Principles of Digital Communication
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Lecture – 12
Random Variables and Random Processes: Properties of Random Process

So, welcome to next lecture on Random Processes. Today's is lecture 6, in the unit on random processes and in the last lecture we defined what is a random process? And we have looked at certain examples of random processes. And what was the most interesting example that we saw in the last lecture it is that a random process can be expressed in terms of orthogonal functions and random variables ok.

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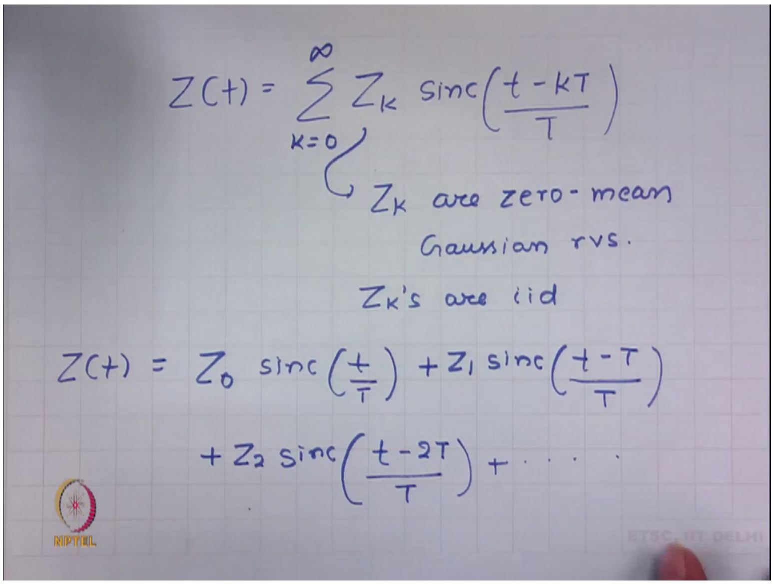

So, we continue with that idea was so strong, so, let us assume that now, I have a random process which is again expressed in terms of sinc functions. So, we have seen a similar example in the last lecture, but now what I have is these Z Ks are zero-mean Gaussian random variables and let us also assume them to be statistically independent of each other. So, Z Ks are let us say they are iid – independent and identically distributed random variables.

So, what can I say about this Z t? What is this Z t? Of course, I have seen that this provides me a framework to express random processes, but first let me see what does this mean is I am saying Z t is Z 0; so, putting K as 0, let me say K goes from 0 to infinity ok.

So, Z 0 I have sinc t by T then I have Z 1 sinc t minus T by T, then I have Z 2 sinc t minus 2T by T and so on so forth, I can continue right because the infinite terms.

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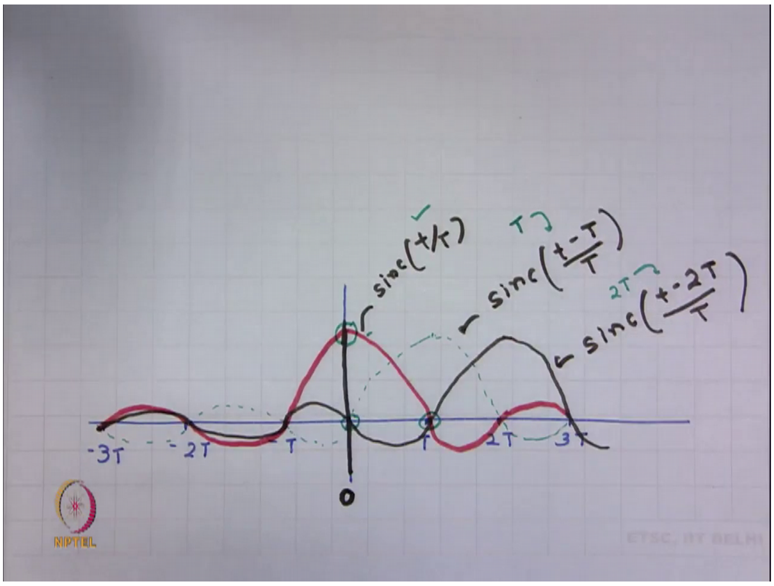

Now, if you look at this and try to draw this if I try to draw this. So, let me have; let me have a sinc function let me first put some markers. Let me draw now the sinc t by T ok. Now, I draw a sinc function which is shifted by T units. So, this will have a maxima here ok. Now, if I shift this sinc by 2T units it would be some function like this. So, this is my sinc t by T, this is my sinc; I am shifting my sinc by T units to the right and this is my sinc t minus 2T by T ok.

So, now if I investigate let us say at 0-th time what do I see? So, if I am interested in the sum at this time instance what you would see is I have only the contribution from the red curve and the contribution from green and black is 0, right. So, I just have contribution from red, contribution from green is 0, contribution also from the black curve is 0. So, at 0th time instance I have contribution only from sinc t by T.

