Nonlinear and Adaptive Control
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Lecture - 02
Preliminaries

Welcome everyone to lecture 2 of this course on adaptive control. Last time we had introduced the idea of adaptive control and we had discussed a couple of examples where adaptive controllers are advantageous over traditional fixed gain controllers and also mentioned that for this course you need to have some prerequisite background in nonlinear systems, nonlinear control or Lyapunov stability theory.

Having said that in this class I will talk about some preliminaries, which will be required for you to take this course forward; however, I encourage that you can go back and read about these preliminaries on your own in more detail. So in this class I will talk about the fundamental concepts involved, okay, so let us start with the stability of equilibrium points.

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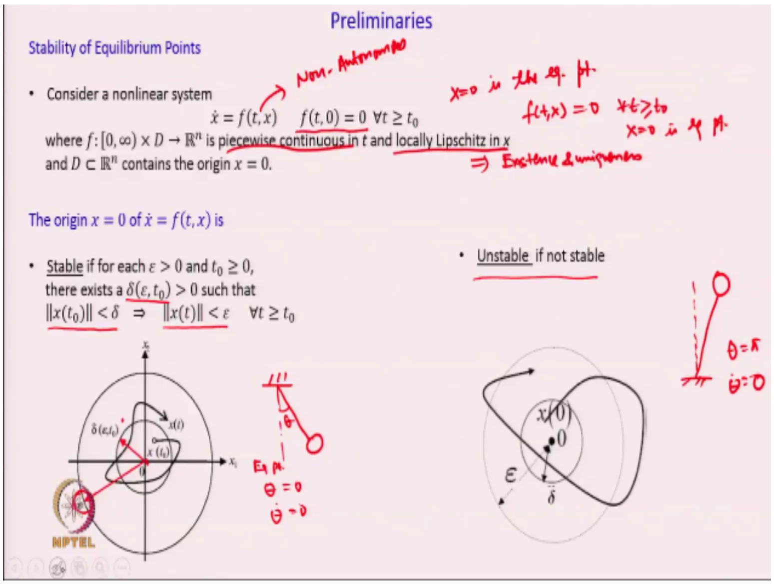

So as I mentioned these stability notion is tightly tied to the equilibrium points of a system, so normally there are various notions of stability that you might have heard about for example bounded input, bounded output stability which talks about the stability of the system where bounded inputs result in bounded outputs. This notion of stability is slightly tied to the stability of equilibrium points, which is small pertubation about the equilibrium point result in small pertubation in the state.

So let us consider a nonlinear system x dot = f of t, x; where f of t, 0 is = 0 for all time >/= t0. So here as you can see this function f is an explicit function of time and so this is an example of a non-autonomous system and the fact that f of t, 0 = 0 suggest that x = 0 is the equilibrium point. So the idea is that you substitute the right hand side of the equation to 0 and the value of x for which this is satisfied for all time gives you the equilibrium point.

So in this case x = 0 comes out to be the equilibrium point and for simplicity we will consider without loss of generality that x = 0 is equilibrium point for a nonlinear system. I had mentioned in the last class that you could always do a change of variables and even if you have a nonzero equilibrium point you could convert that back to 0. So this function f is piecewise continuous in t and locally Lipschitz in x.

So both these properties they suggest that the solutions exist and are unique. So existence and uniqueness of solutions is guaranteed if the function f is piecewise continuous in t and locally Lipschitz in x and this domain D on which x is defined contains the origin x = 0 alright. So let us look at the first definition.

So the origin x = 0 of the system is stable if for each epsilon which is positive and initial time >/= 0 there exist positive constant delta which is dependent on epsilon and the initial time t0 such that if the initial state at time t0 starts within a ball of radius delta then for all future time the state trajectories lie within the ball of radius epsilon. So this is illustrated by this diagram here where x = 0 is the equilibrium point.

And the system starts at some point x of t0, which lies within this delta ball and for all future time these state trajectories they lie within the epsilon ball. So this just suggest that the state trajectories lie close to the equilibrium point if the initial conditions are sufficiently close to the equilibrium point, okay. A thing to note here is that the constant delta depends on epsilon as well as the initial time t0.

For autonomous system that is for systems where the function f is not explicitly dependent on time t this constant delta is only a function of epsilon. So the task here is for each epsilon if you can find a delta which satisfies this relation then we say that the equilibrium point is stable. However, if this condition is not true we say that the equilibrium point is unstable that is even if there exist one epsilon for which there exist no delta such that if the initial condition start within the delta ball they are constrained to be within epsilon.

So even if there exist one epsilon for which this condition is violated we say that the equilibrium point is unstable. So an example of a stable equilibrium point that we had discussed last time as well was that of a pendulum. So for this system we consider the equilibrium point to be theta = 0 and theta dot = 0 that is equilibrium point that we are considering.

You can see that if we perturb a system slightly from the equilibrium point and suppose we assume that there is no friction in the system then for all future time the pendulum keeps oscillating about the equilibrium point and we can say that the system is stable. However, if we consider the other equilibrium point which is vertically upward equilibrium point for the pendulum.

So here theta = pi and theta dot = 0, so for this case for any epsilon that we choose it is not possible to find a delta such that if we start within, if the initial condition start within some delta ball they stay within the epsilon ball for all time. So this is an example of an equilibrium point which is unstable.

