

Information Theory, Coding and Cryptography
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Module – 33
Space Time Codes
Lecture – 33

Hello and welcome to our next lecture on Space Time Codes.

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Information Theory, Coding and Cryptography

Outline

- Concept of Space Time Codes
- Alamouti Code
- Diversity
- Examples

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Let us start with a brief outline of today's talk; we will start with the concept of space time codes. We had originally looked at multi antenna systems and we would justify the use of that in the context of space time codes. Then we would look at the classic Alamouti code, then we will talk about diversity and diversity gains plus we will have some examples to illustrate the concepts.

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Recap

- Combined Coding and Modulation
- Trellis Coded Modulation
- Free distance
- Ungerboeck's design rules
- Performance Evaluation

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So, let us see what we have done in the previous lecture; we have looked at TCM schemes, looked at combined coding and modulation, defined free distance, design rules, performance evaluations and finally, we went over to space time trellis codes; where we started using 2 antennas. So, that was a logical extension of space trellis coded modulation going on to space time trellis code.


In today's lecture, we would look at the space time block codes, where we do not have so, to say trellis to help us encode the sequences.

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Wireless Channels

- An inherent characteristic of a wireless channel is **multipath propagation**, which leads to signal fading.
- One of the ways to improve the capacity of wireless channels is to use **multiple antennas** at the transmitter and receiver.
- However, in many real-world systems, the receiver is required to be small by design (cell-phones, personal digital assistants, smart-phones etc.).
- In such physically small receivers, it is impractical to put multiple antennas.
- The different antennal elements, in a multi-antenna system, need a minimum separation to work well.

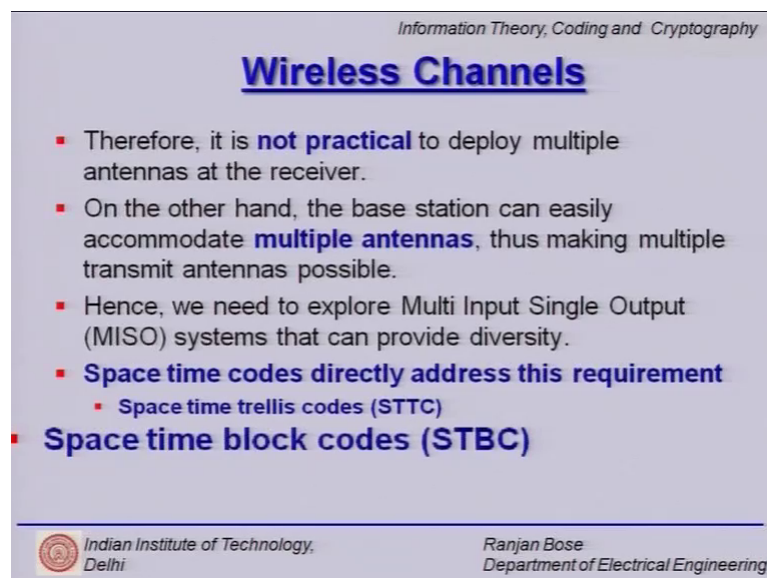
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But before we do so, we start with a quick motivation; we start with wireless channels because this is where we will be using space time codes frequently. So, the inherent characteristic of any wireless channel is multipath propagation; we have seen this before and this leads to signal fading.

Now, one of the ways actually an easy way to improve the capacity of wireless channel is to use multiple antennas and the transmitter and receivers ok. So, multiple antennas can provide several things including the diversity gain that we talked about in the last class, but the problem is that the receiver could be your handset and you do not have enough space to put multiple antennas. So, it is logical to put more number of antennas at the base station.

So, what we would like to do is we would have to also have a minimum separation between the antenna elements for them to be feeding relatively independently. So, the requirement of spacing between antenna elements and the place, a space available on the receiver handset restricts putting multiple number of antenna elements at the receiver end, but from the base stations perspective this can easily be done.

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Information Theory, Coding and Cryptography

Wireless Channels

- Therefore, it is **not practical** to deploy multiple antennas at the receiver.
- On the other hand, the base station can easily accommodate **multiple antennas**, thus making multiple transmit antennas possible.
- Hence, we need to explore Multi Input Single Output (MISO) systems that can provide diversity.
- **Space time codes directly address this requirement**
 - Space time trellis codes (STTC)
- **Space time block codes (STBC)**

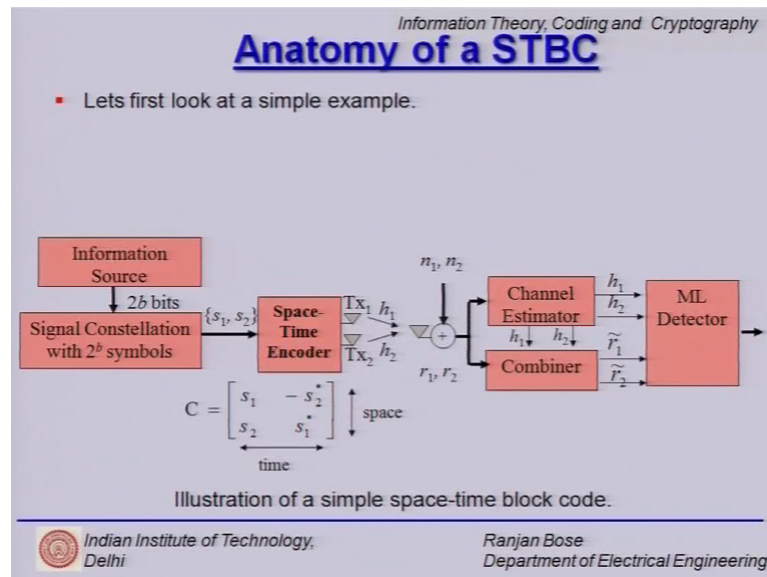
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So, as we mentioned not practical to deploy multiple antennas at the receiver, but on the other hand base stations can easily accommodate multiple antennas. So, this is the basic premise with which we will start, we will have multiple antennas at the transmit site. So, we are going to look at multi input single output or MISO systems.

Now space time coding basically allows different symbols to be sent from the different antenna elements in different time slots. So, we have already looked at Space Time Trellis Codes; STTC as a natural extension of trellis coded modulation and today we would delve into space time block codes; that is the subject for today.