Similarly if I look at the T-th instance the black and red curves are 0, their value is 0 at T and only I have contribution from green curve; that means, at T-th instance I only have contribution from sinc t minus T by T. Similarly I can go on and see that at 2T-th time instance I have contribution only from this term. So, let us look at this expression again. So, I have a Z t which is combination of various sinc terms and certain random variables.

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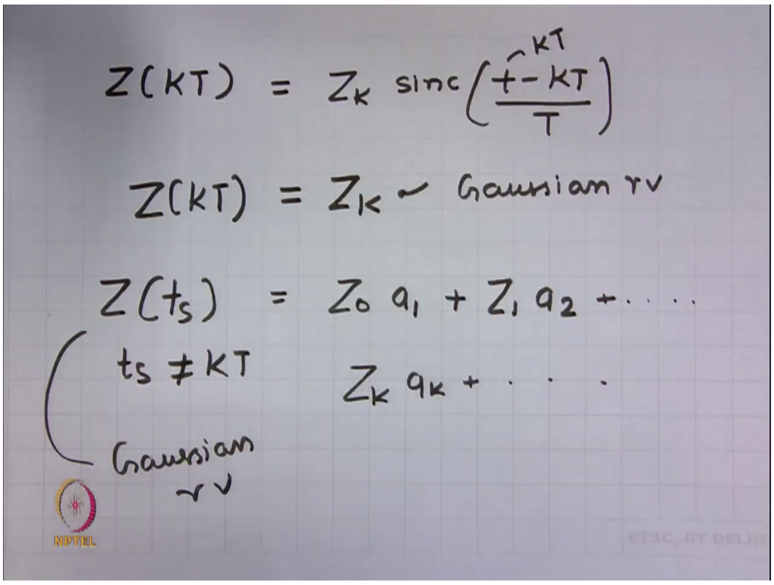

So, let me ask if I am interested in the value of this random process at K T time if I am interested in Z of K T right, where T is the let us say the sampling interval. So, now, this K Tk is some integer, so, this I would have contribution only from the sinc whose maxima lies at K T. This is a sinc whose maxima lies at K T. If I substitute t as K T here right what I would end up with is Z K. Now, important point is so, what I am saying is if I have a random process which I have thought by this expression if I am taking the value of or if I am sampling this random process at K T times what I would end up with is Z K and Z K is a Gaussian random variable. So, if a sample the process at K T time what you would get is a Gaussian random variable.

Let us know sample this process at some other time let us sample it as t s time, where t s is not K T because we have seen what happens if the sampling time is K T. If we do not sample this what you would have is Z 0 and you would have some contribution from this term, right. I do not know what that contribution is, but you would have some contribution because these terms goes to 0 only at K T times, right; at other time instances they are not 0. So, I would have some contribution let me say that contribution as a 1.

Similarly, I would have some contribution from second sinc and let me call this as a 2. Similarly I can go on and on I will have some contributions from all sincs. Now, what you would see is Z t s; what is Z t s? It is a random variable because I am looking down the random process at a specific time instance and this we have said is a random variable. So, Z t s is a random variable and now this random variable as you can see is nothing, but it is a linear combination of these random variable Z naught, Z 1 and Z K and they are assumed to be independent. So, Z t s is also Gaussian random variable ok.

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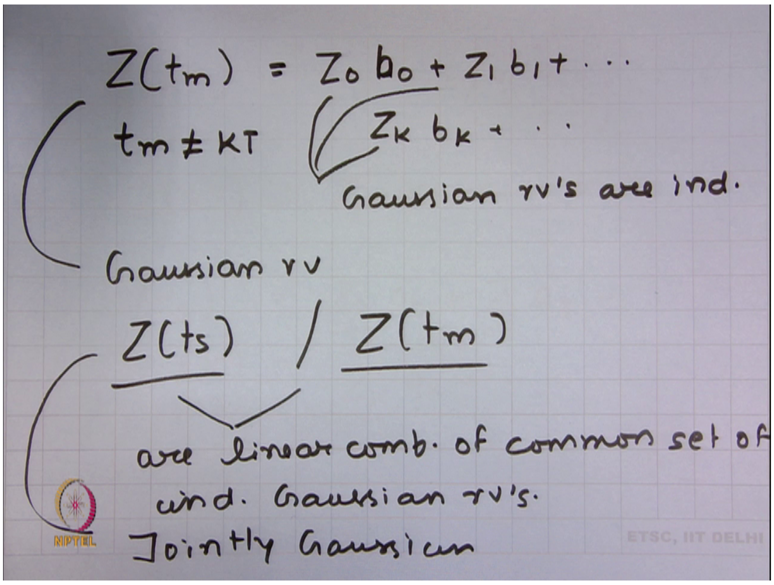

So, if I choose to sample this process at another time instance let us say t m, where t m is not K T what I would end up with is again let me make this as b 0, b K ok. So, what you would see is Z t m is again a linear combination of Z naught, Z 1 and Z K and what are these? These are as I have assumed they are Gaussian random variables more ever they are independent of each other, right.

