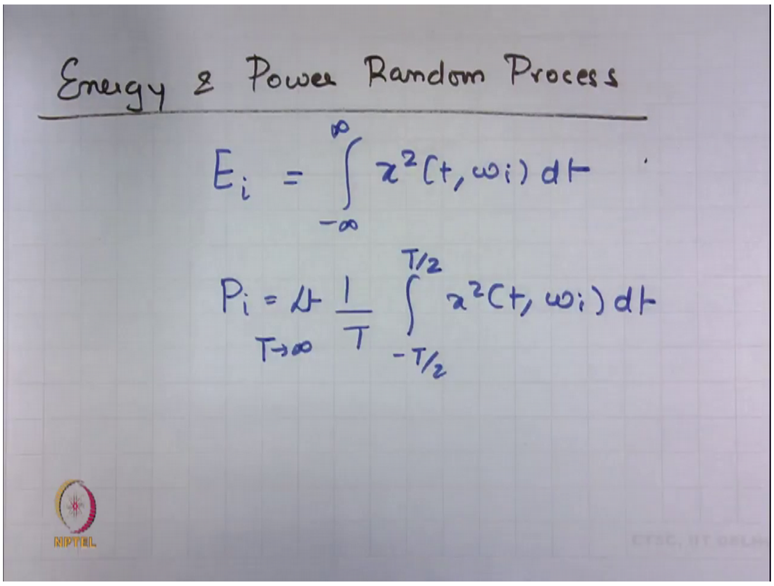

Next we discuss the another class of random processes that are energy and power random process. Let us see. So, we use the basic knowledge from signals and system course and we can define the energy of a function; let us a sample function sample function is also a function by this right. So, energy of a sample function can be calculated like this right.

Now, the power of a sample function can be calculated using this expression right. So, you take a sample function and to calculate the energy you have to square this first and then you have to integrate it from minus infinity to plus infinity. Now suppose if we are interested in calculating the energy of a random process, what is that? You need to take the expected value of energy across all possible realization.

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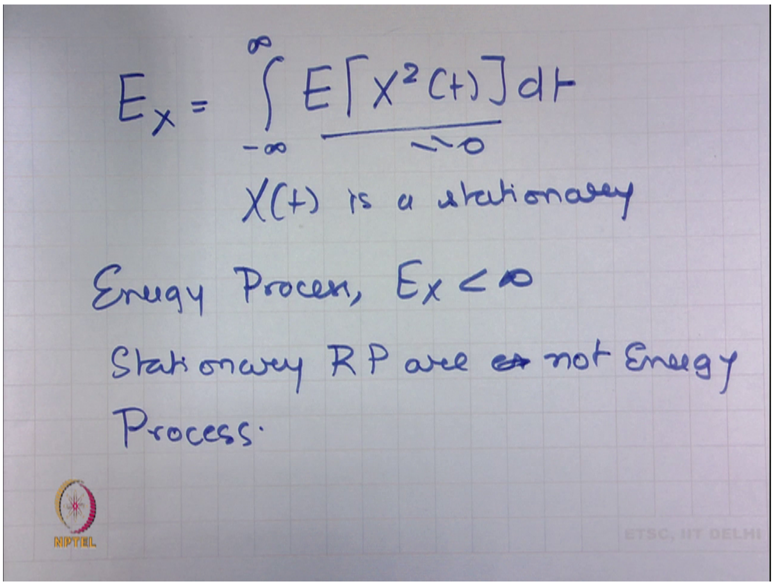

So, if I am talking about the energy of a random process, it can be defined simply like that, but now you have to take the expected value along all possible realization. So, what does this mean is that you calculate this energy and then you multiply with what is the probability for this outcome to happen. So, similarly you do it for all sample functions and this we can write in one step like this ok.

Now, as we have already stated if X t is a stationary process; if x t is a stationary random process, then this quantity exists from minus infinity to plus infinity. And if we are believing that the process is an energy process, what do we mean by that is the energy is finite. So, for a energy process E x must be finite; that means, it must be less than infinity. If this process has to be energy process and this quantity is valid from minus infinity to plus infinity you know that this quantity must be 0 with probability 1 is not it.

This being finite this range is infinite. So, only way in which this quantity can be finite when this is 0 with probability 1 otherwise it is going to be infinity does we conclude that a stationary random process are not energy process; are not energy process and that is why you define either effectively stationary or effectively wide sense stationary process.

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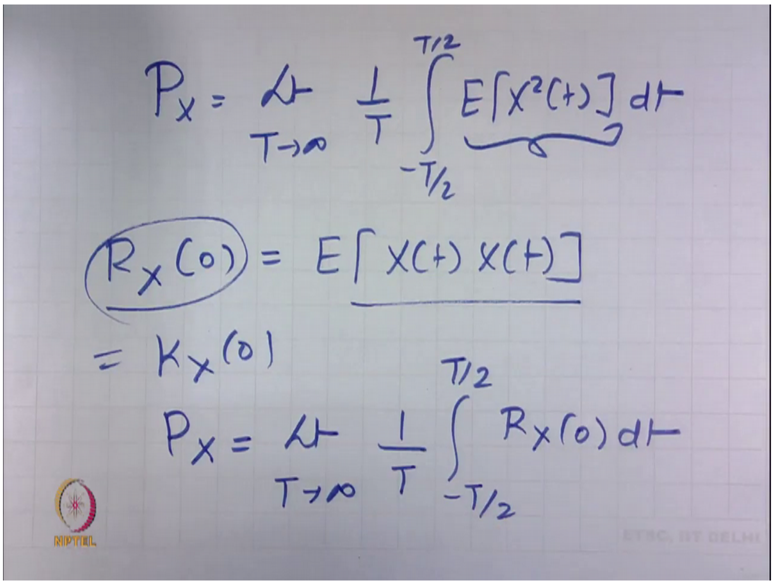

