

Information Theory, Coding and Cryptography
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Module – 34
Space Time Codes
Lecture – 34


Hello and welcome, to our next lecture on Space Time Codes.

(Refer Slide Time: 00:32)

Information Theory, Coding and Cryptography

Outline

- Real Orthogonal Design
- Generalized Real Orthogonal Design
- Complex Orthogonal Design
- Generalized Complex Orthogonal Design
- Examples

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2

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Let us start with a brief outline of today's lecture. What we will study today are real orthogonal designs for space time block codes, then we will look at the generalization of real orthogonal designs, then we will move on to the domain of complex orthogonal design and then we would like to generalize that concept. Finally, we will look at some examples.

(Refer Slide Time: 00:57)

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Recap

- Concept of Space Time Codes
- Alamouti Code
- Diversity
- Examples

3

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Let us do a quick recap as to what we have done so far. We have already introduced the concept of space time codes and how it gives diversity gain. We specifically looked at Alamouti code as a very interesting example of space time block codes. We dealt into diversity gain and coding gain and we looked at some examples along the way.

(Refer Slide Time: 01:22)

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Anatomy of a STBC

- Lets first look at a simple example.

$$C = \begin{bmatrix} s_1 & -s_2^* \\ s_2 & s_1^* \end{bmatrix}$$

time space

Illustration of a simple space-time block code.

3

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Let us revisit the anatomy of a space time block code STBC for short and what we realized was that an information source pumping in bits of information what we do is we take 2b bits at a time and of this first b bit is used to select one signal from the signal

constellation and then the next b bits are used to pick up another signal s_2 from the signal constellation. So, I have s_1 corresponding to the first b bits and s_2 corresponding to the next b bits, thus we have a tuple s_1, s_2 which is coming out from this signal constellation block and this is fed to the space time encoder.

In this example we have two transmit antennas therefore, we picked up 2 symbols s_1 and s_2 . What they do is load symbol s_1 onto the element 1 and load s_2 onto antenna element 2. So, Tx 1 sends out s_1 Tx 2 sends out s_2 . There is only one receiver because we have argued that in most cases it is difficult to put multiple antennas on the receiver, but you can have multiple antennas. In this example we are only looking at one single receiver antenna element. So, the s_1 goes through a channel gain of h_1 whereas, s_2 undergoes a channel gain of h_2 and we receive it and of course, we have additive white Gaussian noise added up n_1 .

But, this we do in the first time slot as a name suggests it is a space time block code. So, the space element is coming from the antenna and the time is the different time slots that we are going to send out. So, we sent out s_1 and s_2 from the two antenna elements in the first time slot. Now, for those same $2b$ bits we have picked up s_1 and s_2 , but in the second time slot what we do is we send out $-s_2^*$ and s_1^* . s_1 and s_2 belong to a complex constellation. So, we have the complex conjugates here.

So, what we do is we send out from antenna element 1 $-s_2^*$ and s_1^* this is just an example I can have other ways of doing it and then we send out. These two go through the channel gains and we get noise n_2 added and so, we have two received signals in time slot 1 and time slot 2 of course, we need the estimates of the channels h_1 and h_2 and then we have a combining strategy which goes through the ML decoder.

(Refer Slide Time: 04:19)

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Example

- A frame of data lasts **two symbol periods**.
- During symbol period 1, antenna 1 and antenna 2 transmit s_1 and s_2 respectively.
- During symbol period 2, antenna 1 and antenna 2 transmit $-s_2^*$ and s_1^* respectively.
- For better visualization, we can write as

	Antenna 1	Antenna 2
Time period 1	s_1	s_2
Time period 2	$-s_2^*$	s_1^*

Alamouti Code

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So, in this example we have talked about the space antenna 1 and antenna 2 and time; time period 1 and time period 2, so, we have this matrix of symbols that we want to transmit and this is kind of the code the space time block code that is being used.

(Refer Slide Time: 04:39)

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Example

- We note that

$$\mathbf{X}^H \mathbf{X} = (|x_1|^2 + |x_2|^2) \mathbf{I}_2$$

where \mathbf{I}_2 is a 2×2 identity matrix.

- The channel from antenna 1 to the receiver (h_1) and antenna 2 to the receiver (h_2) can be modeled as

$$h_1 = \alpha_1 e^{-j\phi_1}$$

$$h_2 = \alpha_2 e^{-j\phi_2}$$

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Now, we make some observations because if you represent \mathbf{x} with those matrix then $\mathbf{X}^H \mathbf{X}$ Hermitian \mathbf{X} happens to be $|x_1|^2 + |x_2|^2$ plus \mathbf{I}_2 , where \mathbf{I}_2 is the identity matrix.

Now, the channel gains h_1 and h_2 are complex and they can be modeled as $\alpha_1 e^{j\phi_1}$ and $\alpha_2 e^{-j\phi_2}$.

(Refer Slide Time: 05:11)

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Example

- It is assumed that the fading channel coefficients h_1 and h_2 are constant across two consecutive symbol periods (is that a fair assumption?).
- The received signal over the first symbol period, denoted by r_1 , can be expressed as

$$r_1 = h_1 s_1 + h_2 s_2 + n_1$$
- Similarly, the signal over the second symbol period, denoted by r_2 , can be expressed as

$$r_2 = -h_1 s_2^* + h_2 s_1^* + n_2$$

where n_1 and n_2 are AWGN samples for the two symbol periods respectively.



