**Introduction to Electronic Circuits**
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**Lecture no 15**
**Module no 01**
**Impedence Function, Poles Zeros and their Applications**

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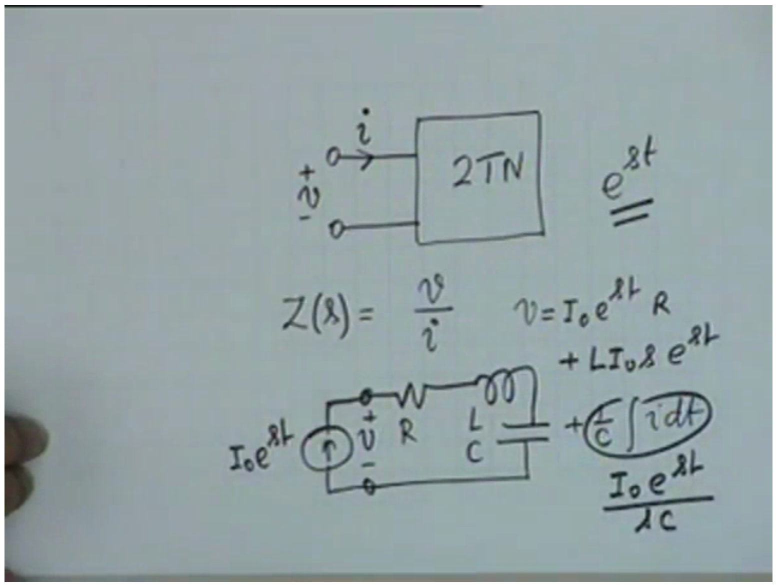


On impedance function, Poles and zeros and their application. We have already introduced the concept of impedance function as the ratio of voltage to current of a 2 terminal network whose excitation is exponential whose excitation is of the form e to the power s t, excitation could be either voltage or a current. And we have defined under this condition Z of s is the ratio of V by i with either V or i, one of them is an excitation, the other is a response of the form e to the power s t. As an example if we have well one point that needs to be mentioned here and noticed by you is that when you speak of excitation of e to the s t, we imply that the network under consideration when used excitation e to the s t and the response is also e to the power s t imply that the network under consideration is initially relaxed alright.

When we calculate the impedance function we do not take account of stored charges in inductor or stored energy I am sorry stored charges in capacitors or stored energy in inductors alright. So Z of s when we say V is exponential, i is also exponential well this is what it means. As an example if I take a simple let us say a simple RLC circuit, resistance, inductance and capacitance series circuit when the if i let say if I excite it by means of a current source I, which is I 0 e to the power s t then you know that the voltage V across this shall be some of the voltages across R, across L and across C and therefore V shall be equal to I 0 e to the power s t times R, this is the voltage drop across resistance + L di dt that is I 0 s e to the power s + 1 by C here is the question of initial relaxation, we are not taking into account any initial charges.

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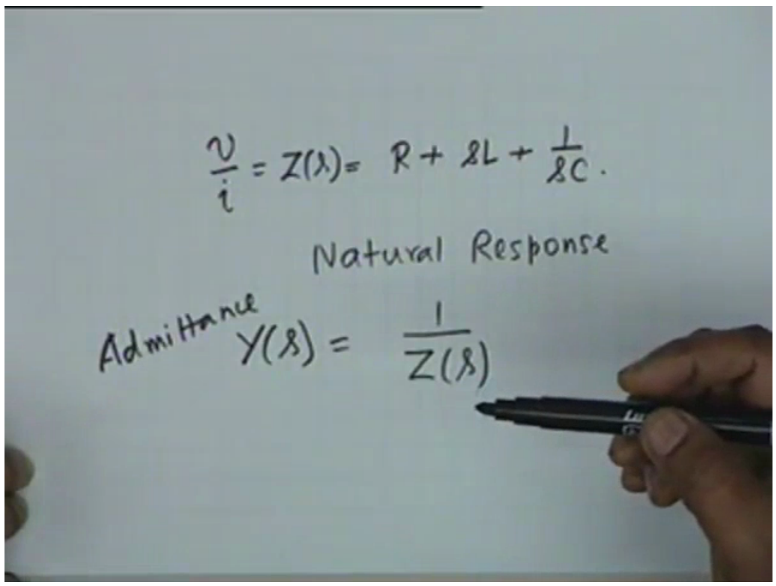