Let us see if they are power process. So, let us see the power of a random process, it can be defined like this expected value of X square t d t and now you know that what is this quantity? What is this? So, you know that autocorrelation function of a random process. It is a stationary random process of the autocorrelation function will be one argument autocorrelation function. So, this would be something like expected value of X t times X t is not it.

So, this quantity is nothing, but it is the autocorrelation function evaluated at zero. If it is a 0 mean process you can also write this as this there is no difference between autocorrelation function or auto covariance function; if it is a zero mean process. So, we have chosen to write this as this. So, we allow for a more general case. So, what does this reduces to? Then I can write P x is limit t tends to infinity 1 by T ok.

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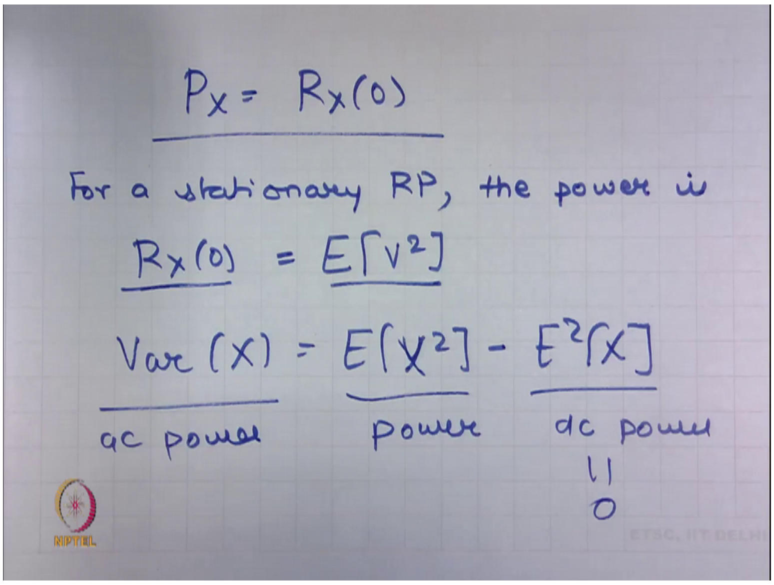

Now, this quantity boils down to P x comes is nothing, but this why is this? So, because this is not a function of time. So, this can be pulled out of integration and then what you get is power in a random process is nothing, but its autocorrelation function evaluated at time t equals to 0 right and of course, we are only talking about these stationary random processes. So, we write for a stationary random process power is R xX 0 right. So, this is an important observation in terms of random variable, this would be nothing, but E V square is not it.

Now I think it is a good idea at this time to revisit the expressions or to understand the properties of this autocorrelation function little bit more carefully and let us summarize what are the interesting properties that these autocorrelation function passes. But before that let me just point this out to you that what is the variance of a random variable. It is nothing, but expected value of minus E square X this we have already seen that a variance of a random variable is nothing, but the second moment minus the square of the first moment and this quantity is the power in the process right this is the dc power and this is the ac power.

So, variance tells us what is the a c power of that random process. So, when you are calculating the variance of a random process this tells you the ac power of a random process. When you are calculating the variance of a process with zero mean; that means, this quantity is 0 the process with zero mean means that it has 0 dc power, then in that case the variance is nothing, but it is the power in the process because dc power is 0 right. So, for a zero mean random process the variance also tells us the total power in the process ok.

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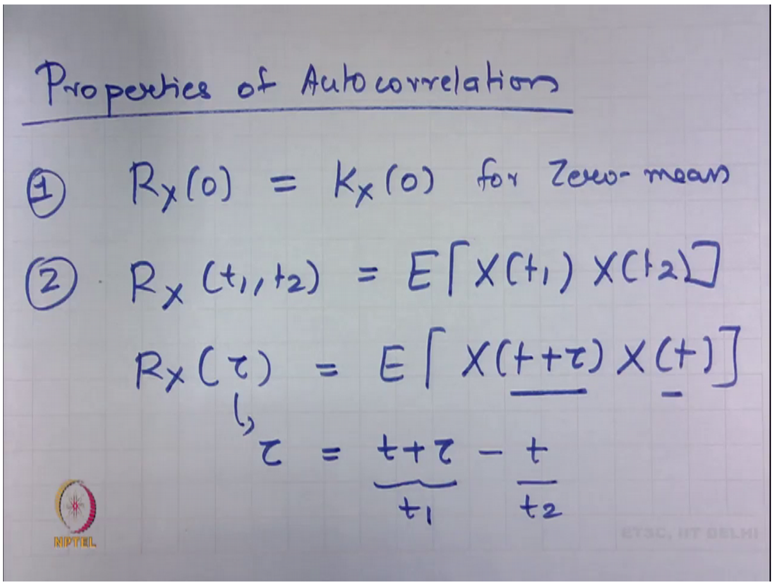