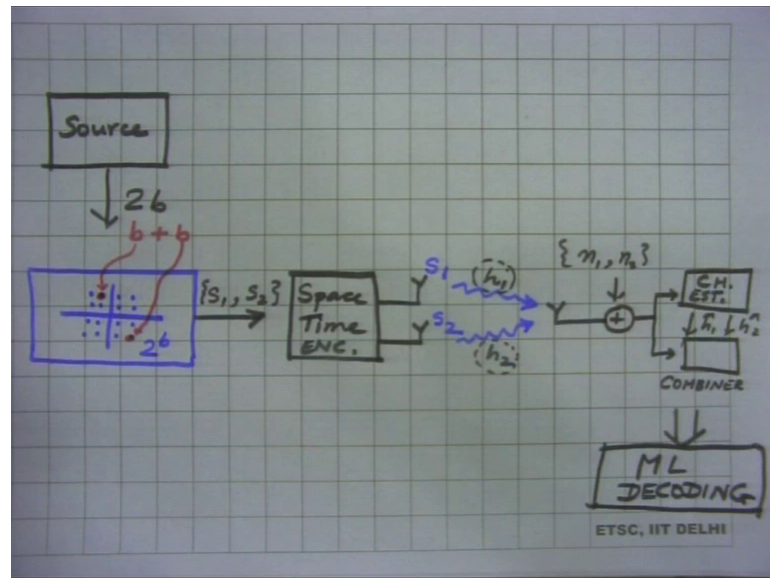
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Now let us start with the anatomy of a space time block codes shortly we will discuss the details. So, we are talking about STBC; Space Time Block Codes let us start from the first block the information source. And this source is generating enough bits and what we do is we take 2 bits at a time and in general we can start with b bits at a time. So, if you club it into 2 b bits then we can look at a signal constellation with 2 b symbols.

So, let us quickly look at it here.

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So, the source will generate $2b$ which is nothing, but b and b ; now what we do is we have a constellation this is predefined, this constellation has a certain number of points in the constellation diagram.

So, suppose we choose any constellation and in this there are 2 raised power b points. So, for the first b , I will go ahead and pick any one point and for the second b ; I will go ahead and pick another point. Because b bits will corresponds to any one of points in the constellation diagram and once we pick up these 2 points then we say that this is your S_1 and this one is your S_2 .

So, we have a pair S_1, S_2 pair this pair has to be sent to the space time encoder. So, this is a space time encoder and this guy has 2 antenna elements and what do these 2 antenna elements do? Well they have got these 2 symbols S_1 and S_2 . So, they will take S_1 and S_2 and transmit it out.

But we observed that the receiver may not have the luxury of multiple antennas. So, we have only a single antenna at the receiver right and the receiver receives S_1 and S_2 simultaneously plus there will be some noise also coming. So, we will have n_1 and n_2 for the 2 signals S_1 and S_2 and this will then be fed to your channel estimator right and of course, I would also have based on the channel estimation a combiner.

So, I will get the estimates of the channels now what are these channels? Well we can represent it as h_1 and h_2 these are complex; so, these represent my channel gains alright. So, these are complex numbers and the signal S_1 gets multiplied with h_1 and gets added with noise n_1 and similarly S_2 gets multiplied by h_2 and then gets added with n_2 . So, the estimates of these to \hat{h}_1 and \hat{h}_2 have to be given to the combiner and here after that we will take this for maximum likelihood decoding.

So, if we go and look at the slides once again we have the same concept exemplified here. You have this information source you take $2b$ bits at a time, separate out these $2b$ bits into b bits and b bits and each one picks up a point in the constellation diagram. So, I have a pair S_1 and S_2 which is fed to the space time encoder. Now this antenna spacing is providing the space we have not talked about that time part yet.

So, we have now the time slot; so, in time slot 1, we send out S_1 and S_2 . So, if you see there is a matrix right want if you want to look at this as a time axis; then this there will be symbols along this and this will be this space. So, this is corresponding to antenna element 1 and S_2 corresponds to antenna element 2.

But what we do is we then also do something more interesting in the next time slot. So, this is the axis and I already have S_2 and S_1 available with me and they are complex because they are coming from a constellation diagram. So, they have a amplitude and phase; I do a simple computation, I do minus S_2^* for example, and S_1^* . So, I take this complex conjugate multiplied with a negative and again loaded this guy on to antenna element 1 and the second person on to antenna element 2 and retransmit.

So, please note this $2b$ bits resulted in the transmission of S_1 and S_2 from antenna elements 1 and 2 in the first time slot. And then now we have not taken any more bits in the second time slot what we have done is we have sent something different based on S_1 S_2 ; so, that is a rule. So, regardless of what S_1 and S_2 are the next 2 symbols to be transmitted simultaneously will be minus S_2^* and S_1^* .

This is just an example, this is a strategy this is my coding strategy and therefore, this can represent my space time block code. Why this will work? How good is it? Why are we doing it in this all of these questions will be answered in the subsequent slide, but the method is pretty easy; it is mechanical, but the beauty lies in the decoding part.


So, let us go on and see what really is the gain that we get from doing this kind of encoding.

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Example

- Consider a wireless system consisting of two transmit antennas and one receive antenna.
- The information source generates bits, which are mapped to symbols from a signal constellation (MPSK, MQAM etc.).
- The signal constellation can be real or complex constellation.
- To transmit b bits/cycle, we use a modulation scheme that maps one symbol from a constellation with 2^b symbols.
- The output of the mapper, $\{x_1, x_2\}$ is input to a space-time coding block.
- The encoding takes in two time slots (the time axis) over two antenna elements (the space axis).
- In each encoding operation, the encoder takes two symbols s_1 and s_2 and transmits according to the following **coding strategy (Alamouti Code)**, depicted by the matrix

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So, let us look at a simple example consider a wireless system with 2 transmit antennas and 1 receive antennas. So, these 2 transmit antennas could very well be seated on the base station and receive antenna is your handset. And we have to have some constellation diagram signal constellation available to us it could be MPSK, MQAM what have you right and this constellation could be real or complex ok. So, there is no restriction that it has to be complex.

Now, we will transmit b bits per cycle right and we will use a modulation scheme that maps one symbol from the constellation with 2^b symbols and output will be $x_1 \times 2$ and as we mentioned this encoding takes 2 time slots right.

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Example: Alamouti Code


$$X = \begin{bmatrix} x_1 & x_2 \\ -x_2^* & x_1^* \end{bmatrix}$$

where x_1^* is the complex conjugate of x_1

- For example, if symbols s_1 and s_2 are selected, we map $x_1 \rightarrow s_1$ and $x_2 \rightarrow s_2$ and obtain the codeword matrix as

$$C = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}$$

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And we have this coding strategy; so, this particular way of coding has a name based on the person who first proposed it is called the Alamouti code and this is the depiction for the Alamouti code. So, if you have this as the coding strategy then the codeword matrix C as we just now observed could be written as follows S 1 S 2 minus S 2 star S 1.