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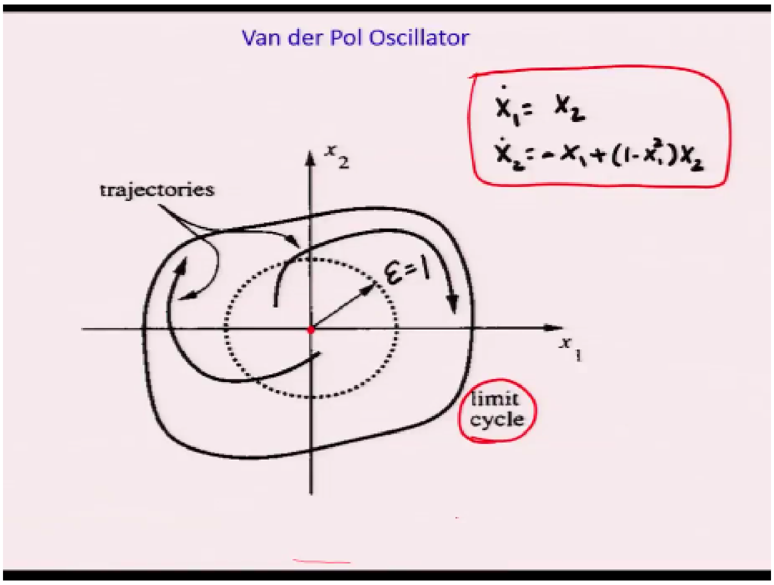

Let us take another example of a Van der Pol oscillator. So the equations of the Van der Pol oscillator are given by this it has 2 states x1 and x2 which are related like this. If you draw the face portrait of the system this is what it comes out to be, the origin is the equilibrium point and for any state trajectory which starts with a nonzero initial condition it will eventually converge to this limit cycle.

So if we construct an epsilon ball of radius 1, which lies within the limit cycle then can we find a delta ball wherein my initial conditions lie such that for all future time the trajectories stay within this epsilon ball of radius 1, the answer is of course no. So because these trajectories eventually will escape from this epsilon ball and converge to the limit cycle. So this is an example of system whose equilibrium point is unstable.

So you can also see that the trajectories since they converge to the limit cycle they always stay bounded that if the trajectories do not grow to infinity; however, the system is still unstable. So there is a difference between stability and boundedness.

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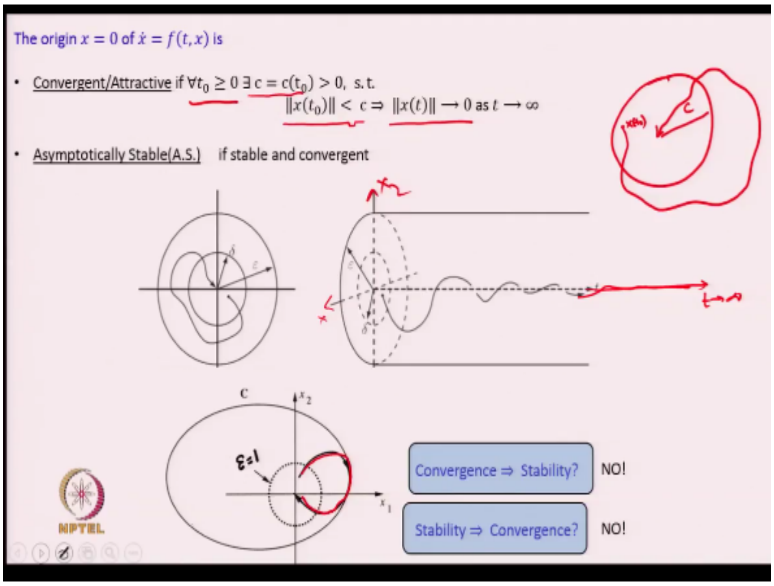

Okay, so the next is the origin x = 0 of a system x dot = f of t, x is convergent or attractive if for all initial time t0 >/= 0, there exist a constant c which is dependent on the initial time such that if the initial conditions start within this ball of radius c then as t turns to infinity x converges to 0. So this is an example of the equilibrium point being convergent or attractive. That is if this is the ball of radius c and if the initial condition start somewhere inside it then eventually they will converge to the equilibrium point.

So note here that the property of convergence or attractiveness is different from the notion of stability where we have to find a delta for each epsilon, here the delta epsilon condition is not to be is not required to hold rather the only condition is that the state trajectories eventually converge to the equilibrium point. The second is asymptotic stability, so the equilibrium point is said to be asymptotically stable if it is both stable and convergent and we just discuss what stability is.

And we also discussed what convergence is. So if the equilibrium point is both stable and convergent then the equilibrium point is said to be asymptotically stable. So this illustrates the asymptotically stable equilibrium point where the initial condition start within some delta ball and for all future time they stay within this epsilon ball so this has to be satisfied for all epsilon.

So if it is indeed true we say that the equilibrium point is stable and in addition to that we also see here that the state trajectories eventually converge to the equilibrium point. So this equilibrium point is not just stable, but it is also convergent or attractive. So it is also illustrated here where you have a 3 dimensional part of the states x1 and x2 and times. So here as t goes on the trajectories eventually converge to the equilibrium point.

Okay, so let us take another example here of a system where the equilibrium point is the origin. So here this is special system where if the state trajectory starts from any non-initial condition then they will reach a curve c and then eventually they will come back to the equilibrium point okay. So can we say that the system is convergent? Yes. It satisfies the definition of convergence that we just mentioned.

Is this system stable? No, because if we consider this epsilon ball of radius 1 we see that any initial condition will eventually leave this ball and reach the curve c. So we cannot constrain the state trajectories to lie arbitrarily close to the equilibrium point even if we start very close to the equilibrium point. So convergence in general does not imply stability, okay, so how about the reverse, does stability imply convergence?

So that is also not true, a system in equilibrium point maybe stable. The convergence may not hold that is the state trajectories may not converge to the equilibrium point.