Now, let us look at the properties of autocorrelation function. So, some properties which we already know that R X 0 is nothing, but it is the K X 0 if the process is zero mean ; that means, for zero mean process the autocorrelation function and auto covariance function now same is not it. Then the second point is by definition let us write what is R X t 1 t 2 by definition it is nothing, but it is the expected value of X t 1 times X t 2

Now, let us take this as R X tau. What is this? We can think in terms of expected value of X t plus tau into X t because we have said in case of the single argument this tau is nothing, but it is the difference between t 1 minus t 2. So, here you can see that t 1 is t plus tau and t 2 is t. So, t 1 minus t 2 is tau; that means, if the process is a stationary that is important if the processes stationary or a wide sense stationary, then its autocorrelation function can we thought in terms of a single argument and in that case R X tau represents this thing.

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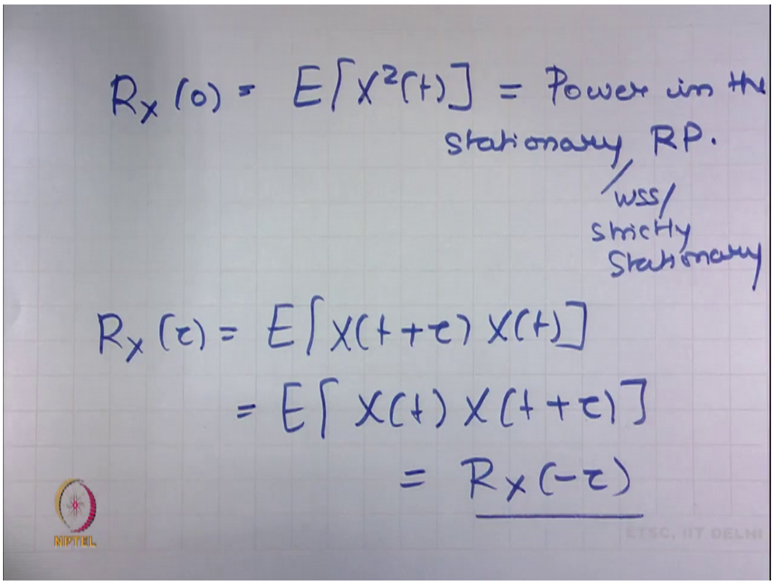

So, from this also we can infer that R X 0 is nothing, but it is the expected value of X square t clear. Now from this we have already said that this is the power in the random process a stationary random process ok. Stationary or let us say wide sense stationary or strictly stationary does not matter.

Now, if we write this again we get expected value of X t plus tau times X t. Now you see that this is also same as this quantity does not matter; the order does not matter in this case is quite obvious and this is nothing, but R X minus tau by definition. So, what you learn from this is R X tau must be same as R X minus tau.

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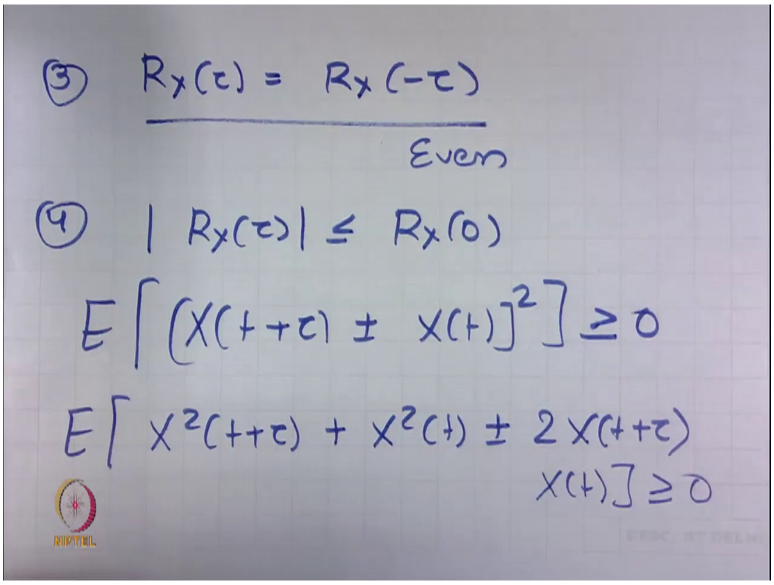

From this we can conclude that R X tau is same thing as R X minus tau; that means, my autocorrelation function is an even function ok. So, let us summarize the three important properties that we have done property number one; that autocorrelation or auto covariance function for a zero mean random processes one in the same thing second important thing that we have said is that we have R X 0. So, let us write this R X t 1 t 2 by definition is this quantity for a stationary process we can say that R X t 1 t 2 is nothing, but R X t 1 minus t 2. So, for a stationary process R X tau can be thought like this and from this we can conclude that R X tau must be same as R X minus tau; that means, autocorrelation function is an even function.

Let us try to prove the 4th property which is the mod of R X tau should be less than or equal to R X 0 and this can be proven by using the simple idea that if we are interested in calculating the expected value of this quantity, it must be strictly non negative right. Because we are calculating the expected value of the square of a quantity; expected value of a positive quantity must always be nonnegative.