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Example

- A frame of data lasts **two symbol periods**.
- During symbol period 1, antenna 1 and antenna 2 transmit s_1 and s_2 respectively.
- During symbol period 2, antenna 1 and antenna 2 transmit $-s_2^*$ and s_1^* respectively.
- For better visualization, we can write as

	Antenna 1	Antenna 2
Time period 1	s_1	s_2
Time period 2	$-s_2^*$	s_1^*

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Alamouti Code

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So, this method of doing space time coding can be expressed in the form of a table. Here on this axis we have the space antenna 1 and antenna 2, they are spatially apart here we have the time period 1 and time period 2. So, this is the time axis and as we mentioned in

the first time slot we use antenna 1 to send out S 1 and use antenna 2 to send out S 2. And then in time period 2, we do minus S 2 star and S 1 star load them onto the antennas and transmit. So, this is in a nutshell the depiction of the Alamouti code ok; so, far it is pretty easy.

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Example

- We note that


$$\mathbf{X}^H \mathbf{X} = (|x_1|^2 + |x_2|^2) \mathbf{I}_2$$

where \mathbf{I}_2 is a 2 x 2 identity matrix.

- The channel from antenna 1 to the receiver (h_1) and antenna 2 to the receiver (h_2) can be modeled as

$$h_1 = \alpha_1 e^{-j\phi_1}$$

$$h_2 = \alpha_2 e^{-j\phi_2}$$



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Alamouti Code

But now let us make some interesting observations. Why has it been chosen like this? What is so, great about putting a minus S 2 star and S 1 star here? So, the first observation is that X hermitian X, where X was given by this is just showing a general coding strategy. And if the exam symbols were S 1 and S 2 then coding strategy will put S 1 and S 2 in the case of x 1 and x 2. So, x represents the code and this is the code word matrix. So, X hermitian X is x 1 squared plus x 2 squared right absolute value squared into this identity matrix.

Now as we mentioned after we transmit the symbols they go through the wireless channel. So, the symbol S 1 gets a channel gain h 1 it is complex can be represented easily by alpha 1 e raised power 1 and h 2 is the channel gain encountered by symbol S 2 equal to alpha 2 e raised to power minus j phi 2.


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Example

- It is assumed that the fading channel coefficients h_1 and h_2 are constant across two consecutive symbol periods (is that a fair assumption?).
- The received signal over the first symbol period, denoted by r_1 , can be expressed as
$$r_1 = h_1 s_1 + h_2 s_2 + n_1$$
- Similarly, the signal over the second symbol period, denoted by r_2 , can be expressed as
$$r_2 = -h_1 s_2^* + h_2 x_1^* + n_2$$

where n_1 and n_2 are AWGN samples for the two symbol periods respectively.



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Alamouti Code

So, let us make some more assumptions it is assumed that the fading channel coefficients h_1 and h_2 are constant across 2 consecutive symbol periods; it is not a bad assumption at the data rates we have the symbol rates that we employ the channel does not change significantly between the 2 time slots. So, is that a fair assumption? Yes it is a fair assumption.

Now, the received signal over the first 2 symbol period denoted by r_1 can be represented as r_1 is $h_1 s_1$ plus $h_2 s_2$ plus n_1 . Now if you look at the second symbol period right. So, this is time slot 1 because in the time slot 1 both transmit antenna are transmitting simultaneously. So, transmit antenna 1 sends S_1 transmit antenna 2 sends S_2 both get multiplied by the respective channel gains and in the first time slot I have got the noise n_1 .

And similarly in the second time slot I have got noise n_2 , but what has happened for the next time slot, the received signal that I get in the second time slot is I sent out from antenna 1 minus s_2^* and from antenna 2 s_1^* . So, this should be $s_1 s_1^*$ and we have this r_1 and r_2 .

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At the receiver

- The aim of the receiver is to extract the transmitted symbols (x_1 and x_2) from r_1 and r_2 .
- The receiver combines the received signal according to the following **combining scheme**

$$\tilde{r}_1 = h_1^* r_1 + h_2 r_2^*$$
$$\tilde{r}_2 = h_2^* r_1 - h_1 r_2^*$$

- Note that the receiver must have an estimate of h_1 and h_2 in order to implement the combining scheme.
- Both h_1 and h_2 are complex.
- Hence we obtain

$$\tilde{r}_1 = (|h_1|^2 + |h_2|^2) s_1 + h_1^* n_1 + h_2 n_2^*$$
$$\tilde{r}_2 = (|h_1|^2 + |h_2|^2) s_2 - h_1^* n_2 + h_2 n_1^*$$

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Alamouti Code

Now the aim of the receiver is to extract the transmitted symbols x_1 and x_2 from r_1 and r_2 that is the aim; so, the receiver requires some kind of a combining scheme. So, we have a strategy in place based on the channel estimates. So, we assume that we have some knowledge about the channel gains h_1 and h_2 and the combining scheme r_1 tilde is $h_1^* r_1 + h_2 r_2^*$ and r_2 tilde is equal to $h_2^* r_1 - h_1 r_2^*$.

Now, what will all of this do? Because r_1 already we know is given by this and r_2 is given by this. So, this and complex conjugate would probably help us simplify things ok, but we make this observation that h_1 and h_2 is known. So, there is a channel estimator which tells us the channel gains and of course, h_1 and h_2 are complex. So, now, if you plug in the values of r_1 and r_2 let us see what we can get for our one tilde n_2 tilde. So, r_1 tilde is if you do the math and it is not to complicate it you get h_1 absolute value squared plus h_2 absolute value squared into s_1 plus $h_1^* n_1 + h_2 n_2^*$ and similarly r_2 tilde has a expression available for it.

So, what does it bias? Well the first interesting observation is r_1 depends only on s_1 . Earlier if you note everything was coupled the received signal clearly depended on s_1 and s_2 ; why? Both the transmitter antennas were sending together. So, it is, but obvious that you will get a jumble, you will get a mixture of s_1 and s_2 multiplied by the channel gains, but I do some smart processing at the receiver and suddenly some magic happens and this r_1 only depends on s_1 .

Now we tell you the advantage of that and r_2 only depends on s_2 of course, I have the estimates of h_1 and h_2 and these are nothing, but some values multiplying s_1 and s_2 , but this is the most interesting fall out of this mathematics that has come out; r_1 tilde depending only on s_1 r_2 tilde depending on only an s_2 this has big ramifications.