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Alamouti Code

And, we can multiply it out to get the first received signal in time slot 1 as a r_1 , this is simply s_1 with channel gain h_1 plus s_2 with channel gain h_2 plus n_1 . Similarly, in the second time slot we have $-h_1 s_2^* + h_2 s_1^* + n_2$ here it should be s_1 . So, we have this received signal r_2 in the second time slot. Now, the observation right here is that r_1 depends both on s_1 and s_2 and r_2 also depends on s_2 and s_1 , simultaneously.

(Refer Slide Time: 05:47)

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At the receiver


- The aim of the receiver is to extract the transmitted symbols (x_1 and x_2) from r_1 and r_2 .
- The receiver combines the received signal according to the following **combining scheme**

$$\tilde{r}_1 = h_1^* r_1 + h_2 r_2^*$$
$$\tilde{r}_2 = h_2^* r_1 - h_1 r_2^*$$

- Note that the receiver must have an estimate of h_1 and h_2 in order to implement the combining scheme.
- Both h_1 and h_2 are complex.
- Hence we obtain

$$\tilde{r}_1 = (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*$$
$$\tilde{r}_2 = (|h_1|^2 + |h_2|^2) s_2 - h_1^* n_2 + h_2 n_1^*$$

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But we have this interesting combining scheme wherein we have \tilde{r}_1 equal to h_1^* complex value, complex conjugate r_1 plus h_2 into r_2^* , which means that if we have the knowledge about the channel gains h_1 and h_2 which are complex we can use this combining scheme to get \tilde{r}_1 and similarly, \tilde{r}_2 .


But, if you workout this then \tilde{r}_1 comes out to be this expression and \tilde{r}_2 correspondingly gets this one. The interesting observation is \tilde{r}_1 only depends on s_1 . So, suddenly with this combining scheme we have been able to decouple the decoding. So, \tilde{r}_1 is only dependent on s_1 and \tilde{r}_2 is only dependent on s_2 . So, we can use the maximum likelihood decoding strategy and recover using the single symbol decoding.

(Refer Slide Time: 06:47)

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Design Criteria for STBC

- The **Rank and Determinant Criteria** for designing space-time block codes is given by
- In order to achieve maximum diversity, the matrix $A(C^i, C^j)$ should be of full rank for any two code words, $C^i \neq C^j$.
- The smallest value of r , over any pair of code words, provides a diversity gain of rM . [**rank criteria**]
- In order to achieve maximum coding gain, the minimum determinant of the matrix $A(C^i, C^j)$ should be maximized for any two code words, $C^i \neq C^j$. [**determinant criteria**]

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In the last class, we therefore, came up with this rank and determinant criteria for designing space time block codes and what we came up with is the rank and the determinant criteria. So, we defined this matrix A and we said that it should be full rank for any two code words C^i not equal to C^j and the smallest value of the r rank over any pair of code words provides a diversity gain of r into M , where M is the number of received antennas.

This is the rank criteria and the determinant criteria says that in order to achieve maximum coding gain, right. So, we have already talked about the diversity gain, but in order to achieve the maximum coding gain the minimum determinant of the matrix A should be maximized for any two code words which are not equal.

(Refer Slide Time: 07:50)

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Alamouti Code - revisited

- It is a coding scheme with $N = 2$ transmit antennas and the transmitted code word is

$$\mathbf{C} = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}$$

- Consider a different pair of symbols with the corresponding code word matrix

$$\mathbf{C}' = \begin{bmatrix} s'_1 & s'_2 \\ -s'_2^* & s'_1^* \end{bmatrix}$$

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So, we quickly revisit the Alamouti scheme with N is equal to two transmit antennas and this is the code further Alamouti scheme and if we have another pair of symbols and this matrix is denoted by C prime as follows.

(Refer Slide Time: 08:08)

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Performance of Alamouti Code

- It is clear from the figure that the Alamouti code provides a much better performance due to the **diversity gain** (observe the slopes of the curves).
- The SEP decreases as S^{-2} , where S is the SNR.
- Thus, the effect of the diversity gain becomes more pronounced at **higher SNRs**.
- This is evident from the fact that the performance gap **widens** as the SNR increases (SEP decreases).

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Then, we can go and find out how the diversity gain is obtained from the Alamouti code. So, diversity gain is obtained simply by looking at the slopes of the BER curves. So, on the x axis we can have the SNR, y axis we have the symbol error rate, if you will and

then if you look at with and without Alamouti code you see a distinct change in the slope which indicates the a diversity gain.

(Refer Slide Time: 08:43)

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Single Symbol Decoding

- We have seen in Alamouti Code that the optimal decisions can be made based on single symbol at a time
- So, what is so unique about this code?
- The received pair may be succinctly written as

$$\begin{pmatrix} r_1 & r_2^* \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \end{pmatrix} \Omega + \begin{pmatrix} n_1 & n_2^* \end{pmatrix}$$

- Where

$$\Omega = \Omega(h_1 \quad h_2) = \begin{bmatrix} h_1 & h_2^* \\ h_2 & -h_1^* \end{bmatrix}$$

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Now, a few words about single symbol decoding; We have realized the uniqueness of this Alamouti code because this pair r_1, r_2^* is dependent only on s_1 and s_2 independently.

(Refer Slide Time: 09:02)

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Single Symbol Decoding

- Note that complex conjugate of both sides has been taken prior to rewriting in the matrix form.
- Upon multiplying both sides by Ω^H , we obtain

$$\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} r_1 & r_2^* \end{pmatrix} \Omega^H = \left(|h_1|^2 + |h_2|^2 \right) \begin{pmatrix} s_1 & s_2 \end{pmatrix} + N$$

where N can also be represented as Gaussian noise.

- This actually represents two separate equations that can be used for decoding the two transmitted signals using simple ML decoding.