So, if I have a linear combination of independent Gaussian random variables we have seen that the resulting random variable is also Gaussian. So, from this I can conclude that Z t m is also Gaussian random variable ok. So, we have seen the Z t s and Z t m both are Gaussian random variables, but there is something more hidden in them that is they are linear combinations of the common set of independent Gaussian random variables. So, these are linear combination of common set of independent Gaussian random variables, right. So, we will see later that these random variables are also thus jointly Gaussian, they are jointly Gaussian random variables.

What is jointly Gaussian we will study a lot about this jointly Gaussian, but this is a first introduction to water jointly Gaussian random variables. The jointly Gaussian random variables are linear combination of common set of independent Gaussian random variables ok. So, we have concluded some examples of random processes and we will see more examples of random processes as we go along.

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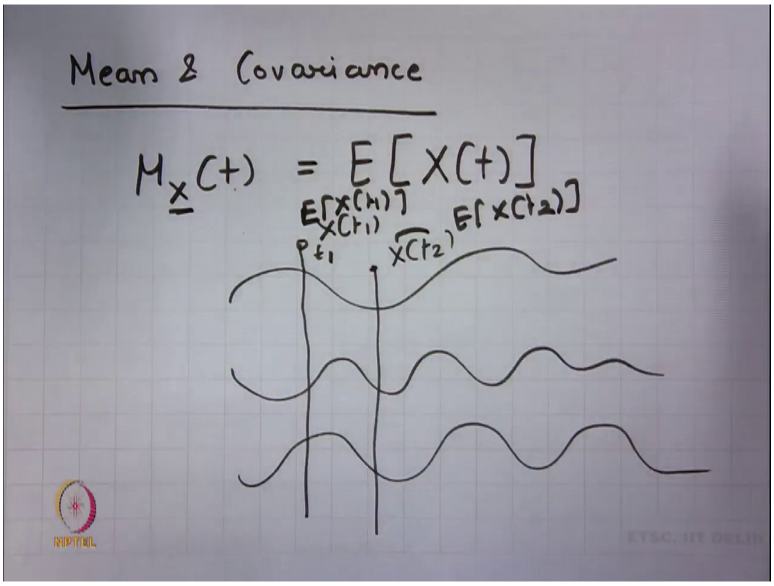

But, now I think its a time to cover the properties of these processes. So, let us talk about the mean and covariance, properties of random processes: so, we have done this for the case of random variables and now we would like to do it for random processes. So, the mean of a random process X t; so, mean of a process. So, X here stands for the process X of t. So, mean of this process is nothing, but it is the expected value of random process X t ok, this is very simple. In the mean of a random variable we were taking the expected value of a random variable, here the mean of a random process we have to take the expected value of a random process. So, let us see how does a random process look like? So, random process is made up of these various sample functions.

Now, if I look down these ensemble at a puc t 1. So, what I would get is a random variable right if I look down this ensemble at a particular time instance I get a random variable, so, this is X t 1. Similarly, I can look this town at and a puc t 2, I get another random variable. So, when I am calculating the expected value I get a certain expected value at this time instance this is mu. So, this is E of X t 1 and this expected value is E of X t 2. Now, as you can see that expected value is a function of time, right; expected value is what random variable you have in here and the random variable itself changes with time, right as you scan this time axis you are ending up with different random variables. And so, now, the expectation is a function of time. In case of random variable it was not a function of time because we were just looking down at a particular time instance, so, time was not varying here.

But, in the case of the random process you have to scan these time instances. So, you end up with different random variables as you move along the time instance and thus the expected value is also a function time good.

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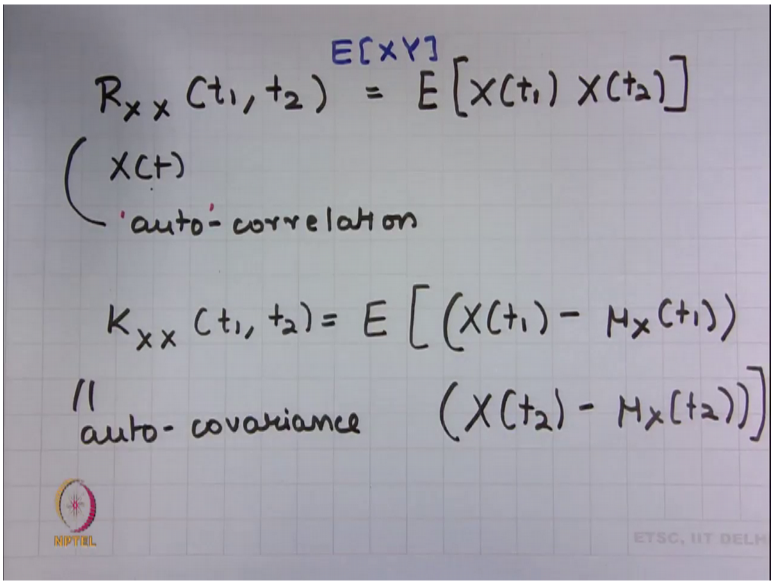

So, let us now define correlation in case of random processes. So, let me first explain the notation. Here I have a random process X t, so, X represents the random process X t. I sample this random process X t at time instance t 1, I get a random variable X t 1, I sample this random process at time instance t 2, I get a random variable X t 2 and I then find the expectation of the product of these two random variables. Remember, we have defined that the correlation is nothing, but finding the expected value of the product of two random variables.