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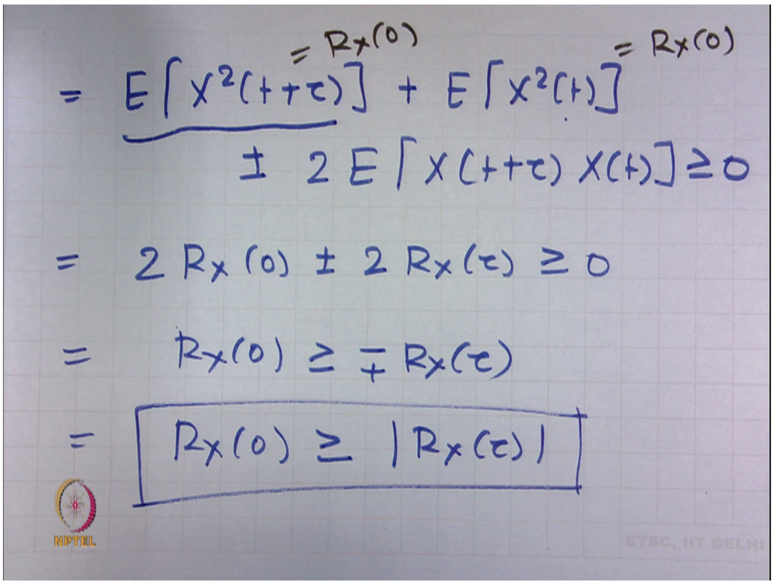

Now if you think about this; this is following from there I can say that this is expected value of X square t plus tau plus expected value of X square t plus or minus 2 times expected value of X t plus tau X t must be greater than or equals to 0. Now you know what is this right.

So, first of all you appreciate that this quantity must be same as this quantity because my processes stationary process and a stationary process does not depend upon the absolute value of time right. So, this quantity is same as this quantity. In this quantity, we have already seen what is this is R X 0 right. So, this must also be same as R X 0. So, from this we get two times R X 0 plus or minus 2 times. And what is this? This is R X tau by definition must be greater than or equals to 0.

So, from here we get R X 0 must be greater than or equal to minus plus R X tau. So, from here we get R X 0 must be greater than or equal to mod of R X tau. So, this is an important result; that means, the peak value of autocorrelation function lies at t equals to 0. So, these are some important properties of autocorrelation function. So, far we have completed the discussion of various kinds of random processes we have defined what is the stationary random process, wide sense stationary random process, ergodic random process energy and power random processes of course, stationary and energy random process does not exist.

Then we have seen certain important properties of autocorrelation function. Now we are going to do something very interesting and that is the interaction of these random processes through LTI system.

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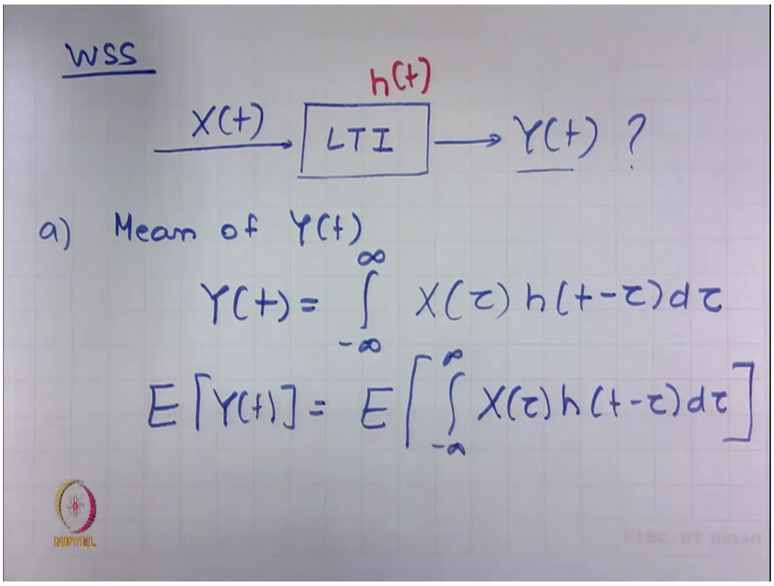

And for this we assume a wide sense stationary process and we ask the question what happens when this wide sense stationary process passes through an LTI system. So, we have an LTI system.

And let us assume that at the input of this LTI system, we have a wide sense stationary process which is X of t and at the output of this LTI system we already know must be a random process and let us call that random process is Y of t and the question is how can we think about these Y ts and X ts right is there some interaction between them. So, the first question that we ask is what happens to the mean.

So, the first question mean of Y t. Let us see if we can calculate this now let me write first the expression of Y of t this can be obtained by taking the convolution of the random transpose with an impulse response. Let me assume that the impulse response as you normally do is h of t that is an impulse response of this LTI system. If we assume the impulse response to be h t we get the output as this times h of t minus tau d tau right. So, this is a normal convolution that you do ok.

Now, so, we want to find the expected value. So, expected value of Y t has to be obtained. So, we put an expectation operator minus infinity to plus infinity X of tau h of t minus tau d tau. And now what we do normally is we take this expectation operator inside pull out this integration.