Because at the end my job is to guess what was transmitted s_1 and s_2 ok. So, we have to do a search because this is a maximum likelihood decoding and it is now a question of matter of reducing the search space.

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Information Theory, Coding and Cryptography

At the receiver

- The beauty of the eq. is that \tilde{r}_1 depends **only** on s_1 , and not on s_2 .
- So the detection can be carried out with respect to this **single quantity** !
- Similarly, \tilde{r}_2 depends **only** on s_2 , and not on s_1 .
- The detector uses the maximum likelihood decision rule on each of the signals separately

$$\tilde{x}_1 = \arg \min_{x \in S} \left| \tilde{r}_1 - (|h_1|^2 + |h_2|^2) s \right|$$

$$\tilde{x}_2 = \arg \min_{x \in S} \left| \tilde{r}_2 - (|h_1|^2 + |h_2|^2) s \right|$$



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Alamouti Code

So, the beauty of the equation is that r_1 tilde depends only on s_1 and not on s_2 . So, the detection can be carried out only with respect to the single quantity. So, when I going to search and find out which is the most likely symbol transmitted, I only do on the possible points in the constellation diagrams s_1 . Similarly for s_2 I only do search only for s_2 ; if they were coupled then I have to have the number of searches s_1 and s_2 together. And that would require me to search if there are 64 points in the constellation diagram, then there the pairs number of pairs would be 64 into 64 as opposed to only 64.

So, that is the biggest advantage that we are going to get here; the detector users maximum likelihood decision and so, x_1 tilde is nothing, but this is the maximum likelihood r_1 tilde we just now minus this h_1 absolute value square plus h_2 squared s . Similarly x_2 estimate is minimum over all the possible constellations points r_2 tilde

minus h_1 squared plus h_2 to absolute value squared. So, this is just the maximum likelihood ml decoding.

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Information Theory, Coding and Cryptography


A different example

- Consider a wireless system consisting of two transmit antennas and one receive antenna,
- Let us employ a **different combining scheme** that leads to values that are a mixture of the transmitted signal.
- For example:

$$\begin{aligned}\tilde{r}_1 &= ar_1 + br_2^* + \tilde{n}_1 \\ \tilde{r}_2 &= cr_1 + dr_2^* + \tilde{n}_2\end{aligned}$$

- where a, b, c and d are some coefficients.
- In this case,

$$\tilde{\mathbf{r}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{bmatrix}$$

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So, that is the beauty of the Alamouti scheme; we have really reduced the complexity the decoding complexity. Now will it work in general? Let us consider a slightly different example; again we have 2 transmit antennas and 1 receive antennas, nothing is different except that this time we employ a slightly different combining scheme and this will lead to a mixture of the transmitted signals.


So, for example, let \tilde{r}_1 be $ar_1 + br_2^*$ plus noise and \tilde{r}_2 could be $cr_1 + dr_2^*$ plus this noise part. So, if a, b, c and d are some coefficients then the vector $\tilde{\mathbf{r}}$ can be expressed as $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \tilde{n}_1 \\ \tilde{n}_2 \end{bmatrix}$.

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Information Theory, Coding and Cryptography

A different example

- Let us say,
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
- Then, the ML decision rule can be written as
$$[\tilde{x}_1 \quad \tilde{x}_2] = \arg \min_{\mathbf{x} \in S^2} \|\tilde{\mathbf{r}} - A\mathbf{x}\|$$
- This process of minimization is basically a search for a vector of length 2.
- If the constellation has M points, then the computational complexity for the search is $\mathcal{O}(M^2)$.
- If, the number of transmit antenna is increased to N , the computational complexity for the search will be $\mathcal{O}(M^N)$.
- For the Alamouti code, this increase in the complexity is avoided because of the use of **an orthogonal encoding matrix**

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So, if a matrix can be represented as a b c d then the ml decision rule can be written $\tilde{x}_1 \tilde{x}_2$ is minimum over this vector \mathbf{x} over S^2 .

So, now this is the process of minimizing is a search for a vector of length 2 as opposed to a single symbol search. So, if the constellation has M points then the computational complexity of the search is of the order m^2 . If the number of transmit antennas is increased to n the computational complexity of the search will be m raised power n . So, early we only talking about 2 therefore, I was sending symbol s_1 and s_2 , if there were 3 transmit antennas then the search will be over a 3 tuple is there n transmit antennas in the search will over m raised power n .

So, this is the biggest problem the complexity will forbid the use of any general arbitrarily designed scheme. This A has to be designed very carefully, but for the Alamouti scheme that we saw this increase in complexity is avoided because of the use of an orthogonal encoding matrix. So, that was the beauty of the Alamouti code the coding matrix was orthogonal therefore, it could couple the s_1 and s_2 . So, this is the crux this is the reason why it is such an efficient decoding algorithm.


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Generalization

- In the previous example, we have used 2 transmit antennas and two time periods, leading to a 2X2 code matrix
- In general, consider a wireless system that uses N transmit antennas during T time periods.
- Then the space-time code can be represented by an $N \times T$ matrix

$$C = \begin{bmatrix} C_{1,1} & C_{1,2} & \Lambda & C_{1,N} \\ C_{2,1} & C_{2,2} & \Lambda & C_{2,N} \\ \text{M} & \text{M} & \text{O} & \text{M} \\ C_{T,1} & C_{T,2} & \Lambda & C_{T,N} \end{bmatrix}$$

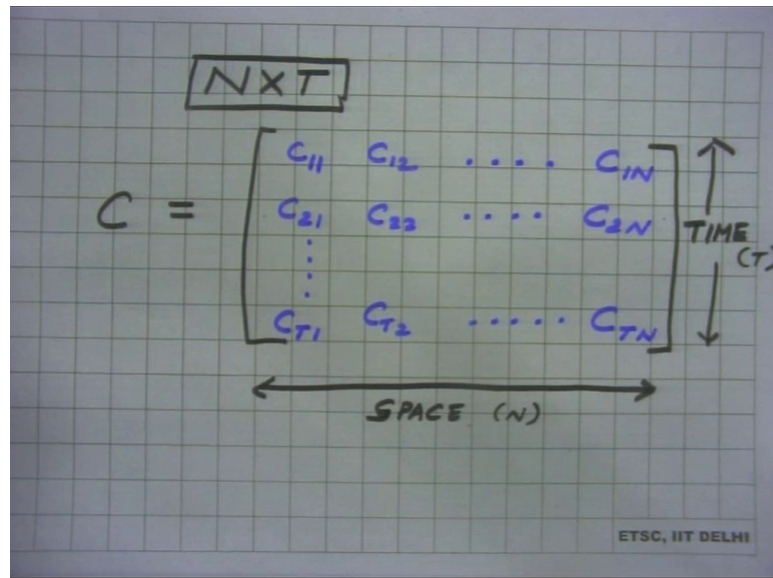
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Now, in the previous example we have used 2 transmit antennas and 2 time periods leading to a 2 cross 2 code matrix right this is obvious space time. In general we can consider a wireless system that uses N transmit antennas, clearly the gain with increasing the number of transmit antennas is phenomenal. So, we have again a certain number of time periods and certain number of N transmit antennas.