Here in this case we are obtaining the two random variables by looking at a process at time instance t 1 and at the time instance t 2 and because these two random variables are formed by sampling the same random process this is also referred to as autocorrelation. So, auto specifies that the random variables obtained are obtained by sampling the same random process ok.

Let us now define auto covariance. So, what is the difference between covariance and correlation we have already seen in case of covariance we have to subtract the mean from the random variable. So, auto covariance can be obtained by subtracting the mean of the random variable X t 1, we denote that with this. So, this quantity represents that it is the mean of the random variable obtained by sampling the random process X t at time instants t 1 ok. So, we get this and then we have to obtain the second random variable and subtract from the second random variable its mean, we multiply these two things we take it is expectation and this will give us auto covariance. Again, auto means that the two random variables are formed by sampling the same random process ok.

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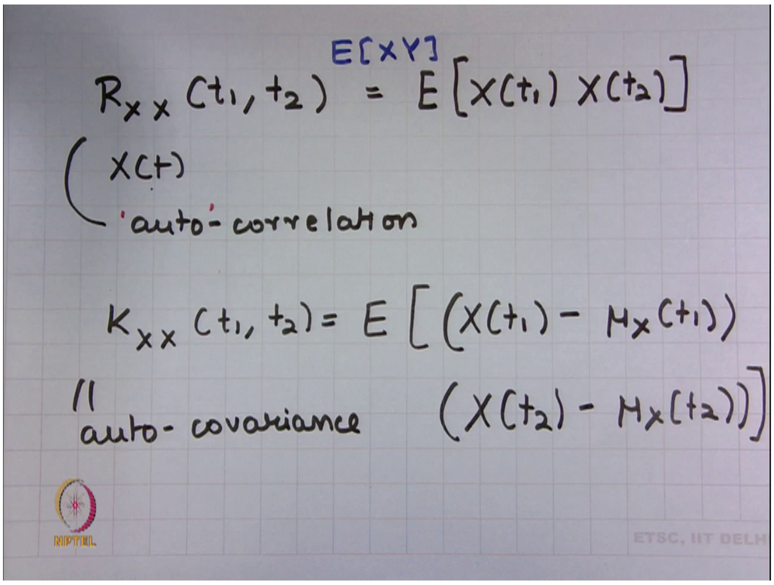

So, if there is auto there also has to be a cross, so we can define cross correlation. So, this is cross correlation and this can be obtained by forming two random variables, but now the main difference is that the first random variable is obtained by sampling this random process X t at time instance t 1. The second random variable is obtained by sampling this random process yt at time instance t 2 and these two random variables are formed by sampling two different random processes and hence the term cross correlation, we can similarly define cross covariance ok.

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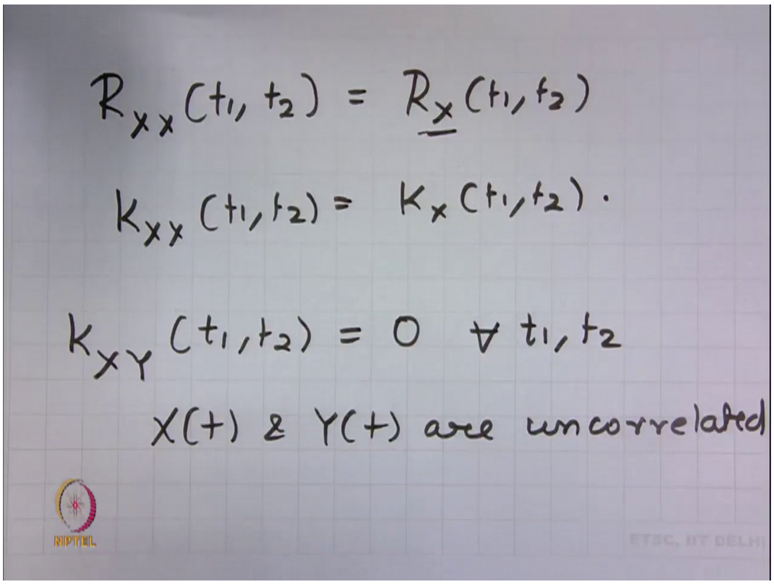

Now, sometimes in this case, when I am obtaining the two random variables by looking down at a single random process there are certain integrities. So, I can also write this as this because it is clear that I am looking down as a single process if I am having it just one random process I do not have to repeat this ok. So, this is also annotation that is sometimes used, so, you can also find these notation in different books ok, this is making things compact.