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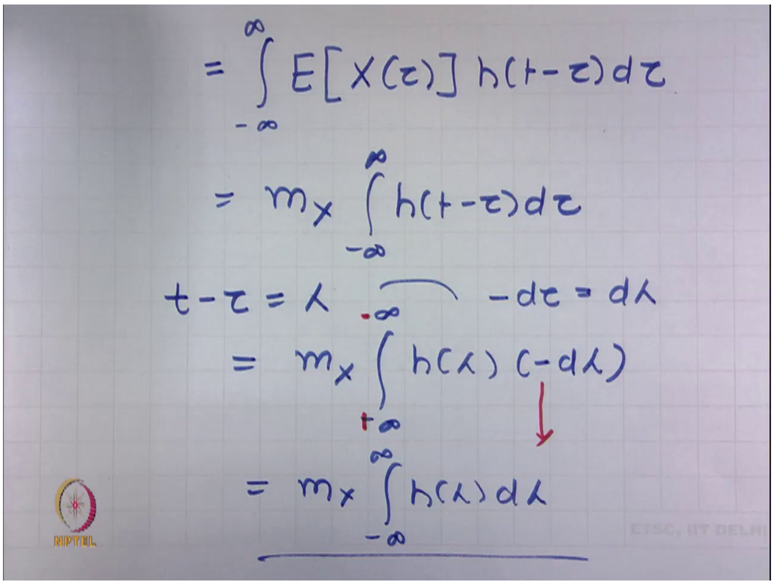

So, this could be written as minus infinity to infinity expected value of; now just see this equation the expected value is done normally over the random part there is no randomness in this right because its completely deterministic. So, if we look at this function its completely deterministic, there is no randomness here randomness is in the random process right.

And normally also you have seen that expected value of random variable times a constant is nothing, but it is a constant times the expected value of a random variable. So, you do not have to put expectation here. Normally this would suffice. So, thinking like this we get expected value of X of tau h of t minus tau d tau and what we have been given is that the processes wide sense stationary.

If the process is wide sense stationary, then this thing expected value of X of tau is a constant right. It is the mean of that process. So, we can write this as m X where m X is the mean of this process times minus infinity to plus infinity h of t minus tau d tau ok.

Now, to make this expression little bit more convenient for you that is what t minus tau as lambda. So, substituting that we get minus infinity to plus infinity h of lambda and from here we get minus d tau is d lambda. Now see this equation carefully because we will be doing this a couple of times. So, what happens, when you have this, you change instead of d tau you put a minus d lambda. Now instead of minus infinity it should become plus infinity is not it. Because when tau becomes minus infinity lambda becomes plus infinity. So, you have plus infinity here and when tau is infinity, lambda is minus infinity ok.

Now, you put this negative sign here negative sign can also be taken care of a flipping the limits right. So, I can get rid off this negative sign by flipping the limits. So, see what happens when you do this what you get is minus infinity to plus infinity h of lambda d lambda ok. So, remember that if you have a negative sign you can take care of this negative sign by just flipping the limits and then you can obtain a better a neater expression than this expression.

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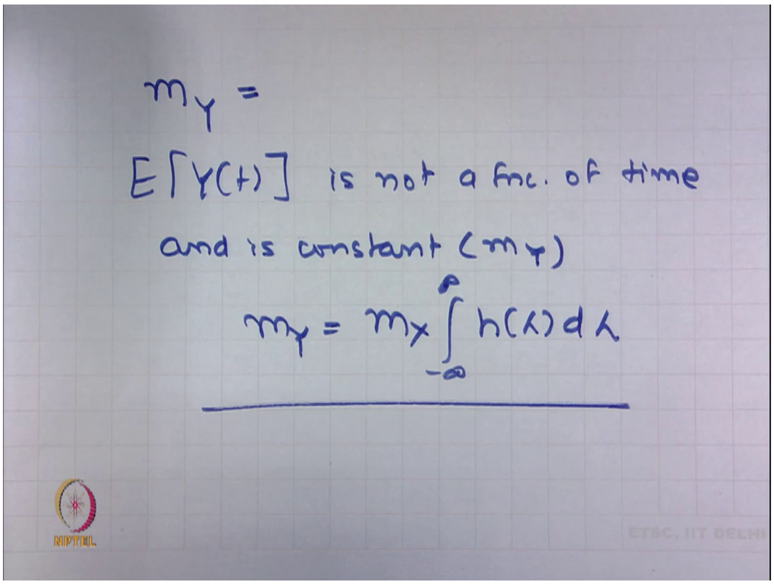

So, let me summarize this that expected value of Y right. So, anyway here this expected value of Y t that we have obtained is not a function of time its not a function of time. It is m X is a constant this quantity is also a constant. So, expected value of Y t first of all it is not a function of time; it is not a function of time; its not a function of time and its constant.

And let us assume that this constant is m Y it is just notational. So, that constant is m X times minus infinity to plus infinity h lambda t lambda right. So, if you know the expected value of input random process; if you know the impulse response, you can find the expected value of the output random process that is one important point then the second point is.