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So, you can have this estimates of x_1 and x_2 as simply as $|h_1|^2 + |h_2|^2$ s_1, s_2 . So, basically what it means is that your x_1 tilde

depends only on s_1 and x_2 depends only on s_2 , ok. So, this is the single symbol decoding which reduces drastically the complexity and the decoder.

(Refer Slide Time: 09:35)

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Single Symbol Decoding


- Further, observe that

$$\Omega\Omega^H = (|h_1|^2 + |h_2|^2)\mathbf{I}_2$$

where \mathbf{I}_2 is a 2x2 identity matrix.

- Alternately, we can represent the **generator matrix** of the code by

$$\mathbf{G} = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & -x_1^* \end{bmatrix}$$

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So, you observe that this sigma sigma Hermitian is nothing, but $|h_1|^2 + |h_2|^2$ squared \mathbf{I}_2 and the generator matrix for the code is simply given by \mathbf{G} . So, throughout this lecture this is how we will denote the generator matrix of the code for different cases. This is a 2 cross 2 complex design, but we will look at real designs and complex designs in our subsequent slides.

(Refer Slide Time: 10:12)


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Single Symbol Decoding

- It is easy to see that

$$\mathbf{G}\mathbf{G}^H = (|x_1|^2 + |x_2|^2)\mathbf{I}_2$$

- As noted before, the beauty of this generator matrix is that it can **decompose the decision problem** at the receiver such that decisions can be made based on a **single symbol** at a time.
- This is possible using **orthogonal design**

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So, what is interesting about this generator matrix is if you take GG^H Hermitian you come up with a scaled identity matrix and this interesting fact allows us to decouple the decoding problem and we do symbol by symbol decoding we have one single symbol at a time which is required to take the decision. And the reason is because we have an orthogonal design. So, GG^H Hermitian is simply this identity matrix multiplied by this term.

So, the key part is this orthogonal design and now, let us focus on what good orthogonal designs are available, how to go about doing it, do orthogonal designs of all size exist or not let us look at these questions.

(Refer Slide Time: 11:15)

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Real Orthogonal Design

- **Definition** A **Real Orthogonal Design** of size N is an $N \times N$ matrix $G = G(x_1, x_2, \dots, x_N)$, with entries consisting of **real elements** drawn from $\pm x_1, \pm x_1, \dots, \pm x_N$ such that

$$G^T G = \left(\sum_{i=1}^N x_i^2 \right) I_N$$

where I_N is a $N \times N$ identity matrix.

- A real orthogonal design exists if and only if **$N = 2, 4$ and 8** .
- We note that G is proportional to an **orthogonal matrix**.

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So, let us now start on real orthogonal design. So, let us start with the definition a real orthogonal design of size N is an N cross N matrix G . So, the moment we are trying to define a space time block code which is a real orthogonal we would simply represent it with a matrix G which is the generator matrix with entries consisting of only real elements drawn from plus minus x_1, x_2 so and so forth, till x_N such that $G^T G$ is summation I is equal to $\sum_{i=1}^N x_i^2$ times this identity matrix of size N cross N .

So, what is very interesting is it can be shown that only when N is 2, 4 or 8 do you have a really orthogonal design that this is possible otherwise you simply cannot have this

condition being satisfied. We also note that G is proportional to an orthogonal matrix, ok. So, I can have a proportionality a constant also ahead in front of you.

(Refer Slide Time: 12:26)

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Example


- Real orthogonal design for $N = 2$ is

$$G_2 = \begin{bmatrix} x_1 & x_2 \\ -x_2 & x_1 \end{bmatrix}$$

- For $N = 4$ is

$$G_4 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ -x_2 & x_1 & -x_4 & x_3 \\ -x_3 & x_4 & x_1 & -x_2 \\ -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$$

- and for $N = 8$ is

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Let us look at these possibilities for N is equal to 2, 4 and 8. So, if you have generator matrix G_2 as follows then you can satisfy yourself by taking G_2 transpose into G_2 and you will get that there will be terms only along the diagonal and there will be 0's off diagonal.

Similarly, this is the design for N is equal to 4. So, please note the convention is the same this axis is a space axis, this axis is the time axis. So, if I were to implement a real life system with G_4 I will divide my implementation into four time slots. In the first time slot I will send out x_1 through antenna element 1, x_2 through antenna element 2, x_3 from antenna element 3 and x_4 from antenna element 4 and then wait for the next time slot wherein I will send out minus x_2 , x_1 , minus x_4 and x_3 and so and so forth for the four time slot.

So, in four time slot I have been able to send out 4 symbols x_1 , x_2 , x_3 and x_4 . So, therefore, the rate is also 1, I have not compromised on the rate, right, but what is interesting is if you take G_4 transpose into G_4 you will again get the values along the diagonal and there will be 0 terms off diagonal. So, again this is an orthogonal design which means you can happily go ahead and do single symbol decoding. So, the decoding complexity is again quite low.


(Refer Slide Time: 14:31)

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Example

Real orthogonal design for $N = 8$ is

$$\mathbf{G}_8 = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 & x_8 \\ -x_2 & x_1 & x_4 & -x_3 & x_6 & -x_5 & -x_8 & x_7 \\ -x_3 & -x_4 & x_1 & x_2 & x_7 & x_8 & -x_5 & -x_6 \\ -x_4 & x_3 & -x_2 & x_1 & x_8 & -x_7 & x_6 & -x_5 \\ -x_5 & -x_6 & -x_7 & -x_8 & x_1 & x_2 & x_3 & x_4 \\ -x_6 & x_5 & -x_8 & x_7 & x_2 & x_1 & -x_4 & x_3 \\ -x_7 & x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 & -x_2 \\ -x_8 & -x_7 & x_6 & x_5 & -x_4 & -x_3 & x_2 & x_1 \end{bmatrix}$$

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If you look at for N is equal to 8 again this is a real design and it is a real orthogonal design you can verify \mathbf{G}_8 transpose into \mathbf{G}_8 will again give you a only non-zero elements along the diagonal. And this is again the simple strategy I have got 8 antenna elements right and here are 8 time slots and this precisely tells you what to send at what time slot on what antenna, ok. Again, single symbol decoding is possible. Unfortunately, beyond eight we have no designs possible, it is possible to prove it.