Now if this and then the time slots is T, then we have an N cross T matrix which will represent the code just like when 2 transmit antennas and 2 time slots were there; so, N was 2 and times loss T were 2; so, you had a 2 cross 2 matrix.

So, in general we will have this representation again this is the epsilons dot dot dot.

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So, if we look at in general if you look at the code for this N cross T encoding matrix, then C will be written as C_{11} , C_{12} to C_{1N} , then C_{21} , C_{22} , C_{2N} and you can keep going and you have T time slots. So, T_1 , C_{T2} , C_{TN} and if you see here we have N . So, this is the space axis and this is a time axis right. So, this is space and you have N and this is T ; so, this gives you the N cross T .


So, this is a general notation for a space time encoding matrix we can represent it.

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Designing good codes

- We saw that a space-time block code is essentially a mapping from input bits to the transmitted symbols.
- These symbols are transmitted simultaneously from the different antennas.
- The aim of the space-time decoder is to correctly guess the transmitted symbols.
- An error occurs when one code word is wrongly mistaken as another code word.
- Suppose a wireless system uses N transmitted antennas during T time periods.

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So, now the question is; what is a good space time block code? How do we design one? We need some design rules. So, we saw that a space time block code is essentially a mapping from input bits to a transmitted symbols. So, we have to play the game there, these symbols are transmitted simultaneously from the different antenna elements ok.

And therefore, they couple at the receiver; the aim of the space time decoder is to correctly guess the transmitted symbols. And error occurs when one code word is wrongly mistaken as another code word. So, suppose the wireless system use uses N transmitted antennas and T time periods, then the code word we have already looked at looked at as C 1.

Now the error is set to be the decoder is set to make an error if it decides that a different code word was indeed transmitted. So, even though C 1 was transmitted we say no it is a C 2 that is sent out. So, if you now look at; so, we already have C 1 which was sent out.

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$$\text{SENT } C^1 = \begin{bmatrix} C_{11}^1 & C_{12}^1 & \dots & C_{1N}^1 \\ C_{21}^1 & C_{22}^1 & \dots & C_{2N}^1 \\ \vdots & \vdots & \dots & \vdots \\ C_{T1}^1 & C_{T2}^1 & \dots & C_{TN}^1 \end{bmatrix}$$

$$\text{DECODE } C^2 = \begin{bmatrix} C_{11}^2 & C_{12}^2 & \dots & C_{1N}^2 \\ \vdots & \vdots & \dots & \vdots \\ C_{T1}^2 & C_{T2}^2 & \dots & C_{TN}^2 \end{bmatrix}$$

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C 11 C 12 so and so forth; C 1N C 21, C 22 up to C 2 N and going on for C T 1, C T 2 up to C T N, this one was transmitted, but what we received is C 2. So, we can differentiate it as C 2 11, c and so and so forth as opposed to what was sent. So, I can put a superscript here and I can distinguish these 2.


. So, this C 1 was sent and this is what we decode and wrongly. So, there is an error in the decoding. So, let us look at the slides again.

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Pairwise Error Probability

- The probability of **erroneously** decoding codeword C^2 when codeword C^1 was transmitted is called the pairwise error probability, and is denoted by $P(C^1 \rightarrow C^2)$.

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The probability of erroneously decoding code word C^2 when it is coded with C^1 was transmitted it is called the pair wise error probability. So, this is denoted by $P(C^1 \rightarrow C^2)$ was sent and C^2 received and this is the notation we will be using.

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Information Theory, Coding and Cryptography

Pairwise Error Probability

- Suppose, the codebook contains K code words.
- We can use the union bound to find the upper bound on the probability of error that the code word C^1 was transmitted and erroneously decoded.
- The upper bound is given by

$$P(\text{error}|C^1) \leq \sum_{i=2}^K P(C^1 \rightarrow C^i)$$

- This will be used determine the **code design criteria**.
- To calculate the pairwise error probability, we assume a fixed, **known channel matrix, H** .
- The average error is calculated by averaging over the distribution of H .

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Now, suppose the codebook contains K code words. So, we can use union bound to find the upper bound on the probability of error that the code word C^1 was transmitted erroneously decoded. And the union bound is simply the probability of error given C^1 is less than all possible. So, i is equal to 2 to K because i is equal to 1 represents the correct

decoding, C^1 goes to C^i . So, this pair wise error probability the upper bound on that will be used for our code design criteria.

So, to calculate the pair wise error probability; we assume a fixed known channel matrix H . So, the average error is calculated by averaging over the distribution of H . So, for different H we will have a different kind of a average error calculation.

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Information Theory, Coding and Cryptography


Pairwise Error Probability

$$P(C^1 \rightarrow C^2) \leq \left(\frac{1}{\prod_{i=1}^N (1 + \lambda_i E_s / 4N_0)} \right)^M$$

- where, E_s/N_0 is the SNR, M is the number of receive antennas and $\lambda_n, n = 1, 2, \dots, N$ are the eigenvalues of the matrix $A(C^1, C^2)$, defined as

$$\begin{aligned} A(C^1, C^2) &= (C^2 - C^1)^H (C^2 - C^1) \\ &= D(C^2, C^1)^H D(C^2, C^1) \end{aligned}$$

↳



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So, after some mathematics we can show that the pair wise error probability is upper bounded by this expression ok. Now here N of course, is a number of transmit antennas right and we have M as a number of receive antennas. So, we have a MIMO system in place and $\lambda_n; n$ is equal to 1, 2, 3 up to N are the eigenvalues of the matrix $A(C^1, C^2)$. Now what is this matrix? Let us define this. So, this matrix will play a central role in our design criteria. So, $A(C^1, C^2)$ is defined as this C^2 this was a matrix minus C^1 this was the sent matrix hermitian into C^2 minus C^1 . So, do we define it like this ok.