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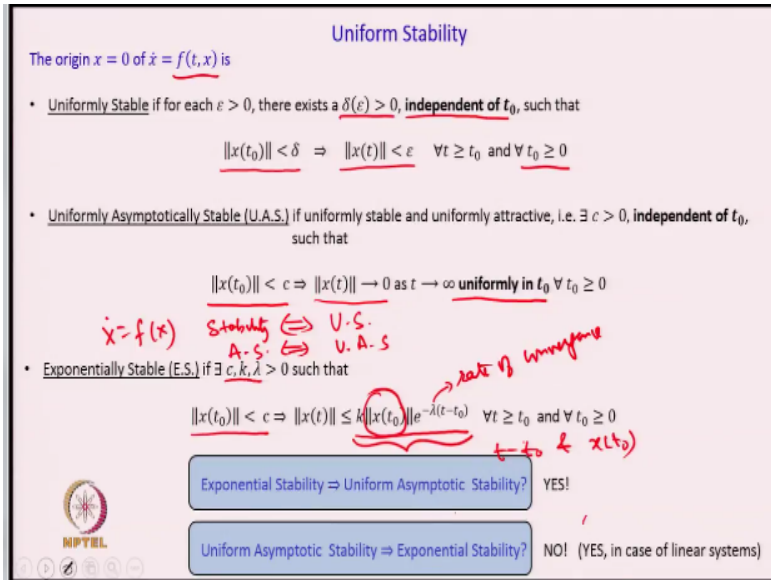

So next we go to the concept of uniform stability, so this concept is very critical for systems which are nonautonomous that means where the function f is explicitly dependent on time t. So the origin x = 0 is uniformly stable if for each epsilon greater than 0 there exist a delta which is dependent on epsilon which is a positive constant. Independent of t0 such that if the initial condition start within a delta ball they stay within an epsilon ball for all future time.

And for all values of the initial time >/= 0. So the important thing to note here is that the delta that we have used here is independent of the initial time t0. So in general for non-autonomous systems the stability behaviour is also dependent on the initial time that is if we start the system at some initial time it will have some stability behaviour. If we start the system at some other initial time it will have some other stability behaviour.

When we say that a system in the equilibrium point is uniformly stable what we mean is that the stability behaviour is independent of the initial time and equilibrium point is uniformly stable. Okay so the second concept here is uniformly asymptotically stable. So an equilibrium point is uniformly asymptotically stable if it is both uniformly stable and uniformly attractive. So we just talked about uniform stability.

So what does uniform attractiveness means, it just means that if there exist a constant, a positive constant Phi which is independent of t0 such that if the initial condition start within a ball of radius c then the state trajectories eventually converge to the equilibrium point as t tends to infinity uniformly in t0 that is the convergence is also uniform with respect to the initial time t0.

So this is very important and this is a stronger notion of stability than just saying that an equilibrium point is asymptotically stable. The fact that we can say that the equilibrium point stability behaviour is independent of the initial time is a very strong notion. For autonomous system, that means systems of the form x dot = f of x stability and uniform stability are the same notions.

Similarly, asymptotic stability and uniform asymptotic stability are the same notion. Because here the solutions x of t of the systems are not explicitly dependent on the initial time t0 and so the stability behaviour is anyways independent of the initial time t0. So both these notions are equivalent in the case of autonomous systems. Yet stronger notion of stability is exponentially stable.

So in the uniform asymptotically stable case this condition does not talk about the rate of convergence towards the equilibrium point it just states at the state trajectories asymptotically converge to the equilibrium point. So exponentially stability demands that the state trajectories converge exponentially fast towards equilibrium point.

So the condition is like this if there exist positive constant phi, k and lambda such that if the initial condition start within some region of radius c then the trajectories x of t for all time are </= an exponentially decaying function where the rate of convergence is given by lambda and this quantity is also dependent on the initial condition x of t0. So as we can see here this term is not explicitly dependent on t0, rather it is dependent on t - t0 and x of t0.

So what we can conclude from here is that exponential stability implies uniform exponential stability. So we do not have when we say the equilibrium point is exponentially stable. It implies that it is uniformly exponentially stable because the definition itself is independent of the initial time t0. Now the next question is does exponential stability imply uniform asymptotic stability, the answer is yes.

So in fact it is a stronger notion of uniform stability, it also defines that the rate of convergence is exponential. The next question is the reverse. Does uniform asymptotic stability imply exponential stability. So the answer is no in general because as I mentioned asymptotic stability does not demand the rate of convergence. However, this relation is true in the case of linear systems that is uniform asymptotic stability of a linear system implies exponential stability of the linear system.

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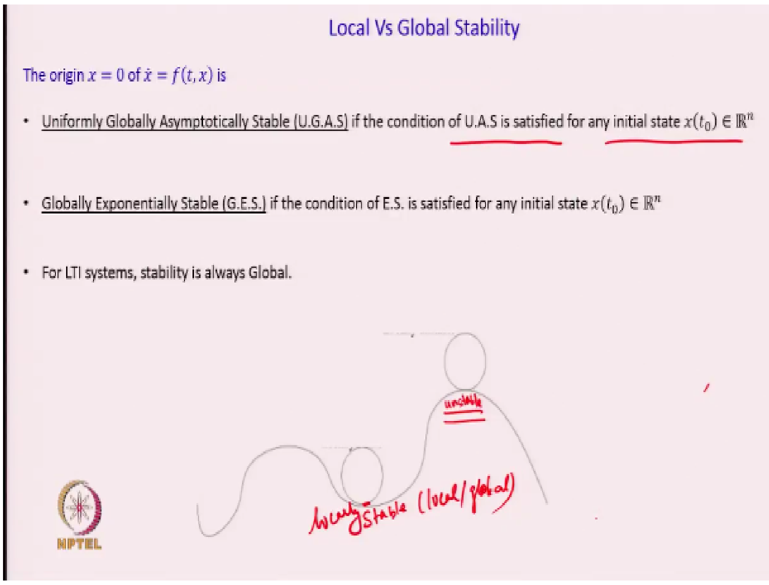

Alright another very important notion in analyzing stability of nonlinear systems is local versus global stability. For linear systems stability is always global. However, for nonlinear systems we need to talk about local stability so whatever notions of stability that we have done so far were all local. So let us look at what global stability means. So the origin is uniformly globally asymptotically stable.

Or in short we say it is UGAS if all the conditions of uniform asymptotically, asymptotic stability that we discussed in the previous slide are satisfied for any initial state x of t0. So if you go back here we stipulate that x of t0 start within some region of radius c. So we say that this kind of stability is local that is initial conditions have to be within some local region of radius c. However, if we say that c = infinity that is initial conditions can be chosen anywhere in the state space.

We say that stability is global. Similarly have global exponential stability that is if the condition of exponential stability is satisfied for any initial state in the state space we say that the equilibrium point is globally exponentially stable. So if we go back here if we say that c = infinity that is the initial conditions can be anything in the state space and this condition is true then the equilibrium point is going to be globally exponentially stable.