(Refer Slide Time: 15:14)

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Beyond Real Orthogonal Design

- It is known that the real orthogonal designs exists only for $N = 2, 4$ and 8 .
- In order to have more designs, we generalize the orthogonal designs to **non-square real matrices** of size $T \times N$.
- In these designs, the number of **time periods (T)** is not necessarily equal to the number of transmit **antenna elements (N)**.

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So, it is known that real orthogonal designs exist only for N equal to 2, 4 and 8. So, how do we go beyond that? We have to generalize the orthogonal designs to non square real matrices of size T cross N earlier it was N cross N , but suddenly I want to have T which means I want to increase the number of time slots. So, it might take a hit on the rate. So, the number of time periods is denoted by T and the number of antenna elements is N and T is not necessarily equal to N .

So, clearly for the N is equal to 2, 4 and 8 cases, the generator matrix was a square matrix, N was equal to T , but since we have no more such matrices for larger values of N we now resort to T cross N . So, these are non square matrices.

(Refer Slide Time: 16:17)

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Generalized Real Orthogonal Design

- A **Generalized Real Orthogonal Design** is a $T \times N$ matrix with entries consisting of **real elements** drawn from $\pm x_1, \pm x_2, \dots, \pm x_K$ such that

$$\mathbf{G}^T \mathbf{G} = \left(\sum_{i=1}^K x_i^2 \right) \mathbf{I}_N$$

- where \mathbf{I}_N is a $N \times N$ identity matrix and K is a constant that represents the number of distinct symbols being transmitted.
- The rate R is defined as

$$R = \frac{K}{T}$$



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So, what is the definition? A generalized real orthogonal design is a T cross N matrix. So, if you have to design a generalized real orthogonal design we have to just come up with a matrix which is T cross N , but all the elements of this matrix all the entries are real drawn from plus minus x_1 plus minus x_2 up to plus minus x_K such that $\mathbf{G}^T \mathbf{G}$ is again equal to \mathbf{I} is equal to $\sum_{i=1}^K x_i^2 \mathbf{I}_N$, where \mathbf{I}_N is a N cross N identity matrix, ok.


So, again only the diagonal terms exist rest are nonzero, rest all off diagonal terms are 0 only the diagonal terms are nonzero. But, clearly, now the rate has to be defined and rate is K over T , right. So, T is of course, larger than K and therefore, it is possible that the rate will be less than 1.

(Refer Slide Time: 17:26)

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Generalized Real Orthogonal Design

- A generalized real orthogonal design with rate $R = 1$ is called a **Full Rate Orthogonal Design**.

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But of course, in real life we would like the rate to be as close to 1, if not 1. So, a generalized real orthogonal design with rate R is equal to 1 it is called a full rate orthogonal design, ok; so, if we are still in the domain of generalized real orthogonal design, but we are not talking about full rate design.

(Refer Slide Time: 17:54)

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Generalized Real Orthogonal Design

- Since space time block codes from orthogonal designs have $T = K = N$, the rate $R = 1$, and they form a special case of the **generalized real orthogonal design**.
- Generalized real orthogonal designs provide **full diversity** and **separate decoding of symbols**.
- A real space time block code is defined as one that uses G as the **transmission matrix**.
- Let us assume that the transmission is being carried out using a constellation consisting of 2^b symbols.

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Since the space time block codes from orthogonal designs have to have T is equal to K is equal to N and the rate R is equal to one and they form a special case of the generalized real orthogonal design. So, the already the case of N is equal to 2, 4 and 8 the orthogonal

designs there were a special case of the generalized real orthogonal designs with rate R equal to 1. Generalized real orthogonal designs provide full diversity and separate decoding of symbols. So, the diversity is there as discussed earlier and you have single symbol decoding. So, separate decoding of symbols exists. So, receiver complexity, receiver time is both reduced.

A real space time block code is defined as one of as one that uses G as a transmission matrix. So, we have already looked at the properties of G . G transpose into G should be a matrix with only the diagonal elements as nonzero. So, let us assume that the transmission is been carried out using a constellation consisting of 2 raise power b symbols we have discussed this before.


So, this generalized real orthogonal design we will now make it GROD standing for Generalized Real Orthogonal Design. So, let us now talk about the steps for generalized real orthogonal design.

(Refer Slide Time: 19:33)

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Steps for GROD Design

- Pick up a block of Kb bits from the input bit-stream.
- Based on these Kb bits, select the K symbols from the constellation: (s_1, s_2, \dots, s_k)
- Substitute $x_k \rightarrow s_k$.
- in the matrix G and generate the codeword $C = G(s_1, s_2, \dots, s_k)$.
 Note that C is a matrix of size $T \times N$.
 The transmission is done row-wise.
- Each row (of length N) is transmitted in one time period using N **antenna elements simultaneously**.
- At time period $t = 1, 2, \dots, T$, transmit the t^{th} row of C using the different antenna elements, $n = 1, 2, \dots, N$.



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So, pick up a block of Kb bits coming from the input stream. So, we have no worries getting enough bits to process today we are generating humongous amounts of bits, but what we do is we take up a block of K into b bits. Remember, the constellation has signals coming from 2 raise power b points in the constellation.