So, what is this C^2 minus C^1 ? It is like the distance, but remember what are the elements of this matrix? Well these are symbols in the constellation diagram. So, it is nothing, but a distance; so we represent it with D ; so, this is the distance between C^2 and C^1 ; $D(C^2, C^1)$ hermitian and distance between C^2 and C^1 ; so, this is a definition of this A matrix.

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Information Theory, Coding and Cryptography

Pairwise Error Probability


- Where $D(C^2, C^1)$ is the **difference matrix** (error matrix).
- Let r be the rank of the matrix $A(C^1, C^2)$.
- At high SNR, the one in the denominator of (6.18) can be neglected in order to obtain the following upper bound

$$P(C^1 \rightarrow C^2) \leq \frac{1}{\left(\prod_{i=1}^r \lambda_i \right)^M (E_s/4N_0)^{rM}}$$

- As discussed earlier, the error probability can be expressed as

$$P(C^1 \rightarrow C^2) \approx \frac{c}{(G_c S)^{G_d}}$$

where S is the SNR, G_d is the **diversity gain** (or diversity order) and G_c is the coding gain and c is some constant.

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So, and this D is the difference matrix. Now, let us talk about the rank of the matrix A right; so, at high SNR what we can do is you can neglect the previous equation the denominator and so, the pair wise error probability is upper bounded by 1 over; this are the eigenvalues product M and this is the indicator for SNR; $E S$ is energy per symbol and power noise power is denoted here and raised power $r M$, M is the number of receive antennas, small r is the rank rank of this matrix A .


So, if you recall we had earlier said to the pair wise error probability; if represented in general as follows some constant over $G_c S$ raised power G_d . Then G_d is the diversity order of the diversity gain this term diversity gain G_d and G_c is of course, the coding gain alright. So, if you compare these 2; we can quickly see that there is this r into M corresponding to G_d right because S is the SNR here S is energy per symbol over for N naught. So, this is an indicated for the S ; SNR and G_d is $r M$.

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Information Theory, Coding and Cryptography

Observations

- The **diversity gain** of the code is $G_d = rM$, i.e., the product of the rank of the matrix $A(C^1, C^2)$ and the number of receive antenna
- The coding gain is a function of the product of the non-zero eigenvalues of the matrix $A(C^1, C^2)$, or equivalently, the determinant of the matrix $A(C^1, C^2)$.
- **Full diversity** (MN) is possible if the matrix $A(C^1, C^2)$ is of **full rank**.

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So, the diversity gain of the code G_d is rM , but what is r ? r is the rank of the matrix $A(C^1, C^2)$. So, it appears that this matrix plays a critical role and we have to design the C^1 such that this rank is maximized. What is M ? M is the number of receive antennas; earlier M was 1, in that case this diversity gain would be limited to r right.

So, the coding gain is a function of the product of the nonzero eigenvalues of the matrix A , this one where is this coming from? We go back and see that this G_c provides the coding gain. So, this term is essentially the coding gain and it depends on the eigenvalues of this A matrix very interesting.

So, the coding gain is a function of the product of the nonzero eigenvalues the matrix A or equivalently the determinant of the matrix A , which means the full diversity is possible and it will be M into N the matrix A is of full rank.

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Coding Gain Distance

- The **Coding Gain Distance** (CGD) between codewords is the product of the non-zero eigenvalues of the matrix $A(C^1, C^2)$.

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So, we have to design this essentially gives us the method to design good space time block codes. We have to design A with full rank, coding gain we have studied earlier also, but we revisit it.

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Information Theory, Coding and Cryptography

Design Criteria for STBC

- The **Rank and Determinant Criteria** for designing space-time block codes is given by
- In order to achieve maximum diversity, the matrix $A(C^i, C^j)$ should be of full rank for any two code words, $C^i \neq C^j$.
- The smallest value of r , over any pair of code words, provides a diversity gain of rM . [**rank criteria**]
- In order to achieve maximum coding gain, the minimum determinant of the matrix $A(C^i, C^j)$ should be maximized for any two code words, $C^i \neq C^j$. [**determinant criteria**]

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So, the coding gain distance between codeword is the product of the nonzero eigenvalues of the matrix A. So, we see from this expression that the block space time block codes can provide you definitely with the diversity gain and coding gain and depending upon how you have designed this matrix A.

So, let us now summarize these design criteria for the space time block codes. So, of course, 2 things played an important role the rank and the determinant. So, we talk about the rank and the determinant criteria for designing space time block code and what is it? In order to achieve maximum diversity and diversity is so, important in we will take up an example to show the importance.

In order to achieve maximum diversity, the matrix A should be full rank for any 2 code words C_i not equal to C_j alright. And the smallest value of r over any pair of code words provides a diversity gain of r times M; this is called the rank criteria M being the number of received antennas.

Now in order to maximize the coding gain ok; so, we have already talked about the diversity gain. In order to maximize the coding gain, the minimum determinant of the matrix A should be maximized for any 2 code words C_i not equal to C_j . So, this is talking about the determinant and hence it is called the determinant criteria. So, together they are called the rank and determinant criteria for designing good space time block codes ok.

So, we can actually do a simple computers search to look at all possible combinations and whichever gives this high rank and the maximizes the minimum determinant right we get that.

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Information Theory, Coding and Cryptography


Alamouti Code - revisited

- It is a coding scheme with $N = 2$ transmit antennas and the transmitted code word is

$$C = \begin{bmatrix} s_1 & s_2 \\ -s_2^* & s_1^* \end{bmatrix}$$

- Consider a different pair of symbols with the corresponding code word matrix

$$C' = \begin{bmatrix} s'_1 & s'_2 \\ -s'^*_2 & s'^*_1 \end{bmatrix}$$

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So, let us revisit Alamouti code in this slide; so, we have seen that there are 2 transmit antennas right; so, N equal to 2 and the code word we have seen earlier.

Now, let us consider a different pair of symbols with the corresponding code word matrix C prime alright.