As I mentioned for linear time invariant system stability is always global. So either they are globally stable in fact they are globally exponentially stable or they are marginally stable or they are unstable. So let us look at this example where I have a system with multiple equilibrium points, so we can see here that this equilibrium point is a stable equilibrium point, that is if we perturb the ball slightly from it is equilibrium position it will eventually stay within the vicinity of the equilibrium point.

However here if we disturb the ball from it is equilibrium position it will not stay within the vicinity of the equilibrium point. So this equilibrium point is unstable. If I ask you the question about the stable equilibrium point whether this is local or global stability, then the answer here would be local stability because I just mentioned that for global stability the initial condition, the initial state can be anywhere in the state space and it will eventually stay within the neighbourhood of the equilibrium point.

However, if the initial state here is chosen to be the unstable equilibrium position then it will just stay there for all time. So this equilibrium point which is stable is only locally stable. Okay so the stability definitions that we discussed thus far are very rigorous; however, they as you can see they are very hard to verify because it requires that you solve the nonlinear differential equations which is not a straightforward task.

For linear systems it is easy because you could solve the linear systems and you could comment on stability; however, for nonlinear system in general it is not straightforward to solve nonlinear differential equations and so the stability definition that we discussed are hard to verify.

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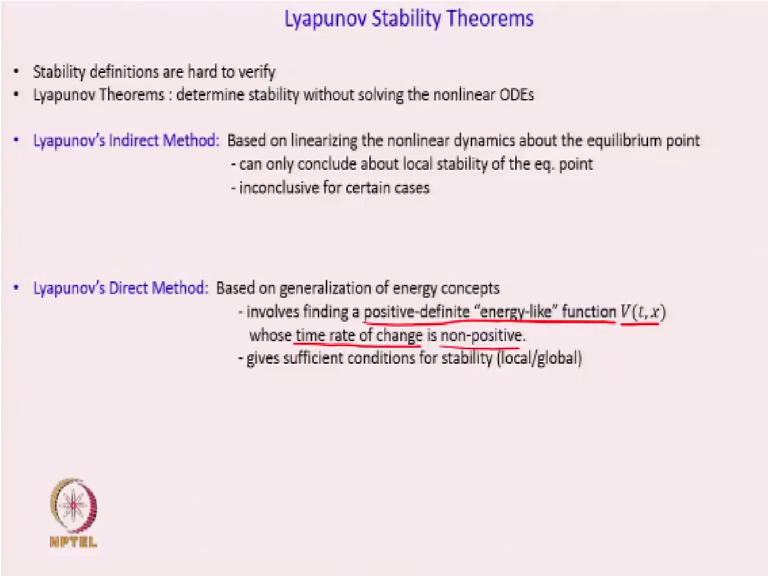

So Lyapunov theorems are useful in this scenario because they help us determine stability without solving these nonlinear ordinary differential equations and that is the real power of these Lyapunov theorems. So Lyapunov proposed 2 methods, one is the indirect method which is given a nonlinear system. You could linearize nonlinear dynamics about the equilibrium point.

However, you could only and then so what we do here is that given any nonlinear system we use the Taylor series expansion about the equilibrium point and we get Jacobian matrix then we can check the Eigen values of the Jacobian matrix if they lie on the open left half plane. We say the system is locally asymptotically stable if any of the Eigen values lies on the right hand plane we say that the system is unstable.

However, if any of the Eigen values are the real part which is 0 that is the any of the Eigen values lies on the imaginary axis we cannot conclude. So although this method is useful because it takes you to the linear region and then you could look at the Eigen values, but it is inconclusive for certain cases and so the Lyapunov's direct method is the more general method for determining stability.

So the basic idea is that it is based on generalization of energy concepts. So it involves finding a positive definite energy like function say V of t, x where t is the time and x is the state, so it involves finding a positive energy like function. So why do I say energy like because I certain situations it is easy to find the t total energy of the system for example for mechanical systems you could find the potential and mechanical energy similarly for electrical circuits it is easy to evaluate the energy of the system.

But for many systems the notion of energy is not very clear; however, we could still come up with a positive definite function of the states and we call that as energy like function and further if the time rate of change of this V of t, x is non-positive. We say that the equilibrium point is stable. So that is the basic idea, it is similar to the notion of the energy of you know of a system, which is continuously dissipated and if a system dissipates energy continuously it will eventually settle close to the equilibrium point.

So this method gives sufficient conditions for stability and it also talks about local as well as global stability, so it is a very general method.

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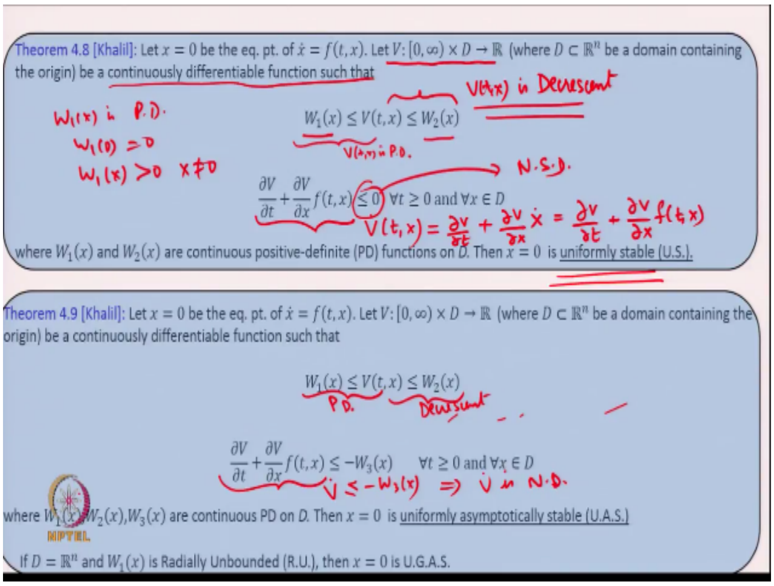

So let us look at the main theorem, which is theorem 4.8 in the book by Khalil, so I encourage that you look at this theorem in more detail. So this theorem says that let us consider x = 0 to be the equilibrium point of the system. So this can be said without loss of generality as I mentioned before and let us consider, let there be a function V, which is defined like this where D is the domain containing the origin and this function is continuously differentiable and it satisfies these relations.