Now, something more interesting; so, in case of random variables we said the two random variables are uncorrelated if their covariance is 0. Here also the two random processes X and Y if their covariance is 0, if the covariance of two random processes is 0, but this has to be 0 for all t 1 and t 2, so, for all time instances for all epochs if the covariance of the two random process is 0, then we say that the random processes are uncorrelated. This is how we define uncorrelated random processes. So, let us understand this covariance with an example.

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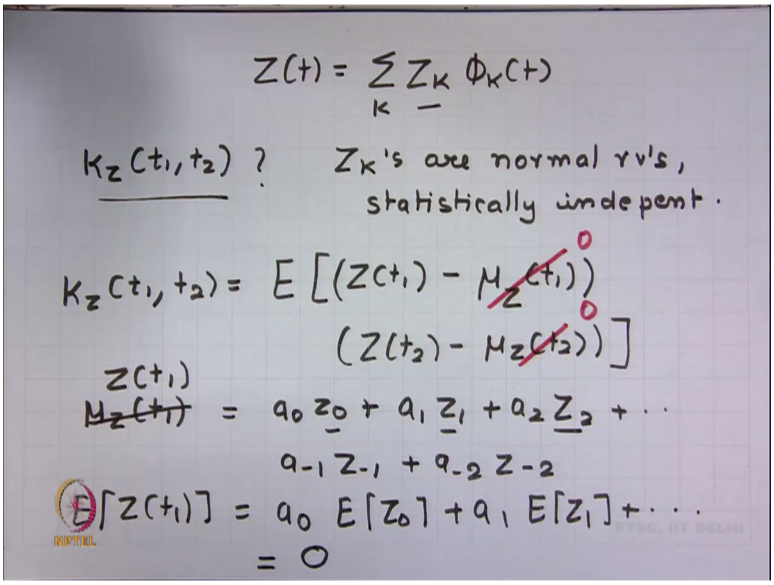

So, let us take a random process which we have taken several times and this is the random process where K takes all possible values, K can go from minus infinity to plus infinity. So, do not lose your sleep over whether K should go from 1 to infinity or from minus infinity to infinity as long as K takes a countable infinite values, it is fine, it does not make a lot of difference. So, from now onwards I will assume K goes from minus infinity to plus infinity and I would just denote that by writing K here; that means, K takes all possible values ok.

So, this is the random process that we have seen several times and now what I am interested in what is the covariance of this random process ok? Now, to simplify the calculation of the covariance let me make certain assumption; so, this is what we have to find and let me assume that Z K or Z K's are normal random variables and they are also statistically independent. So, this is what we assume. So, Z K's are normal random variables; normal random variables means it has a mean 0 and the variance of 1. So, this is given to us and we are interested in finding the covariance of this process.

So, let us start what is covariance? By definition the covariance is; so, you have to collect it random variable by sampling Z at time instance t 1 minus t mean of Z at time t 1 into Z of t 2 minus mu Z t 2. So, you have to find the expectation of this argument. So, let us now see what is mu Z t 1? So, if you recall in the example that we did just before this random variable collected at time instance t 1 would be some function it is a linear combination, so, it would be linear combination of these random variables. So, Z 1, Z 2 and Z 0 is nothing, but these that case you can also include negative part if you like.

So, what I am saying is the random variable, let us say if I collect a random variable at time instance t 1 this is nothing, but it is a linear combination of all these random variables. Now, if I am given that this is a zero mean random variable, this random variables. So, if I am interested in the expected value of this random variable, this is nothing, but a naught times expected value of Z 0 plus a 1 times expected value of Z 1 and so on so forth.

Now, because these random variables are given to us as 0 mean all these expected value is 0. So, from this we can conclude that expected value of Z at time t 1 is also 0 ok. So, now, this terms will vanish; this is 0 and this is also 0.

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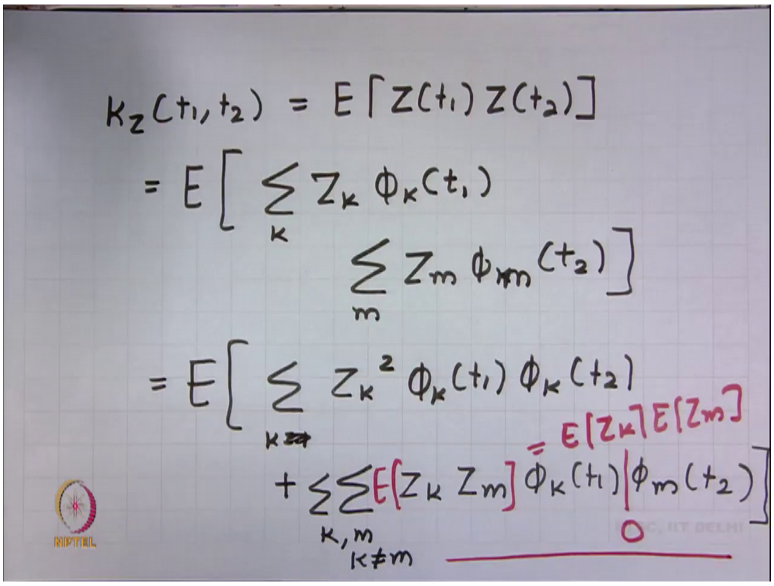