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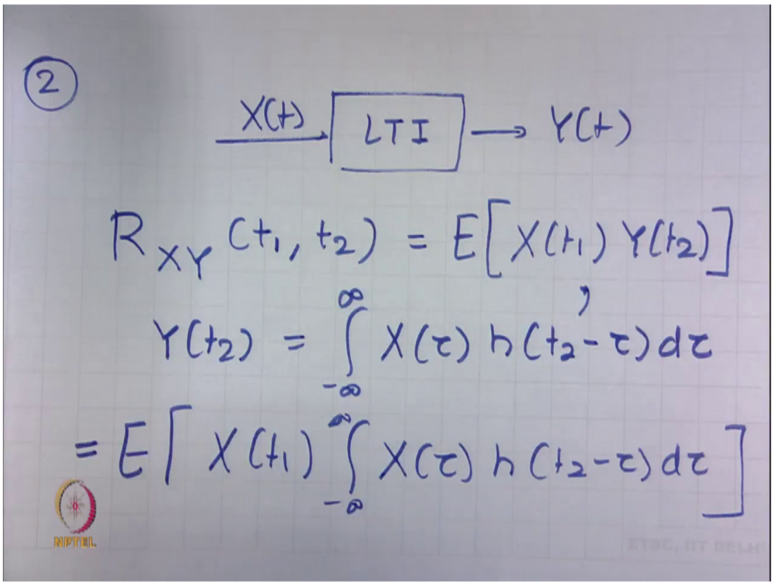

So, if again coming back to that picture so, that is the second key idea that we are going to think about. So if you have an LTI system as before and the input you have a random process X of t and the output, you have a random process Y of t. And now suppose we are asking a question what is the cross correlation function right? So, what is the cross correlation function? Let us say that we want to find it at two time instants t 1 and t 2. I hope you have become comfortable with this notation right. So, we are saying that this actually means that we want to find the expected value of X t 1 times Y t 2 ok. Y t 2 Y t 2 you can think that this is nothing, but X of tau h of t 2 minus tau and d tau right. So, that is what is Y t 2.

Now, plugging this into this expression we get this is nothing, but expected value of X t 1 times this quantity which is X tau h t 2 minus tau d tau ok. Then again the same trick as usual putting this integration to that side taking expected operator inside what we get is this is nothing, but same thing as minus infinity to plus infinity expected value of X t 1 times X tau.

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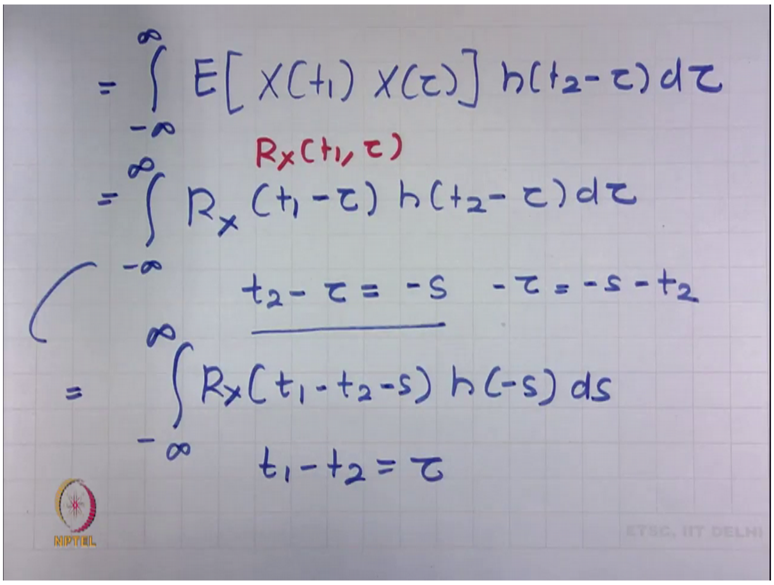

What we had is h of t 2 minus tau d tau i hope its correct let us check. Now, this quantity is nothing, but we can write this as R X t 1 minus tau why can we write this as this because we were given that X t is a wide sense stationary process. So, usually this has to be R X t 1 comma tau. But because it is a wide senses stationary process then absolute values of time can be replaced by just their difference. So, we get this from that that X is a wide senses stationary process

Now, you put t 2 minus tau as minus s. So, what does tau becomes then? So, tau becomes or minus tau let us say becomes minus s minus t 2. So, from this we get how does the limit change? We do not worry about it at this moment. So, R X t 1 minus tau is same thing as minus t 2 minus s this becomes minus s d tau. Let me write it in terms of s becomes d s and the limits remains unchanged. So, using this change of variable I can express this quantity as this quantity. Now if you see this carefully; let us put t 1 minus t 2 as tau.

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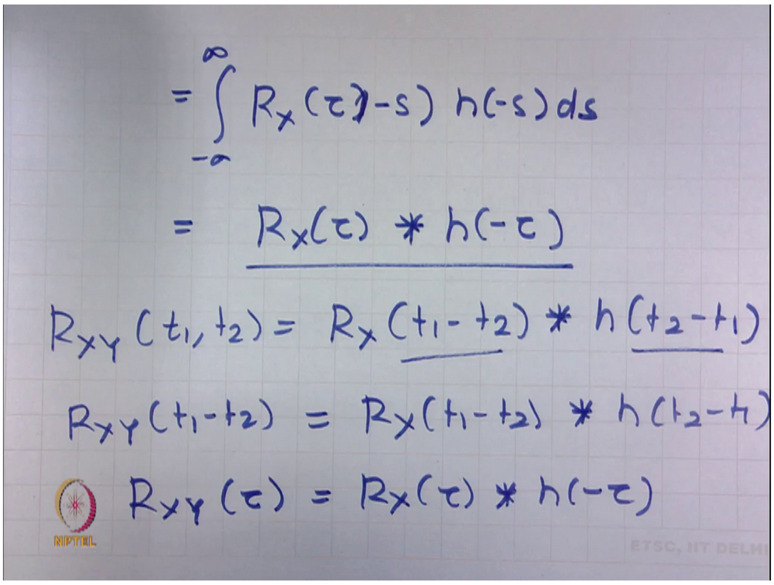

So, this quantity turns out to be, this becomes minus infinity to plus infinity R X tau. There has to be minus s R X tau minus s h of minus s d s now what is this is nothing, but it is the convolution of tau with h of minus tau ok.