So the first relation says that V of t, x is upper and lower bounded by time invariant positive definite functions W1 of x and W2 of x okay. So what this relation means is that V of t, x is positive definite. So if you can find a lower bound of this time varying function V of t, x and this lower bound is positive definite then we say that this function V is positive definite.

So when is W1 positive definite? So W1 of x is positive definite when W1 of 0 is 0 and W1 of x is greater than 0 when x is not equal to 0. This is what is meant by positive definiteness. The second condition where V of t, x is upper bounded by a positive definite function W2 of x this just implies that V of t, x is decrescent, okay so the first condition basically states that the function V is both positive definite and decrescent okay.

Okay so let us move on to the second condition. So the second condition is slightly more involved; however, if we look at it more closely this is simply the derivative, the time derivative of V. So V dot of t, x can be computed by using the chain rule. So we first take the partial of V with respect to t and then we take the partial of V respect to x times x dot which is same as saying.

So we replace x dot by f of t, x. So we in a way we take the derivate along the system trajectories, the fact that we substitute for x dot using the function f, we say that this is the time derivative of the function v along the system trajectories. So if the time derivative of V is </= 0 for all t >/= 0 and for all x on D, we say that the equilibrium point is uniformly stable.

Okay, so the definition that we had discussed previously which was very hard to verify can now be verified by using this Lyapunov theorem. So what this theorem essentially says is that if you could find a function V which is positive definite and decrescent and it is time derivate is </= 0 which in a way means that V dot is negative semi-definite then the equilibrium point is uniformly stable.

Suppose this condition of decrescent was not true then we could only say that the equilibrium point is stable. The uniform uniformity part is coming from the condition that the Lyapunov function candidate is decrescent. Okay so the second important theorem is theorem 4.9 in Khalil, which says, which is very similar to the previous condition so this just means that V of t, x is positive definite.

This means that V of t, x is decrescent and this condition implies that V dot is </= -W3 of x. So W3 is the positive definite function so –W3 is the negative definite function which means that V dot is </= some negative definite function which just implies that V dot is negative definite. So if these conditions are true the equilibrium point is said to be uniformly asymptotically stable.

So for asymptotic stability we see that we have a stricter condition for V dot. In the previous case V dot was only required to be negative semi-definite. In this case V dot is required to be negative definite. So then we look at the last condition that means if this domain D is the entire state space and the function W1 is radially unbounded then the equilibrium point is uniformly globally asymptotically stable.

So the fact that W1 is radially unbounded and the domain D is the entire state space suggest that we could include the notion of global stability here along with UAS. So what does radial unboundedness means, it just means that W1 of x goes to infinity. For all norm of x pertaining to infinity. So this is what we mean by radial unboundedness. So as you can see here we have not solved any nonlinear differential equation, but we have been able to comment on the stability properties of nonlinear systems.

So these are very powerful condition. However, the main challenge here is to find the function V. So how to find this Lyapunov function V to comment on stability properties is a perennial challenge in this area and by intuition, by experience, by some trial and error we could choose suitable Lyapunov function candidates. Not just to analyze stability but also we will see later to design controllers.

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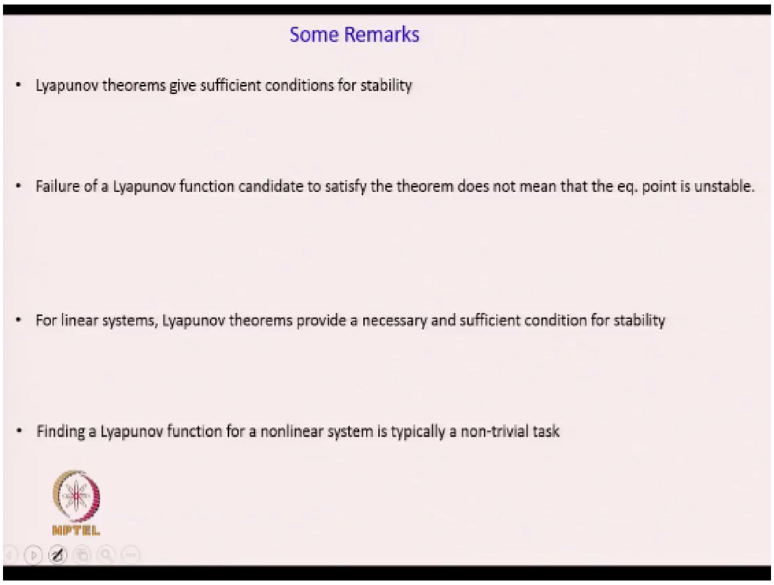

Okay so some more remarks about Lyapunov stability theorems. So the first one is that these are sufficient conditions for stability. So they are not necessary they are only sufficient which means that failure of a Lyapunov function to satisfy the theorem does not mean that the equilibrium point is unstable. So if we choose a Lyapunov function candidate that is you could find a function V of t, x which is positive definite and it is time derivative is at least negative semi-definite then we could say that the equilibrium point is stable.

However, if the Lyapunov function candidate does not satisfy the negative semi-definiteness of V dot we cannot say that the equilibrium point is unstable because these are only sufficient condition for stability. So you could find another Lyapunov function which may satisfy all these conditions. For linear systems Lyapunov theorem provide both necessary and sufficient conditions for stability and as I mentioned finding a Lyapunov function for a nonlinear system is typically a nontrivial task.