So, what we will get now is we have got a simplified expression that K Z t 1, t 2 is nothing, but expected value of Z t 1 times Z t 2; this looks much simpler now. Now, substituting this in this expression we get expected value of summation K Z K phi K t 1 because we are sampling this at time t 1 into summation m Z m phi K phi m t 2. Just see why do we have a different running variable here because if we would have chosen K, then it would not have been possible for us to collect cross terms. So, if you multiply this with this what you would end up with is Z K would be multiplied with some other terms as well.

So, that is why to collect cross terms we need to choose a different running variable than in this case. So, this is also what we did before. So, I can write this as summation K going from 1 to for all possible values of K, Z K square phi K t 1 times phi K t 2 plus summation, where K is not same as m, we can have two summation.

So, what I have written is now let us look at these terms carefully. So, what I am saying is when I multiply this summation with this summation what would happen is there would be same terms that would get multiplied. So, corresponding to that I would have Z K square this then would become phi K t 1, this would become phi K t 2 because m has taken the value which is same as K.

Now, you will also end up with having different terms. So, Z K would also be multiplied by Z m, where m is not same as K. So, you would collect different terms cross terms and you would also collect similar terms and now we have to take the expected value of this. Now, if you look carefully if I take expected operation inside expectation is a linear operation, so, this would be multiplied by this. So, expected value of Z K into Z m and if Z K and Z m are statistically independent this is nothing, but it is expected value of Z K times expected value of Z m, right. An expected value of Z K is 0, because these random variables has zero mean, so, this term would vanish, so, we would not have this term.

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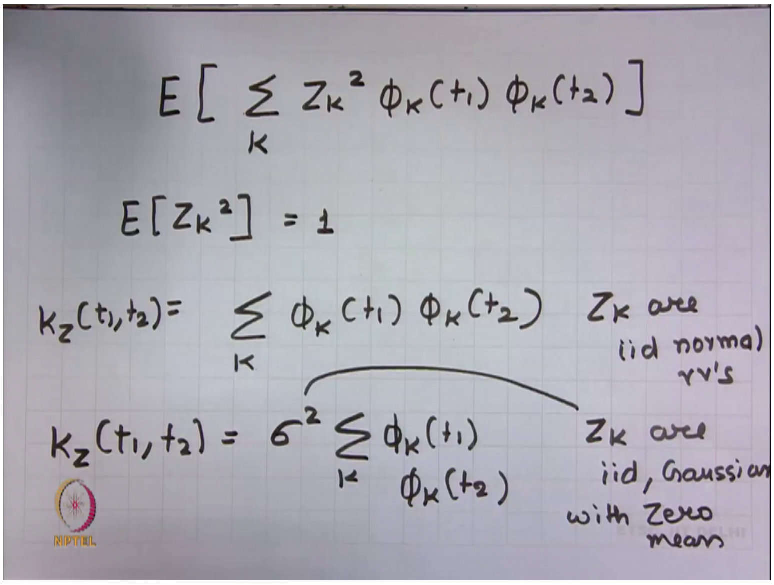

So, what we will be left with is a much simpler expression that expected value of for all values of K. Now, if you look this carefully what is this? So, this is nothing, but it is 1 because we have been given that these are normal random variables and this is nothing, but is the variance of normal random variables and the variance of a normal random variable is 1.

So, what we will end up with is just summation K phi K T 1 into phi K T 2. So, this is the covariance at time instance t 1 and t 2. Now, this we have obtained assuming that Z K are iid normal random variable. You can use the same steps and if we assume that Z K are iid Gaussian with zero mean; mean is 0, but variance is not 1, that is the difference. So, we did for a case where they were iid and normal. So, variance was one here you can generalize it to the case where this is Gaussian. So, still we are assuming zero mean because mean is not so important we can always plug in the mean or take out the mean that is not so important.

So, in this case if I am assuming this case you can prove that the covariance is nothing, but sigma square summation K by K t 1 times phi K t 2. A similar expression the only thing that changes here is instead of 1, I would have sigma square where sigma square will be the variance of these random variables.