So, based on these Kb bits select K symbols from the constellation; so, the first b bits are used to pick up first symbol s_1 , the second b bits are used to pick up s_2 and the last b bits are used to pick up s_k . So, we have now s_1, s_2 up to s_k symbols as coming mapped out of this Kb bits. So, from my generator for the space time block code I substitute x_k to s_k .

So, now we have the code word coming as the generator matrix consisting of s_1, s_2 , up to s_k . Clearly, this generator matrix is of the size T cross N . So, the transmission is done row-wise, which means in the first time slot send out row 1, in the second time slot send out row 2 because the rows represent the time axis the columns represent the antenna element the space axis. So, each row obviously, of length N is transmitted at one time period using the N antenna elements simultaneously.

At the time period t is equal to 1, 2, 3, up to T transmit the T -th row of the C ; the code word C , using the different antenna elements N is equal to 1, 2, up to N . So, this much is pretty clear.


(Refer Slide Time: 21:45)

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Steps for GROD Design

- Thus, the entry $C_{t,n}$ is transmitted from antenna element n at time period t .
- At the end of T time periods, effectively K symbols would have been transmitted, thus justifying the definition $R = K/T$.
- Intuitively, T corresponds to the block length of the code.

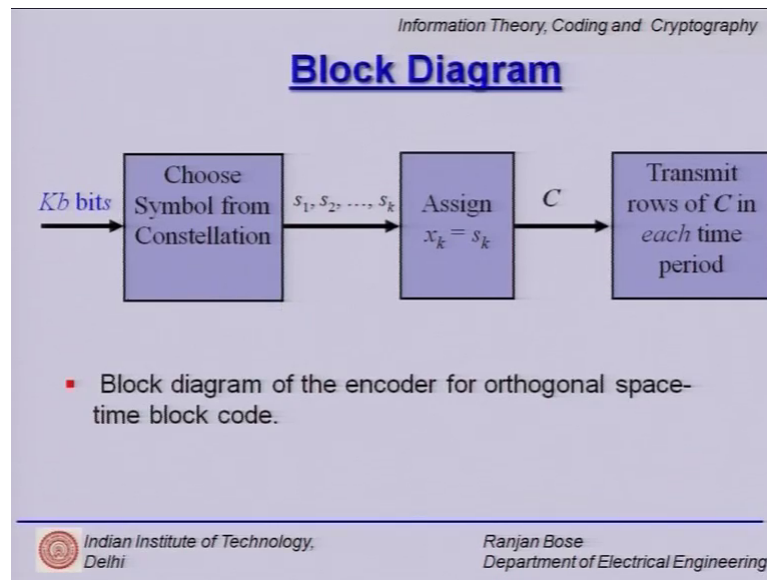
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So, thus the entry $C_{t,n}$ of the code is transmitted from the antenna element n at the time period t , that is the notation. At the end of T time periods effectively K symbols would have been transmitted, thus justifying the rate R is equal to K by T .

So, what does this mean? Intuitively, T corresponds to the block length of the code.

(Refer Slide Time: 22:17)



Let us show it using a block diagram for orthogonal space time block codes. What we have done is taken Kb bits coming at a time divided this Kb bits into b bits then b bits then b bits and so and so forth for each of the b bits I pick up one symbol from the constellation diagram.


So, s_1 corresponds to the first b bits s_2 to the second and so and so forth and then I assign $x_k = s_k$ gives my generator matrix to generate the code C . Now, this is T cross N matrix and then I transmit the rows of C in each time period.

(Refer Slide Time: 23:33)

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Delay Optimal Design

- In order to maximize the rate, it is important to have the **smallest** value of the block length, T .
- This parameter, T , determines the decoding delay of the code, because we cannot start the decoding process until all the codewords have been received.
- This motivates us to define the following.
- **Definition:** An orthogonal design with minimum possible value of the block length T is called **Delay Optimal**.

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Now, let us talk about delay optimal design. Clearly as we increase the size of T, I will use more and more time slots if I use more and more time slots then my delay would increase at the receiver side to get the decoded output. So, let us talk about this delay optimal where we minimize this T. So, in order to maximize the rate it is important to have the smallest value of the block length T, this is obvious.

So, the parameter T determines the decoding delay of the code, because we cannot really start decoding until all the code words have been received, ok. So, we have to wait till the last transmission and after T time slots and we would like to reduce this to the extent possible.

So, what do we do? Let us define the delay optimal design. So, an orthogonal design with minimum possible value of the block length T is called delay optimal.

(Refer Slide Time: 24:20)

Information Theory, Coding and Cryptography


Example

- Consider the following 4 × 3 matrix

$$G = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & -x_4 \\ -x_3 & x_4 & x_1 \\ -x_4 & -x_3 & x_2 \end{bmatrix}$$

- Computing $G^T G$ gives **We have a 4×3 real orthogonal design**

$$G^T G = \begin{bmatrix} x_1^2 + x_2^2 + x_3^2 + x_4^2 & 0 & 0 \\ 0 & x_1^2 + x_2^2 + x_3^2 + x_4^2 & 0 \\ 0 & 0 & x_1^2 + x_2^2 + x_3^2 + x_4^2 \end{bmatrix}$$

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It is better to look at an example to illustrate the point. So, let us consider the following 4 cross 3 matrix. So, N is equal to 3, T is equal to 4, leading it to a T cross N matrix. Remember, this is the space axis. So, we have 3 antenna elements and this is the time axis that I will be using four time slots. So, what this generator matrix tells us is in the first time slot send x 1 from antenna 1, x 2 from antenna 2, x 3 from antenna 3, then wait for the next time slot and transmit minus x 2, x 1 minus x 4 and so and so forth in the 4 time slots.