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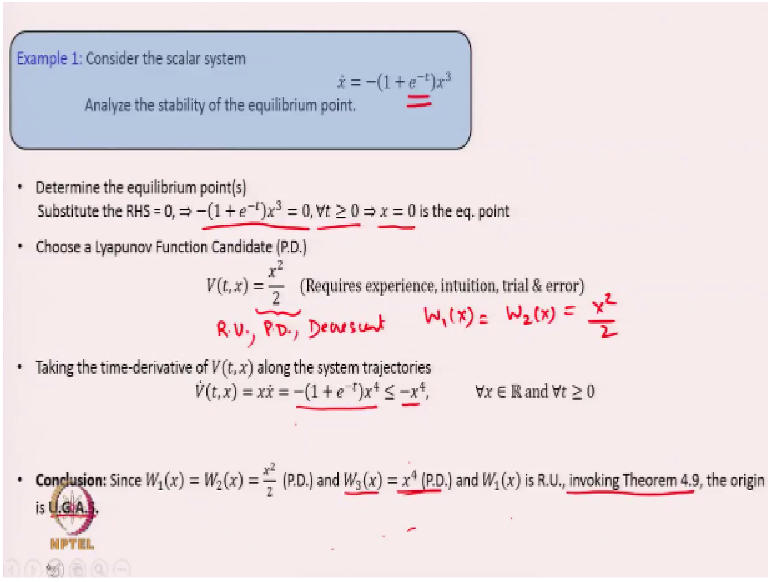

Okay so let us consider some examples, so let us consider this scalar system given by this nonlinear differential equation which is also time varying which means that this is the nonautonomous system and we have been asked to analyze the stability of the equilibrium point, okay. So the first step is to determine the equilibrium points.

So for this we substitute the right hand side of this differential equation to be equal to 0 and we see that if this equation is true for any value of x for all time t >/= 0 and we find that yes it is indeed is true when x = 0. So we say that x = 0 is equilibrium point in this case. Second we choose Lyapunov function candidate. So we choose a function which is positive definite.

So here we have chosen a very simple positive definite function x square over 2. Now choosing this function requires some trial and error, some experience and we will get better at it with time. So let us just right now assume that we choose this as Lyapunov function candidate and let us see if this satisfies any of the conditions that we laid out in the theorems. Okay so this function is positive definite right.

This function is also decrescent because here W1 and W2 are simply equal to the Lyapunov function candidate itself. So it is easy in this case because we have chosen a function which is not time varying. Next we, okay just one more thing. So this function is also radially unbounded. So it is radially unbounded, positive definite and decrescent.

So we take the time derivate of V along the system trajectories that is we differentiated respective time and using the chain rule and then we substitute for the system dynamics x dot

= this fraction site and what we get is this as a V dot. We see here that we have this time dependence in V dot. So V dot is in fact dependent on time although V was not time varying. Now this expression can be upper bounded by –x to the power 4.

So can we conclude anything from here? So as we said we have been able to choose Lyapunov function candidate which is positive definite radially unbounded and decrescent. Further we have also been able to find W3 which is positive definite, that is we can invoke theorem 4.9 and conclude that the origin is in fact uniformly globally asymptotically stable.

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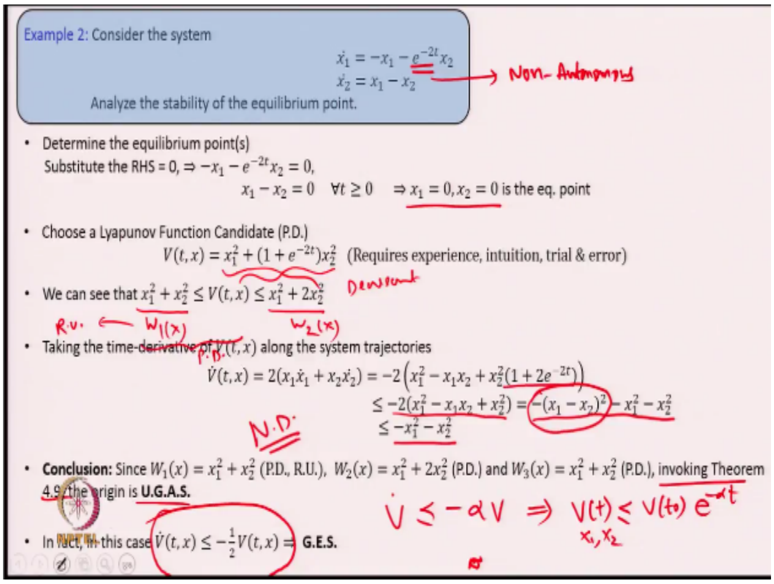

Let us move to the next example, so this is system with 2 states x1 and x2 and again we can see that this is a nonautonomous system and we have been asked to analyze the stability of the equilibrium points. So as we did before we determine the equilibrium point. We set the left hand side to be = 0 and we find that x1 = x2 = 0 which is the origin is the equilibrium point.

So next step we choose Lyapunov function candidate so we try to choose a positive definite function. So let us choose this as our Lyapunov function candidate and how we choose this is you know we can talk about that later, but right now let us just go ahead with this choice. So as you can see here as opposed to last time this Lyapunov function is in fact time varying okay.

So you could try different Lyapunov function candidates and you know eventually converge on a Lyapunov function candidate which helps you conclude about stability. So we say here that this function V can be upper and lower bounded by time invariant positive definite functions, so this is w1, this is w2, so which means that V of t, x is positive definite and this suggest that it is decrescent.

Further since w1 is radially unbounded we can conclude that V is also radially unbounded. So next we take the time derivate of V along the system trajectories, so we do the same thing. We take the time derivate, substitute the dynamics and what we get is, so we get some time dependence here then we further upper bound the system like this and then using some manipulations we convert it to this and this.

So this is a negative term, so we just throw it out and further upper bound this by this expression. So this expression says that V dot is negative definite. So we have been able to choose Lyapunov function candidate which is positive definite, decrescent, and radially unbounded and V dot is negative definite. So we can again invoke theorem 4.9 to prove that origin is uniformly globally asymptotically stable.

Further in this case you could also prove that V dot is </= -1/2 V of t, x. So maybe you can try this and then exercise so with this you can in fact say that the equilibrium point is globally exponentially stable. So wherever we can get this we can say that V of t is </= sum exponentially decaying term and since V is dependent on x1 and x2, we could potentially make similar conclusions about x1 and x2.