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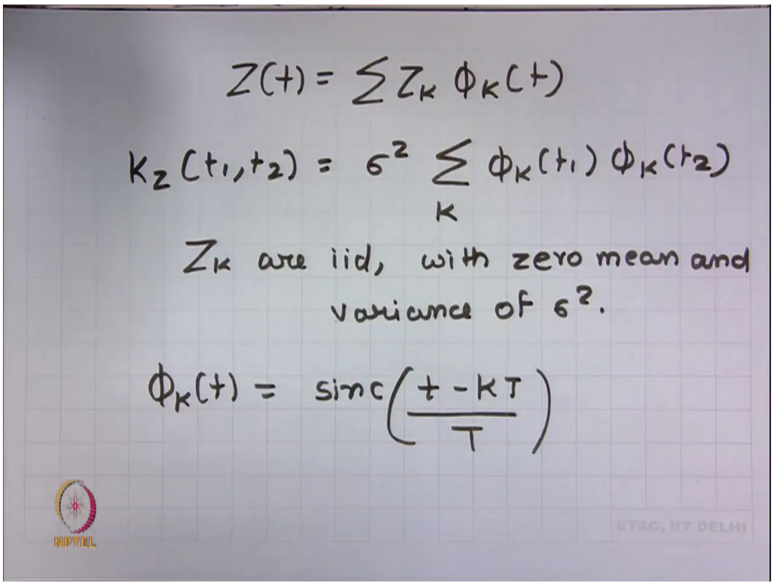

Let me write down this result again that if I have a process, then the covariance function of this process is sigma square times summation K phi K t 1 times phi K t 2 ok. If I am given that Z K are iid with zero mean and variance of sigma square ok.

Now, let us assume that phi K t. So, now, if we are specializing it to the case where phi K t is nothing, but the sinc function. So, as in the previous examples also I have illustrated the case where we assume this phi K t as a sinc function. So, what is the covariance; what is the covariance? If my phi K t takes a sinc form, let us see, it is interesting.

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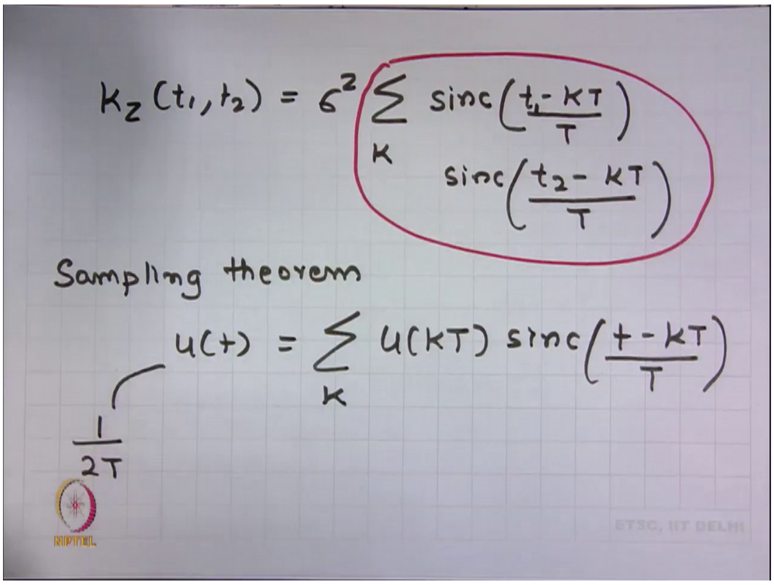

So, we would have K Z t 1 t 2 sigma square summation K would have sinc, it is t minus K T by T. So, this is taking time t 1, so, we would have t 1 here into sinc t 2 minus K T by T.

Simplifying this part requires some trick and the trick that we would use is the sampling theorem. So, if I state again the sampling theorem which we discussed before so, sampling theorem says that if I have a function or a signal which is band limited to 1 by 2T. So, if this function is band limited to 1 by 2T, then I can think about the signal by putting sinc caps around the samples collected at integer multiples of T and again the K takes all possible values ok. So, this is the sampling theorem you can think about the signal from it is sample. So, u KT are the samples of u t collected at integer multiples of T and around these samples you need to put the sinc cap and then you can recover back u t. So, this is the sampling theorem it was if u t is band limited to 1 by 2T.