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Information Theory, Coding and Cryptography

Alamouti Code - revisited

- The difference matrix can be written as

$$D(C, C') = \begin{bmatrix} s'_1 - s_1 & s'_2 - s_2 \\ s_2^* - s_2'^* & s_1'^* - s_1^* \end{bmatrix}$$

- The determinant $\det[D(C^2, C^1)] = |s_1 - s_1'|^2 + |s_2 - s_2'|^2$ is zero if and only if $s_1 = s_1'$ and $s_2 = s_2'$.
- Thus, $D(C^2, C^1)$ is full rank when $C^2 \neq C^1$.
- **Therefore, Alamouti code satisfies the rank criterion and gives a diversity of $2M$, where M is the number of receive antennas.**

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So, the difference matrix is nothing, but C 1 minus C prime. So, if you find out the determinant of this is given by this and this is 0 if and only if S 1 is equal to S 1 prime and S 2 is equal to S 2 prime in all other cases the determinant will be nonzero. Consequently this difference matrix D is full rank right, when C 2 is not equal to C 1. So, that is the beauty of the design; it is a full rank.


So, Alamouti code satisfies the rank criteria and gives a diversity of 2 M, where M is the number of receive antennas. Earlier we had talked about M equal to 1 for Alamouti code; so, only one receive antenna. So, at least it will give a diversity gain of 2, I will just now see what do we mean by a diversity gain of 2 is it good does it help.

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Information Theory, Coding and Cryptography

Alamouti Code - revisited

- We can summarize the following about Alamouti code:
- Maximum diversity: since the code satisfies the rank criterion, it provides the maximum diversity (**diversity gain = 2**).
- Simple decoding : each symbol can be decoded separately, using only linear processing.
- **Full rate:** Alamouti code transmits two symbol in two time periods, thereby providing a rate of **$R = 1$** .

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So, let us understand a little bit more about the Alamouti code; it was one of the milestones. So, maximum diversity since the code satisfies the rank criteria, it provides the maximum possible diversity of 2 when the number of receive antennas is 1. So, diversity gain is 2 it provides symbol decoding; so, single symbol decoding, it is simple each symbol can be decoded separately using a linear processing this we have established because of its orthogonal nature.


And then it is full rate that is another interesting criteria what do we mean by full rate? Well we transferred it to symbols into 2 2 time slots right; so, effectively we are sending 1 symbol per time slot; so, the rate is 1. So, we are not saying that we are slowing down the communication, we did not take more time, we did not slow down the symbol rate ok, we did not increase the decoding complexity, but at the same time we got a diversity gain of 2.

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Information Theory, Coding and Cryptography

Performance of Alamouti Code

- The term '**quasi-static**' implies that for two time periods the channel does not change.
- Lets plot the **symbol error probability (SEP)** versus SNR for $N = 2$ transmit antennas and $M = 1$ receive antenna.
- For comparison, the SEP for a system with only for $N = 1$ transmit antenna and $M = 1$ receive antenna is also plotted.
- Both systems use the **QPSK constellation**.

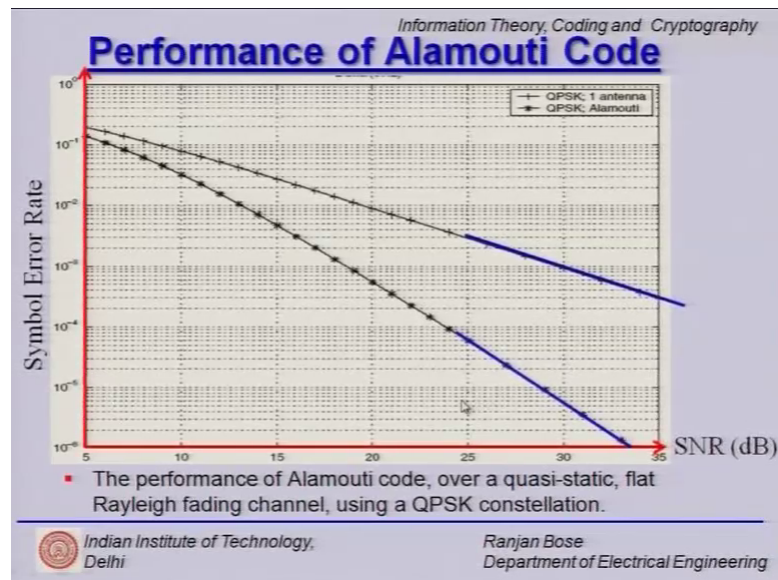
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So, let us quickly look at the performance of Alamouti code. So, we have made this assumption of quasi static implying that 2 time periods are such that the channel does not change significantly; so, H value is maintained.

So, let us plot the symbol error probability versus the SNR for N is equal to 2 transmit antennas an M is equal to 1 receive antenna. So, we had already studied this scheme in detail the Alamouti scheme and let us see how the symbol error probability curve looks like, but how do we compare, did we gain anything? So, we consider the case when it was a single antenna system. So, N is equal to 1 right; so, we plot that curve also and compare how much is a Alamouti giving us let us say both of them are using this QPSK constellation.

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So, let us plot the axis; so, on the x axis we are going to plot SNR, on the y axis will go to plot the symbol error rate. So, if you do so if you see the stuff curve is the single antenna transmit antenna and single receive antenna case, the second curve is Alamouti scheme; both employing this QPSK constellation. And if you can see that there is a shift in the slope, the gradient of this changes indicating the diversity gain and also please note that these 2 diverge, which means as we go at a higher SNR regime the diversity gain increases.


So, it brings home the very interesting point that a diversity gain becomes more and more effective as we go to higher and higher SNRs. Therefore, if you are working with systems which inherently work in low SNRs scenarios, it is really not worth it to look at schemes that provide diversity, diversity gain is best employed when reasonably good SNR is available ok.

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Information Theory, Coding and Cryptography

Performance of Alamouti Code

- It is clear from the figure that the Alamouti code provides a much better performance due to the **diversity gain** (observe the slopes of the curves).
- The SEP decreases as S^{-2} , where S is the SNR.
- Thus, the effect of the diversity gain becomes more pronounced at **higher SNRs**.
- This is evident from the fact that the performance gap **widens** as the SNR increases (SEP decreases).

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So, from the figure we observe that the symbol error probability decreases S is power minus 2 where S is the SNR which confirms that the diversity gain is of the order 2 and the effect of the diversity gives becomes more pronounced at higher SNRs and it as it is evident because the performance gap widens as SNR increases. And if you look at the slopes of the 2 curves alright, you will see that the slope is the indicator of the diversity gain that we get and typically asymptotically you can see.