But, we make a very interesting observation. We just do not have three symbols to transmit we have x_1 up to x_4 . So, in 4 time slots we have been judiciously distributed the symbols such that x_1 , x_2 , x_3 and x_4 all have been placed, such that $G^T G$ again adheres to the definition of the orthogonal design. So, let us compute $G^T G$ and if you do so, for this you will be surprised to find the answer as follows and the observation is that the diagonal elements are non-zero and rest all are 0.


So, we have really in front of us a 4 cross 3 real orthogonal design, it is a generalized real orthogonal design because it is not a square matrix, right.

(Refer Slide Time: 26:22)

Information Theory, Coding and Cryptography

Example

- Observations regarding the 4×3 real orthogonal design.
- Here $K = 4$, $T = 4$ and $N = 3$.
- The rate of this real orthogonal design is $R = K/T = 1$.
- Hence, we have a **full rate, real orthogonal design**.
- This can be used for a system with $N = 3$ transmit antennas.
- It uses $T = 4$ time periods to transmit different symbols from the different antennas as follows:
- Since it sends 4 symbols (x_1, x_2, x_3, x_4) using 4 time slots, the rate is unity.



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So, what are the observations regarding this design? T is equal to 4, N is equal to 3, as we observed K is equal to 4, but the rate is 1, ok. We did not compromise on the rate. So, it is a full rate generalized real orthogonal design. So, we have only used 3 transmit antennas because N is equal to 3, but we have judiciously used the 3 antenna elements in different time slots to send out the message which are single symbol decodable.

Of course, it uses 4 time periods to do. So, the delay increase a little bit, right. Please remember, we never had a 3 cross 3 solution for real orthogonal designs, we had 2 cross 2, 4 cross 4 and 8 cross 8, but when we went to the generalized domain we have a 4 cross 3 solution and that is the interesting observation. So, it sends out 4 symbols using 4 time slots and the rate is unity.

(Refer Slide Time: 27:42)


Information Theory, Coding and Cryptography

Example

$$G = \begin{bmatrix} x_1 & x_2 & x_3 \\ -x_2 & x_1 & -x_4 \\ -x_3 & x_4 & x_1 \\ -x_4 & -x_3 & x_2 \end{bmatrix}$$

	Antenna 1	Antenna 2	Antenna 3
Time period 1	x_1	x_2	x_3
Time period 2	$-x_2$	x_1	$-x_4$
Time period 3	$-x_3$	x_4	x_1
Time period 4	$-x_4$	$-x_3$	x_2

- Since it sends 4 symbols (x_1, x_2, x_3, x_4) using 4 time slots, the rate is unity.

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So, let us start back and analyze it a little bit further. So, you have this three antenna elements sending out this first row in time slot 1, second row in time slot 2, third row in time slot 3 and fourth row in time slot 4 and if you want to depict it more clearly you have this 3 antenna elements and four time slots and you exactly know which one you are sending in which time slots, ok. So, this is actually the recipe for sending out the code based on this design.


So, the important observation is that it is full rate we have not compromised on the rate.

(Refer Slide Time: 28:37)

Information Theory, Coding and Cryptography

Comment

- It can be shown that for **any number of transmit antennas, N** ,
 - there exists a full rate, $R = 1$, real space time block code with a block size $T = \min[2^{4c+d}]$,
 - where the minimization is over all possible integer values of c and d in the set $\{c \geq 0, d \geq 0 | 8c + 2^d \geq N\}$.

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So, it can be shown that for any number of transmit antennas N , there exists a full rate real space time block code with block size T given by a minimum of 2 raise power 4 plus d . So, this rate full rate can be guaranteed provided you have a constraint on this block size T . The minimization is all possible integer value values of c and d .

So, here this is the block size T and if you can have c and d as integers then you can probably come up with this design and we look at an example shortly. This c and d have some constraint c should be greater than or equal to 0 d should be greater than or equal to 0 and $8, c$ plus 2 raise power d should be greater than or equal to N . So, if you can find c and d which can satisfy these conditions then T can be found out by this minimum and then you can find out an example.


(Refer Slide Time: 29:49)

Information Theory, Coding and Cryptography

Example

- Consider the following 8×7 matrix for $N = 7$ transmit antennas

$$G = \begin{bmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ -x_2 & x_1 & -x_4 & x_3 & -x_6 & x_5 & x_8 \\ -x_3 & x_4 & x_1 & -x_2 & x_7 & x_8 & -x_5 \\ -x_4 & -x_3 & x_2 & x_1 & x_8 & -x_7 & x_6 \\ -x_5 & x_6 & -x_7 & -x_8 & x_1 & -x_2 & x_3 \\ -x_6 & -x_5 & -x_8 & x_7 & x_2 & x_1 & -x_4 \\ -x_7 & -x_8 & x_5 & -x_6 & -x_3 & x_4 & x_1 \\ -x_8 & x_7 & x_6 & x_5 & -x_4 & -x_3 & -x_2 \end{bmatrix}$$



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So, let us look at the following example we have a 8 cross 7 matrix for 7 transmit antennas. So, just by looking at this matrix I know that there are 7 columns so, 7 transmit antennas will be there. There are 8 rows so, I will be using 8 time slots. Obviously, in the first time slot I will be sending out this first row, second row in the second time slot and the eighth row in the eighth time slot.

Now, before we proceed any further we can quickly take G transpose into G and we will see whether only the diagonal elements are non-zero rest are all zeros which will give that it is a orthogonal design.