So, what we were investigating originally is R X Y t 1 comma t 2 and we said that this is nothing, but it is R X tau. What is tau? Tau is t 1 minus t 2 convolution with h of t 2 minus t 1. So, the first thing that you need to appreciate is that this is not a function of t 1 and t 2 rather it is a function of the difference right. So, R X Y I can write this as t 1 minus t 2 because it is not a function of absolute values. There is no dependence on absolute values of t 1 and t 2, but rather it is a function of the difference right.

So, we can write this as this and now let us assume that let us assume again that t 1 minus t 2 is tau. So, we can say R X Y tau is R X tau convolved with h of minus tau. So, this is an important property that we have derived. Let us see the third and last important property about these X ts and Y ts and let us see can we get can we conclude the picture.

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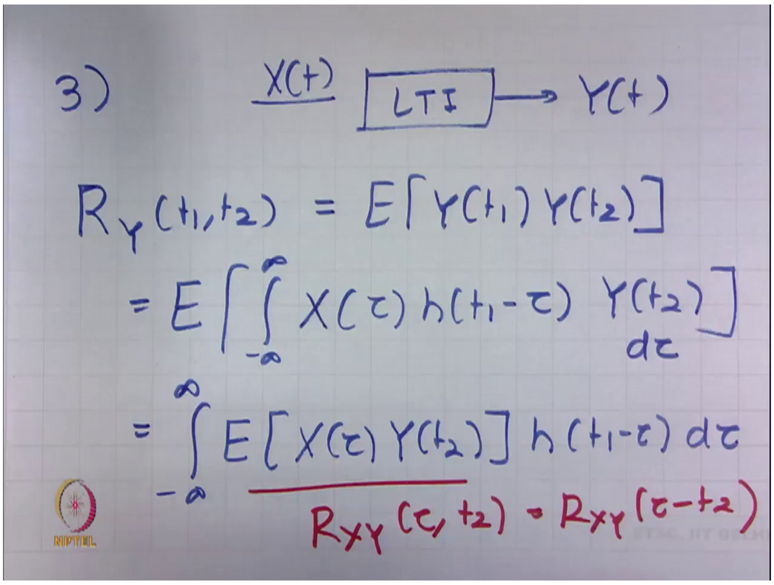

So, now the process remain same. So, we have X t as an input we have an LTI system. We have a Y t process at the output and now the question that we ask is what is the autocorrelation function of this Y t? So, we ask what is R Y t 1 t 2? We do not know yet whether it depends upon the absolute values or whether it will be a function of the difference of these timing instants. And we by definition right this is Y t 1 into Y t 2 is expected value. And this now Y t 1 we know that it can be written as minus infinity to infinity X tau h t 1 minus tau, then you have Y t 2.

Now what are the random variables here? X tau and Y t 2. So, we put it together, we get minus infinity to plus infinity. So, let me do it carefully X of tau we get Y t 2 h t 1 minus tau and there has to be d tau here; we missed it, so, we put d tau. Now what is this quantity? This quantity as we have said is R X Y; now this we have seen it already. So, normally it has to be tau and t 2, but because we have established that this is nothing, but R X Y tau minus t 2 ok.

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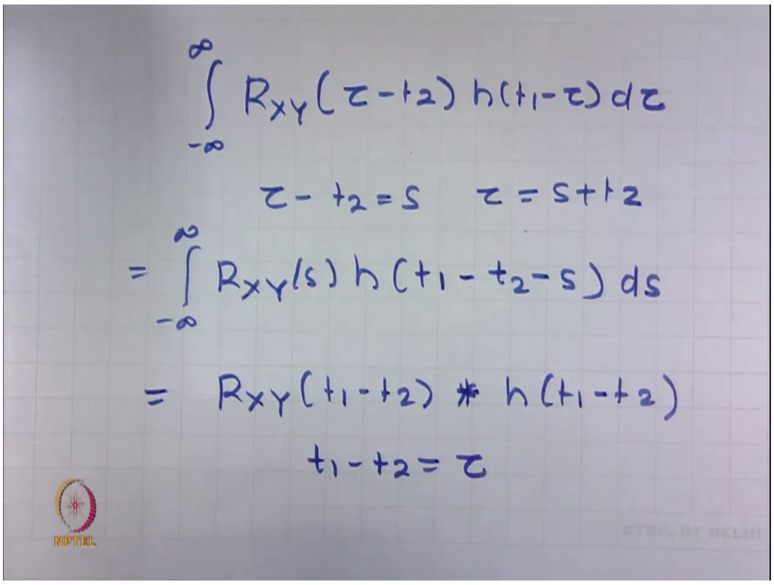

So, we will use this result. So, using this we can say that this is nothing, but R X Y tau minus t 2. What have you got more is h t 1 minus tau d tau. Now let us substitute tau minus t 2 as s. So, we get this as minus infinity to plus infinity R X Y s this will then become h t 1 minus. So, from this we can first write tau is s plus t 2 and this will become t 1 minus t 2 minus s d tau is the s. And as you can see this clearly this is nothing, but it is the R X Y t 1 minus t 2 convolution with h t 1 minus t 2 ok.