We have assumed that there are no charges stored therefore it is 1 by C integral i dt 0 to t alright and therefore this is simply this is simply equal to I 0 e to the power s t divided by s C. In other words, the impedance of the network between these 2 terminals is given simply by Z of s if I take the ratio of V to i which is Z of s which is equal to R + s L + 1 over s C. As you can see now, the impedance concept helps us to determine the so-called natural response of a circuit, we shall see later that the impedance concept also helps us in determining the forced response or steady-state response and shall be demonstrated later.

In the last lecture at the last moment we mentioned that there is an associated concept of admittance Y of s defined as the reciprocal of the impedance. In other words, the admittance of the series RLC circuit for example shall be 1 over R + s L + 1 over s C, very simply related, Y of s can be defined as the ratio of current to voltage when the excitation of 2 terminal network is of the form of e to the power s t with the network initially relaxed that condition in important alright. Now let us take another example, let us say we have a resistance and inductance and a capacitance.

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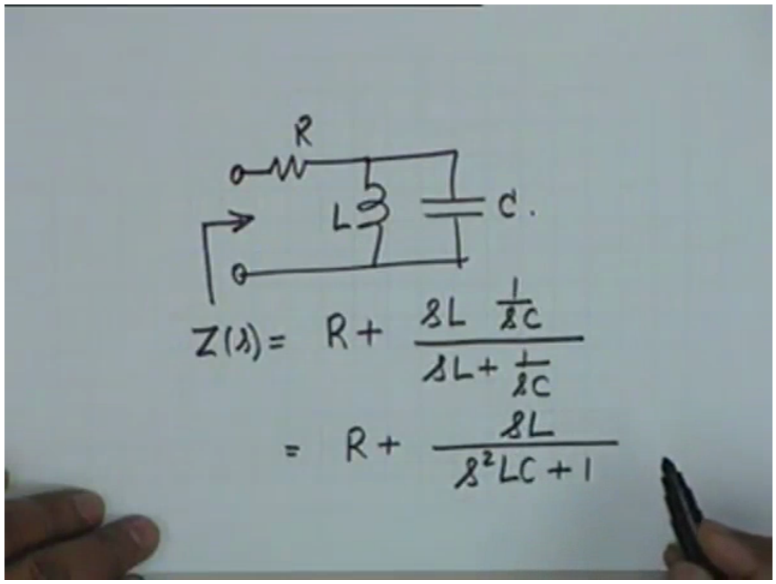


These are the 2 terminals of the network, the resistance inductance and capacitance, so the impedance by looking at these terminals, it is usually denoted by an arrow, the impedance looking at this terminal is therefore given by R + s L multiplied by 1 over s C, combination rules are exactly those of resistances s L + 1 over s C, which I can write as I can simplify this as R + s square L C + 1 and here I shall have simply s L, I can simplify this further.

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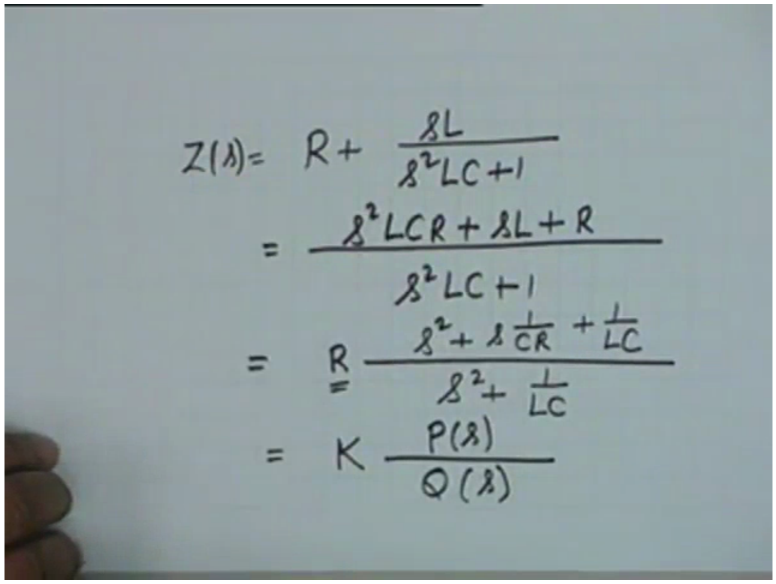