(Refer Slide Time: 30:38)

Information Theory, Coding and Cryptography

Example

- Computing $G^T G$ gives

$$G^T G = \begin{bmatrix} \sum_{i=1}^8 x_i^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \sum_{i=1}^8 x_i^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sum_{i=1}^8 x_i^2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sum_{i=1}^8 x_i^2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \sum_{i=1}^8 x_i^2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^8 x_i^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sum_{i=1}^8 x_i^2 \end{bmatrix}$$

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So, we do that computation $G^T G$ and one can easily verify that you get this expression. So, this will guarantee single symbol decoding.


(Refer Slide Time: 30:54)

Information Theory, Coding and Cryptography

Example

- Thus we have a 8×7 real orthogonal design.
- Here $K = 8$.
- $T = \min[2^{4c+d}] = 8$ for integers $c = 0$ and $d = 3$, and $8c + 2^d = 8 \geq N$.
- The rate of this real orthogonal design is $R = K/T = 1$.
- Hence, G represents a full rate, real orthogonal design which can be used for $N = 7$ transmit antennas.

There exists a full rate, $R = 1$, real space time block code with a block size $T = \min[2^{4c+d}]$, where the minimization is over all possible integer values of c and d in the set $\{c \geq 0, d \geq 0 | 8c + 2^d \geq N\}$.

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But, we also see that 8×7 is a real orthogonal design with K is equal to 8 and you can do T as minimum of 2 raise power $4c$ plus d equal to 8, because the integers which we found was c is equal to 0 and d is equal to 3. So, we have satisfied those conditions and we were able to get this 8×7 matrix. So, not only it tells you that this $T \times N$ matrix exists, but you already have an example of what that matrix should look like.


So, this is the condition we satisfied ourselves with that there exists a full rate R is equal to 1 real space time block code with block size given by this where the minimization is over all possible interval integer values of c and d . In this example you have R is equal to 1, because you had if you see eight time slots and you have x_1 up to x_8 if you carefully observe I am pushing through along x_1, x_2, x_3 up to x_8 in a distributed manner. So, in eight time slots effectively I have sent out eight symbols. So, my rate is indeed 1. So, G represents a full rate real orthogonal design which can be used with 7 transmit antennas.

(Refer Slide Time: 32:37)

Information Theory, Coding and Cryptography

Observations

- Orthogonal designs are not unique.
- Multiplying any orthogonal design G with a matrix U having the property $U^T U = I$ results in another orthogonal design $G' = UG$.
- Deleting a column from an orthogonal design leads to another orthogonal design that can be used to design a STBC **with one less transmit antenna**.
- This is called **shortening**.
- If the original real orthogonal design is delay optimal then the shortened design is also **delay optimal**.



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So, some observations orthogonal designs are not unique, number – 2 multiplying any orthogonal design G with a with a matrix U , right having the property $U^T U = I$ equal to 1. So, results in another orthogonal design ok. So, deleting a column from an orthogonal design leads to another orthogonal design that can be used to design a space time block code with a one less antenna.

So, these are practical issues that we can look at, if we have a one design we can go to another design by deleting a row or a column depending upon how you look at it. So, deleting a column basically reduces one antenna element and you still have a orthogonal design left. This process is called shortening if the original real orthogonal design is a delay optimal then the shortened design is also delay optimal.

(Refer Slide Time: 33:37)

Information Theory, Coding and Cryptography


Complex Orthogonal Design

- A **Complex Orthogonal Design** of size N is an $N \times N$ matrix G with entries consisting of complex elements drawn from $\pm x_1, \pm x_1, \dots, \pm x_N$ and their conjugates $\pm x_1^*, \pm x_2^*, \dots, \pm x_N^*$ or multiples of these by $j = \sqrt{-1}$

$$G^T G = \left(\sum_{i=1}^N |x_i|^2 \right) I_N$$

- where I_N is a $N \times N$ identity matrix.
- A real orthogonal design exists if and only if $N = 2$.

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Now, we changed gears slightly and we say look lot of our signal constellations are complex. So, are there complex orthogonal designs? Of course, we had looked at Alamouti to begin with so, we know the answer is, yes. So, let us define a complex orthogonal design. A complex orthogonal design of size N is an N cross N matrix G with entries consisting of complex elements drawn from plus minus x_1, x_2 up to x_N and their complex conjugates x_1^*, x_2^* up to x_N^* , right or multiples of these by j is equal to under root of minus 1 such that $G^T G$, right should be this.

So, we actually should write G Hermitian G should be written in this form, where I_N is the N cross N identity matrix and what is interesting is a real orthogonal design exists if and only if N is equal to 2. It can be shown that and this great design is that Alamouti code we have looked at so.

(Refer Slide Time: 34:50)


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Example

- Consider the following 2×2 matrix for $N = 2$ transmit antennas

$$\mathbf{G} = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$$

- Since $\mathbf{G}^H \mathbf{G} = (|x_1|^2 + |x_2|^2) \mathbf{I}_2$ it is a complex orthogonal design.
- In fact, it is the **Alamouti code**.
- Since, we have **only one** complex orthogonal design of size 2×2 , it motivates us to explore generalized complex orthogonal designs.

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This is an example of a 2 cross 2 matrix which we have already seen as being an orthogonal design it is the Alamouti code, but this is the only one complex orthogonal design which exists size 2 cross 2.

So, it is obvious that we must graduate to the generalized complex orthogonal design.