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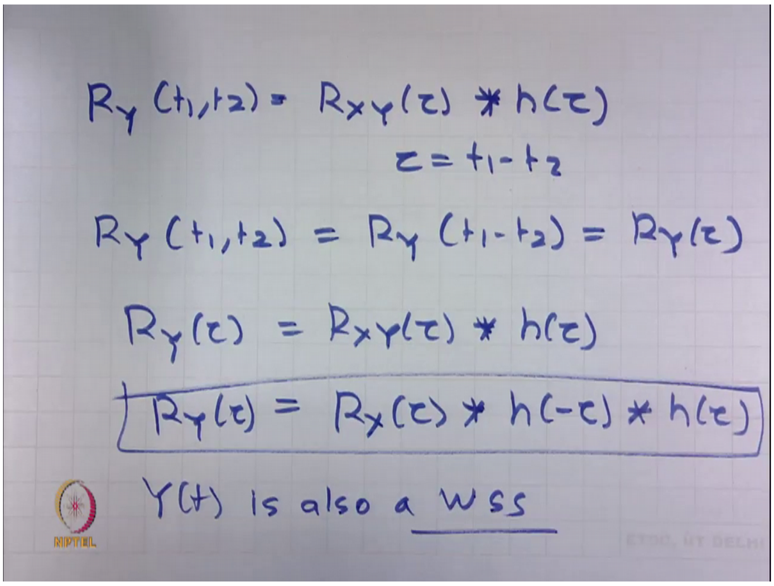

And again thinking that t 1 minus t 2 is tau we can write that this R Y t 1 t 2 is nothing, but it is R X Y tau convolution with h tau.

Now, one thing that you can appreciate is again this is not a function of absolute values of t 1 and t 2 rather its a function of tau and tau is tau is t 1 minus t 2. So, its a function of the difference of these timing instants. So, we can say that R Y t 1 t 2 is nothing, but it is R Y t 1 minus t 2 which is same thing as R Y tau. So, R Y tau is R X Y tau convolution of h tau and we have already proven in property number 3 that R X Y tau is itself R X tau convolution of h of minus tau convolution of h of tau. So, we have obtained an expression for the autocorrelation function of random process Y t.

First of all we have seen now that the autocorrelation function of a random process Y t is not depending on the absolute values of timing instants. As in the first property we have seen that the Y t mean is also constant; that means, the Y t satisfies all properties of wide sense stationary process. From these property 1 and 3 we can conclude that Y t is also a wide sense stationary process right. So, this is an important conclusion that if X t is a wide sense stationary process; if you pass this process through an LTI system, you end up with a wide sense stationary process with autocorrelation function defined like this.

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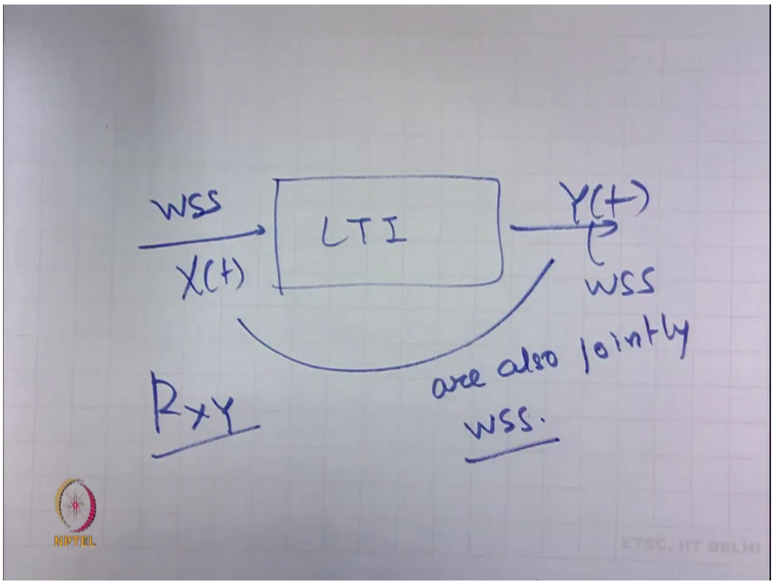

Final picture or today's lecture that if you have an LTI system if you excite this LTI system with wide sense stationary process X t. What do you end up with a random process, Y t which is wide sense stationary right. More ever, if you have defined R X Y when we were defining the cross correlation function, we also said that the cross correlation function is also not a function of absolute values of time, but rather it is a function of the difference of these sampling times. And thus we can say that these two process are also jointly wide sense stationary process right.

So, what are jointly wide sense stationary process? A jointly wide sense stationary process of the process that are individually wide sense stationary as it is in this case and moreover their cross correlation function is not a function of absolute values of time, but rather it is a function of the difference in the sampling time. So, with this we finished the interaction of a random process with LTI system and in today‘s lecture we have also defined various kinds of random processes.

And in the last lecture on random processes we will discuss about how can we think about all this things in terms of frequency domain. So, that is also a very valuable spect of trying to think about random processes in frequency domain.

Thank you.