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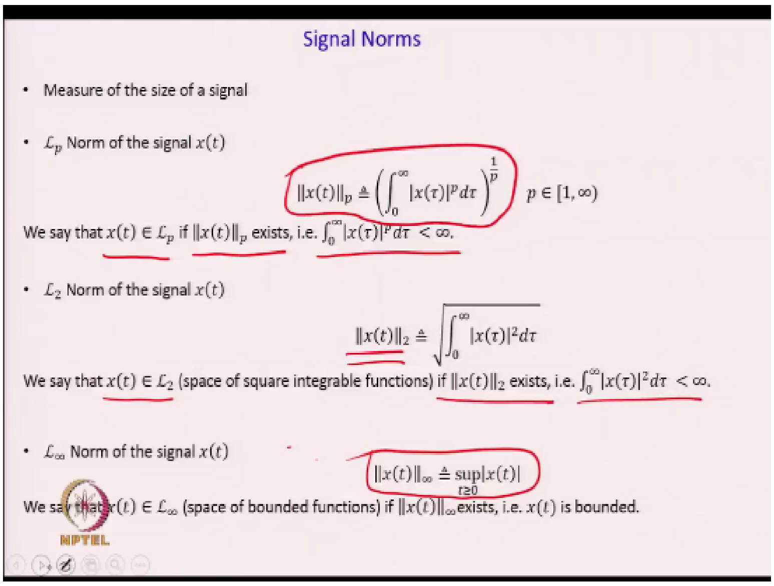

Okay so we move on. So the next thing that I wanted to discuss was signal norms. So given any time varying signal how do we find the size of the signal. So these norms give us a measure of the size of any signal. So let us talk about the Lp norm of the signal x of t which is piecewise continuous. So the Lp norm is defined like this, so here p is any number from 1 to infinity where infinity is not included.

So it is defined as this integral from 0 to infinity of the absolute value of x raise to the power p and then for the whole you take the p'th root of this. So we say that the function, the signal x of t belongs to the space Lp, if it is Lp norm exist, that is that this integral is finite. So this Lp is the space of all piecewise continuous functions for which the Lp norm is finite. Let us look at L2 norm. So we just substitute 2 in p and we get this p2 norm of the signal x of t.

So it is defined similarly. So we say that x of t is L2 if the L2 norm exist that is that this integral is finite. So this is the space of square integralable functions. Now in this definition V of Lp norm we said that p cannot be infinity because L infinity norm of the signal is defined slightly differently, it is defined like this as supremum over all t of the absolute value of x of t.

So we say that the signal belongs to the L infinity space if the L infinity norm exist and that just means that the signal is essentially bounded. So L infinity is the space of all bounded functions. So this will be useful when we do analysis in this course.

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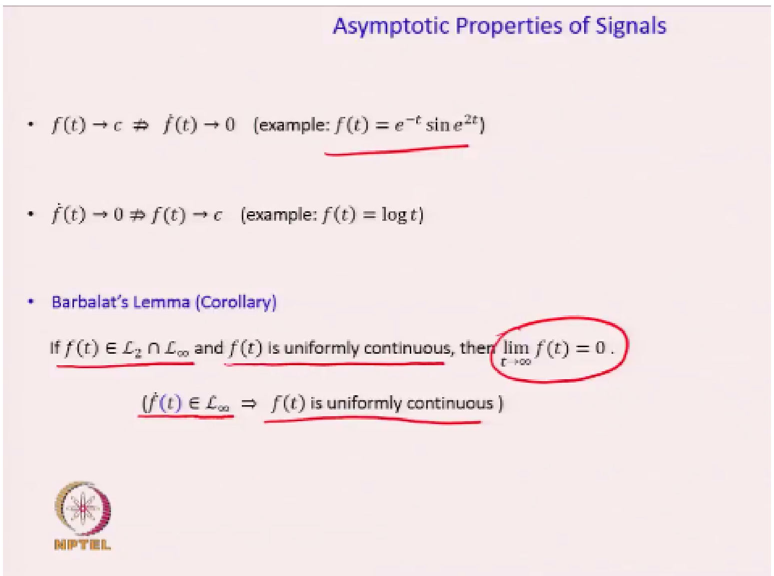

Okay the other concept I wanted to talk about was asymptotic properties of signals. So we say that so there are certain misconceptions about the asymptotic properties of signals. The first one being that if there exists function f which converges to a constant Phi that is not imply that it is derivate converges to 0. So as an example you can consider this and try and prove that this is in fact a misconception.

And secondly we can also look at the reverse which is if the derivate of signal converges to 0 that does not imply that the function itself converges to some constant example we can take this function to the log of t and verify that this is the case. Alright, so very important result that we will be using again and again in this course is the Barbalat's lemma. So what the Barbalat's lemma says, so here in fact I am taking about the corollary of the Barbalat's lemma.

So if the function f of t belongs to L2 and L infinity, so it is both L2 and L infinity. It is both bounded as well as square integralable and the function f of t is also uniformly continuous. Now uniform continuity is a stronger form of continuity or without going into the actual definition of uniform continuity what I want to tell is that it is sufficient to say that the derivative of the function is bounded that just implies that f of t is uniformly continuous.

So it is just sufficient to say that the derivative is bounded. So if the function f is uniformly continuous then we can say that as t tends to infinity, this function goes to 0. So these 3 conditions have to hold that if the function has to be absolutely square integrable, bounded and uniformly continuous then as t tends to infinity this function converges to 0.