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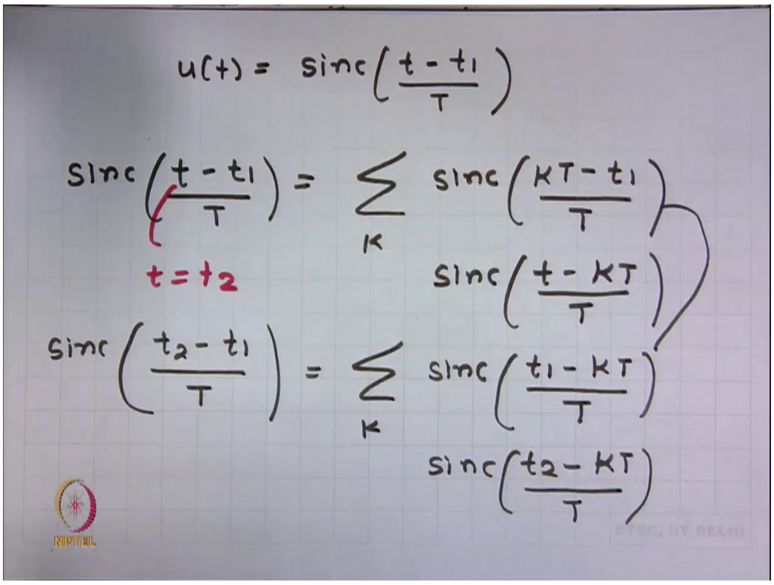

So, using sampling theorem let us assume that u t is sinc t minus t 1 by T. So, using sampling theorem then I can say that this is sinc t minus t 1 by T should be nothing, but sinc I have to substitute t as KT minus t 1 by T into sinc t minus KT by T. So, just using the sampling theorem; so, from sampling theorem I have this and then I have substituted u t as this function, then I get such an expression.

Now, if we look carefully and if I now choose t as t 2 what I would get is sinc t 2 minus t 1 by T is summation K sinc; now, I make one more change here because sinc is an even function. So, whether I write this as KT minus t 1 or t 1 minus K t does not matter. So, I am using the property that sinc is an even function, so, I can write this as this into sinc. Now, this t has been replaced by t 2, so, we would have this.

Now, this was the summation in which we were interested. So, let me go back and let us see. So, I was interested in calculating this summation and now I exactly have this summation and from the sampling theorem I can say that this summation is nothing, but this quantity.

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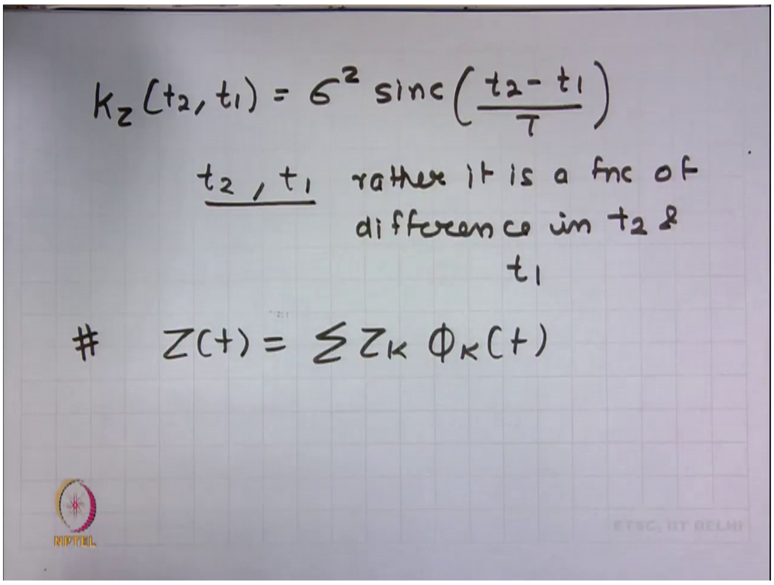

So, covariance becomes sigma square times sinc t 2 minus t 1 by T. So, I am replacing the whole summation this summation with the simpler quantity. So, using this I have got this ok, so, this is the final result. So, we have started with a process let me show you the process again. So, we have started with this process and after starting with this process, assuming that this Z K's are iid with zero mean and variance of sigma square we have calculated this covariance; where covariance we found out in terms of orthogonal functions. Then specializing these orthogonal functions to take sinc form the final answer that we have got is the covariance is nothing, but this.

Now, there is something interesting that has happened here and that is now this covariance function is not dependent on t 2 and t 1. It is not a function the covariance function is not a function of t 2 and t 1 rather it is a function of difference in t 2 and t 1. Such processes are very important processes they fall under the umbrella of what are known as a stationary process, right.

So, we will look about this and then we will define more carefully what I mean by a stationary process, but let me give you some hint that such processes where the covariance is not a function of absolute values of time, but rather it is a function of difference of the time instances. Those classes of processes are really important processes and useful processes and these are the kinds of processes with which we work, ok alright.

So, let us now quickly revisit what we have done today. So, we have started with an example of random process which is a key example of a random process, where the random process is expressed in terms of orthogonal functions and we have also defined the mean and covariance of a random process. And we have looked as what happens if you take this as a random process what happens to the covariance of such a random process. And there we have noted something very interesting that if you take these orthogonal functions as sinc functions, something interesting happens is that a covariance does not depend upon the absolute values of time rather it is a function of difference in time instances.

So, with this we will conclude this lecture and in the next lecture, we will continue with the discussion of random processes and we will they discuss a very important in significant class of random process and that is a Gaussian process.

Thank you.