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Information Theory, Coding and Cryptography


Single Symbol Decoding

- We have seen in Alamouti Code that the optimal decisions can be made based on single symbol at a time
- So, what is so unique about this code?
- The received pair may be succinctly written as

$$\begin{pmatrix} r_1 & r_2^* \end{pmatrix} = \begin{pmatrix} s_1 & s_2 \end{pmatrix} \Omega + \begin{pmatrix} n_1 & n_2^* \end{pmatrix}$$

- Where

$$\Omega = \Omega(h_1 \quad h_2) = \begin{bmatrix} h_1 & h_2^* \\ h_2 & -h_1^* \end{bmatrix}$$

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Now just a few more comments about this single symbol decoding and then we will try to explain intuitively why diversity happens, why diversity gain is happening? So, we have talked about this Alamouti code and the optimal decisions can be made based on single symbols at a time. And what we have is this r_1, r_2 star pair can be simply written as S_1, S_2 times ω ; what is that? Well ω is this h matrix h_1, h_2 right n_1, n_2 . So, we have just repositioned these parameters and we have rewritten the received pair as follows.

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Information Theory, Coding and Cryptography

Single Symbol Decoding


- Note that complex conjugate of both sides has been taken prior to rewriting in the matrix form.
- Upon multiplying both sides by ω^H , we obtain

$$\begin{pmatrix} \tilde{x}_1 & \tilde{x}_2 \end{pmatrix} = \begin{pmatrix} r_1 & r_2^* \end{pmatrix} \omega^H = (|h_1|^2 + |h_2|^2) \begin{pmatrix} s_1 & s_2 \end{pmatrix} + N$$

▷

where N can also be represented as Gaussian noise.

- This actually represents two separate equations that can be used for decoding the two transmitted signals using simple ML decoding.

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So, if you see we have this r_1, r_2 star that we had rewritten and if you multiply it with the hermitian of this ω ; then you simply get $|h_1|^2 + |h_2|^2$ times S_1, S_2 . So, you can see that upon multiplying both sides with this ω^H hermitian; you decouple the decoding part. So, it is effectively 2 separate equations that can be solved separately ok. So, that is the reason why it is a single symbol decoding.

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Information Theory, Coding and Cryptography

Why is there Diversity Gain?

- Suppose, instead of two transmit antennas, there was a single transmit antenna.
- Then the received signal would depend solely on h_1 , the fading channel coefficient.
- In deep fading environment, $|h_1|^2$ would be very small, and the received signal would be dominated by the noise in the system.
- However, if we consider the system with two transmit antennas, the received signal would depend on $(|h_1|^2 + |h_2|^2)$
- For the system to be dominated by noise, the quantity $(|h_1|^2 + |h_2|^2)$ should be very small, which in turn implies that $|h_1|^2$ **both** $|h_2|^2$ and must be very small **simultaneously**.

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Now, let us spend just 2 more minutes on why there is diversity gain intuitively; mathematics we saw slope of the curve we saw, simulation results we saw, but intuitively why are we getting? What is the physical reason? Can we link this mathematics to any physical intuition?

So, let us suppose that instead of 2 antennas there was only 1, then the received signal would depend only on the h the fading coefficient which can vary drastically and what does h do? When the fading is bad the value of h is small. So, h squared is even small right and then the noise will dominate because the signal is just out of the picture, but if we have 2 transmit antennas the receiver signal depends as we have seen on h_1 squared plus h_2 squared we have seen this therefore, here see the received signal pair depends on h_1 squared absolute value and h_2 absolute value squared right.

Now, when will the system be dominated by noise here is the noise. The system here will be dominated by noise if both h_1 and h_2 are small; only then this signal will become insignificant with respect to noise and system becomes dominated by noise both of them not only both of them both of them must simultaneously be small. If one is small the other is large, but the signal is still there both of them must simultaneously drop down, but that is bad because h_1 is for channel 1, h_2 is for channel 2.


And somewhere we said these guys are supposed to be independent, they are independently fading. So, the probability that both h_1 and h_2 ; together simultaneously very small is very rare and that is where we are getting the diversity gain.

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Information Theory, Coding and Cryptography

Why is there Diversity Gain?

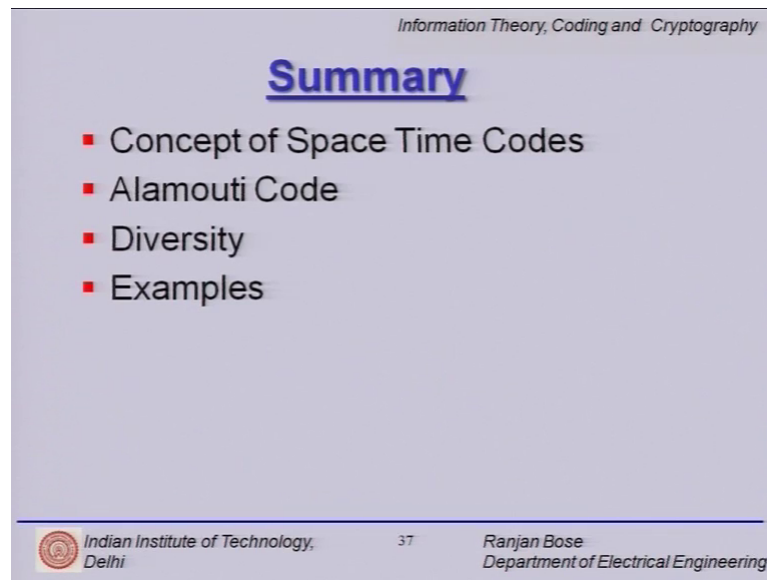
- This suggests that both h_1 and h_2 must be fading at **the same time**.
- However, this is highly unlikely, as we have assumed h_1 and h_2 to be **independent**.
- Thus, the received signal is less likely to be in fade, because of the **diversity** provided by the two independently fading channels.
- Here we have argued, based on intuition, why the system with two transmit antennas provides diversity.

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So, if noise must dominate then h_1 , h_2 must be fading at the same time, but it is contrary to our assumption that h_1 and h_2 are fading independently. This intuitively the received signal is less like to likely to be in fade because of the diversity provided by the 2 independently fading channels. So, thus we have argued purely based on intuition why the system with 2 transmit antennas provides diversity ok. So, there is a very physical feel to the whole thing.

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Information Theory, Coding and Cryptography

Summary

- Concept of Space Time Codes
- Alamouti Code
- Diversity
- Examples

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So, let us summarize what we have learned today; we have introduced this concept of space time codes and we brought it out with the help of this wonderful coding scheme called Alamouti code. We looked at why Alamouti code works and we also looked at this rank and determinant criteria, for designing good space time block codes. Then we spent some time talking about diversity and intuitively explained why diversity happens in this multi antenna systems; we also had some examples.

So, with that we come to the end of this lecture.