(Refer Slide Time: 35:15)


Information Theory, Coding and Cryptography

Generalized Complex Orthogonal Design

- A **Generalized Complex Orthogonal Design** is a $T \times N$ matrix G with complex elements drawn from $0, \pm x_1, \pm x_1^*, \pm x_2, \pm x_2^*, \dots, \pm x_K, \pm x_K^*$ or multiples of these by $j = \sqrt{-1}$ such that

$$\mathbf{G}^H \mathbf{G} = \kappa \left(\sum_{i=1}^K |x_i|^2 \right) \mathbf{I}_N$$

where \mathbf{I}_N is a $N \times N$ identity matrix and κ is a constant.

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So, we define it quickly. Generalized complex orthogonal design similar to the generalized version of the real orthogonal design is a T cross N matrix G with complex elements drawn from 0, x 1, x 2, x 3 and so and so forth up to x K such that G Hermitian

G is some κ times summation of x_i absolute value squared I_N , where I_N is an N cross N identity matrix and κ is a constant.


(Refer Slide Time: 35:48)

Information Theory, Coding and Cryptography

Observations

- It is possible to have $\kappa = 1$ by appropriately **normalizing** the elements of G .
- Multiplying a generalized complex orthogonal design by a unitary matrix leads to another generalized complex orthogonal design, that is, if $U^H U = I$ then, $G' = UG$ is also a **generalized complex orthogonal design**.

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So, it is possible to have κ is equal to 1 by appropriately normalizing the elements of G is still remains an orthogonal matrix and we can multiply as before with the unitary matrix then G prime equal to U times G is also generalized complex orthogonal design. So, multiplication by unitary matrix as in the earlier case does not change it is still remains a generalized complex orthogonal design.

(Refer Slide Time: 36:29)

Information Theory, Coding and Cryptography

GCOD Design

- A **generalized complex orthogonal design** can be used to generate a rate $R = K/T$ space time code with N antenna elements.
- Let us assume that the transmission is being carried out using a constellation consisting of 2^b symbols.
- The steps involved in encoding using a generalized complex orthogonal design are as follows:
 - Pick up a block of Kb bits from the input bit-stream.
 - Based on these Kb bits, select the K symbols from the constellation: (s_1, s_2, \dots, s_k)
 - Substitute $x_k \rightarrow s_k$ in the matrix G and generate the code word $C = G(s_1, s_2, \dots, s_k)$.

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So, let us use the acronym generalized complex orthogonal design GCOD. So, a GCOD design can be used to generate a rate R is equal to K by T space time code with N antenna elements. Again, we do a similar thing we assume that we are using a constellation consisting of 2^b symbols and we look at the following steps as before take a big block of Kb bits divided into b bits then again block of b bits and so and so forth gives each block to pick up the symbols s_1, s_2 up to s_k we have done this before and then substitute x_k for s_k in the matrix G to generate the code C .

So, this G matrix always have the elements x_1, x_2 , up to x_k and once you substitute the G generator matrix generates a codeword, the codeword matrix becomes the C , ok.


(Refer Slide Time: 37:33)

Information Theory, Coding and Cryptography

GCOD Design

- Note that C is a matrix of size $T \times N$ and consists of linear combinations of s_1, s_2, \dots, s_k and their conjugates.
- At time period $t = 1, 2, \dots, T$, transmit the t^{th} row of C using the different antenna elements, $n = 1, 2, \dots, N$.
- Thus, the entry $C_{t,n}$ is transmitted from antenna element n at time period t .
- At the end of T time periods, effectively K symbols are transmitted.

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So, this C matrix is of the size T cross N and it has a linear combination of s_1, s_2 , up to s_k and their conjugates because it is a complex design. So, at time period t is equal to 1 through T you transmit the t -th row as before and then, clearly the $C_{t,n}$ is transmitted from antenna element n at time period t as before it is pretty much the same and we have transmitted effectively K symbols at the end of T time period thereby justifying that rate.

(Refer Slide Time: 38:12)

Information Theory, Coding and Cryptography

Observations

- A complex space-time block code constructed using a $T \times N$ generalized complex orthogonal design provides a diversity of NM for N transmit antennas and M receive antennas.
- It also results in separate maximum-likelihood decoding of its symbols.
- Since there are **three** independent parameters, the number of transmit antennas (N), the number of symbols (K) and the number of time periods (T), the transmission matrix, G is sometimes denoted by G_{NKT} .

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So, some simple observations a complex space time block code constructed using a T cross N generalized complex orthogonal design provides a diversity of N cross M . So, N is the number of transmit antennas M is the number of receive antennas. So, this is the diversity that is provided and it also results in a separate maximum likelihood decoding symbol by symbol decoding is possible.

But, please note there are three independent parameters, the number of transmit antennas N , the number of symbols K and the number of time periods T and so, the transmission matrix generator matrix G is sometimes denoted by G subscript NKT .

(Refer Slide Time: 39:00)

Information Theory, Coding and Cryptography

Summary

- Real Orthogonal Design
- Generalized Real Orthogonal Design
- Complex Orthogonal Design
- Generalized Complex Orthogonal Design
- Examples

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So, let us now try to conclude what we have studied in today's lecture. We started out with the notion of real orthogonal designs which provide single symbol decoding and full diversity and we realize that only N is equal to 2, 4 and 8 is possible for real orthogonal design that motivated us to look at generalized real orthogonal designs, where we are working with a matrices which are not square and so, you can have more number of time slots.

But, still it is possible to have rate R is equal to 1 and with then we defined the notion of delay optimal then we moved on to complex orthogonal design, where the elements in the generator matrix could be complex and we found out that only 2 cross 2 the Alamouti code exists as the only example of complex orthogonal design. Therefore, we moved over to the generalized complex orthogonal design. For both real orthogonal design and complex orthogonal design the generalized versions we defined the steps as to how to go about designing and sending it.

Finally, we looked at the examples for ROD, GROD, COD and GCOD. So, with that we come to the end of this lecture.