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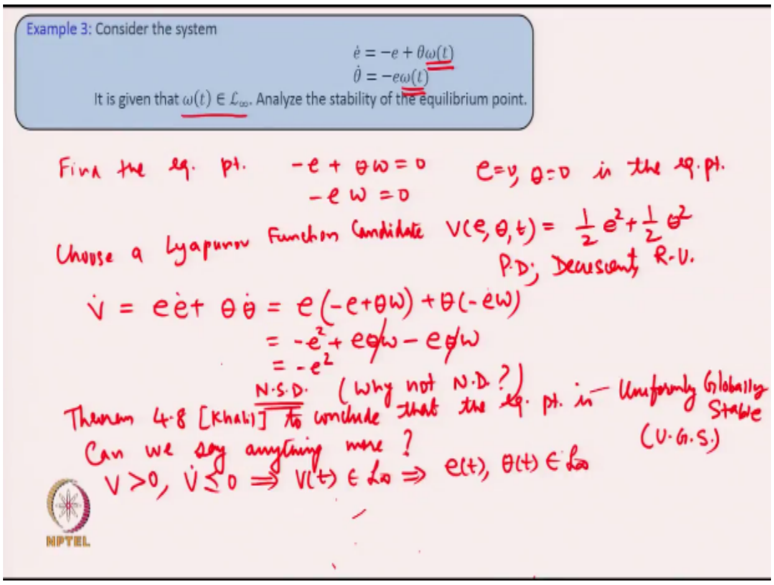

So let us consider another example so here we have a system consisting of 2 states e and theta and omega is given to be bounded and we have to analyze the stability of the equilibrium point. So the first step is always to find the equilibrium points. So we set the right hand side to be = 0 and we say that e = 0 and theta = 0 satisfies this equation for all time. So this is the equilibrium point.

And we have to analyze the stability with respect to this equilibrium point. So let us choose a Lyapunov function candidate V of as 1/2 e square + 1/2 theta square. So this is the most commonly used Lyapunov function candidate the sum of squares, it is the positive definite. It is decrescent, it is radially unbounded, so it satisfies all the nice properties. Let us differentiate this so V dot = e times e dot + theta times theta dot which is = e times.

We substitute the dynamics. So we see that these 2 terms cancel out and we are left with –e square. So what can we conclude about V dot. So V dot is only negative semi-definite. So why not negative definite. Maybe you can take that as an exercise. So V is positive definite, decrescent, radially unbounded and V dot comes out to be negative semi-definite. So we can invoke theorem 4.8 in Khalil to conclude that the equilibrium point is uniformly globally stable, that is U.G.S., okay.

So can we say anything more? So it turns out that we can so let us look at the signals, the signal e of t and let us see if we can use any of our theorems or lemmas that we have discussed. So here V is positive and V dot is </= 0, which implies that V of t is bounded,

okay. So since V is bounded and V is sum of squares of the states e and theta that just implies that e of t and theta of t are also bounded, okay.

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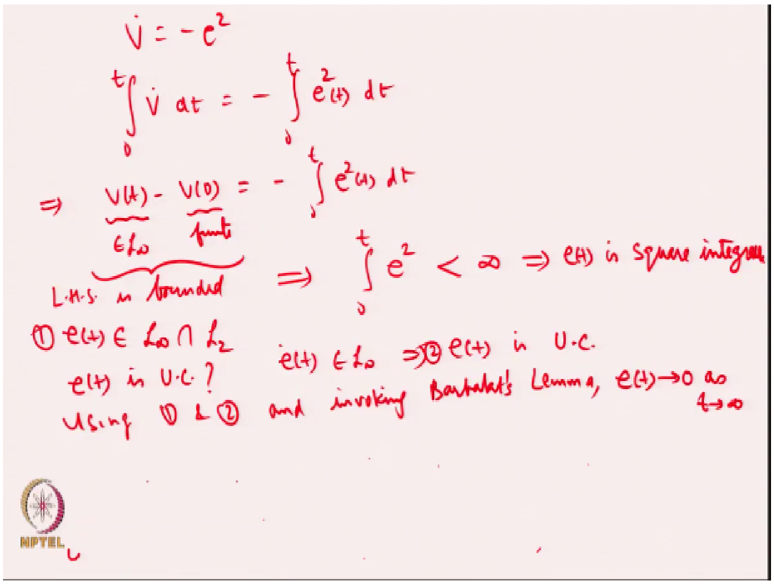

So let us look at the equation so V dot = - e square, so let us integrate this equation both sides from 0 to t. So what we get is V of t – V of 0 = - 0 to t of e square. So we have already proved that V of t is bounded and this is some finite number. So that means the left hand side is bounded which means that the right hand side should also be bounded. So this quantity e square should be less than infinity which means that e of t is square integralable.

So far we have concluded that e of t belongs to infinity and it belongs to L2. So it is both bounded and square integralable. The only condition that we need to prove for e to go to 0 is that if e is uniformly continuous. So to do that we will have to look at e dot and I mention before that if we can prove that e dot is bounded that implies that e of t is uniformly continuous.

So we go back and see what e dot is, so we look at e dot, so e dot is –e + theta omega t, so from here we know that e we have already prove as bounded. Theta also we have proved to be bounded, omega is given to be bounded which means that e dot of t is also bounded. So we have proved that e dot of t is bounded which implies that e of t is uniformly continuous. So we have satisfied all the conditions that is e of t is both an infinity and L2 and e of t is uniformly continuous.

So using 1 and 2 and invoking the corollary of the Barbalat's lemma, we can conclude that e of t goes to 0 as t goes to infinity. So using Lyapunov theorem we could only prove that the system, that the equilibrium point is uniformly globally table; however, then we used another lemma called as the Barbalat's lemma to prove that the state e of t converges to 0. So in this case although we can prove that e of t goes to 0 we cannot prove at least in this scenario it is not possible to guarantee that theta of t also converges to 0.

So that we cannot guarantee so the system cannot be guaranteed at least we cannot prove that the system is asymptotically stable although we have proved that this is uniformly globally stable and one on the states that is e of t converges to 0. So this system in fact is going to be very useful when we look at adaptive systems in general and this will be like a benchmark example for the adaptive systems that we will study in this course.

Alright, so just to summarize in this lecture we discussed the preliminaries which are required to understand this course that is the notion of Lyapunov stability, the different stability definitions, and then we also talked about the Lyapunov stability theorem which enable us to determine stability without solving the nonlinear differential equations and then towards the end we read Barbalat's lemma which is set of conditions that enable us to prove that certain signals converge to 0 as time goes to infinity.

So these will be useful for us as we go on I encourage all of you to go back and read about all the topics that we have discussed in this course in detail. Thank you.