Z of s = R + s L divided by s square L C + 1, I can write this s square L C R + s L + R and this it is usual to write such expressions. Obviously this expression is a rational function, it is a ratio of polynomials 2 polynomials, it is usual to write the polynomials with the highest power term coefficient to unity that is the coefficient to the highest power term is reduced to unity. Obviously that can be done if I take L C R out from the numerator and L C out from the denominator and then I shall have multiplied by s square + s what should I have here, 1 over C R + 1 over L C and here I shall have s square + 1 over L C, is the term rational function familiar to you?

Okay, rational fashion is a ratio of polynomials and polynomial is a finite series with integral powers integral positive powers alright. Polynomial is a finite series with integral positive powers of the variable alright. So I see that this can be written as some constant some constant K which in this case is R, constant means it is independent of s then polynomial in s P of s divided by a polynomial in s Q of s alright so the impedance function is in general of this form.

And you notice that the numerator and denominator both here are quadratics that is $2^{nd}$ order polynomials and $2^{nd}$ order polynomial shall have 2 roots alright, both numerator and denominator have 2 roots and any polynomial can be written in terms of its roots as factors alright, and this is one reason why the numerator here has been reduced to a form in which the coefficient to the highest power is unity so that I can write this in terms of its factors.

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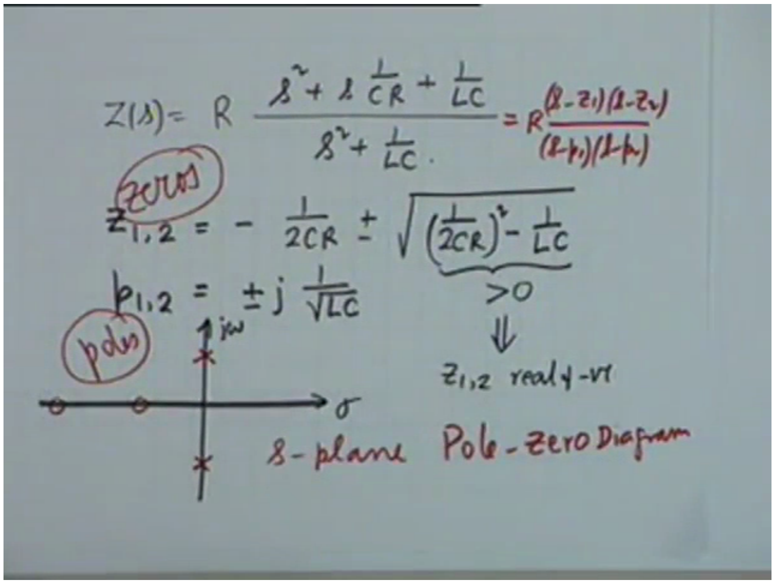


Suppose well if you write this particular Z of s is equal to R s square + s 1 by C R +1 over L C divided by s square + 1 over L C, you notice that the numerator roots numerator roots if I call them z 1 and z 2, small z which I denote by a z with a slash in the middle, this is how we distinguish between cap Z and small z. The roots of the numerator function are given by – 1 over 2 C R + – root of 1 by 2 C R whole square – 1 over L C is that okay? Alright these are the 2 roots of the numerator, and let us say the denominator roots, roots of the denominator polynomial are denoted by p 1 2 small p 1 2 then you see these are purely imaginary that is it will be + – j 1 over square root of L C.

The roots of the numerator can be either real or complex, can they be imaginary? Yes they can also be purely imaginary but if that is so, purely imaginary under what condition? Capital R is infinity. Did I make a mistake somewhere? No, this is okay. Pardon me.

Student: When L is less than 4 C R…

Student: Purely imaginary.

Purely imaginary, okay if it is purely imaginary then you see it is identically equal to 1, you see that the function will be identically equal to 1 because this term will drop out. Capital R infinity means s square + 1 by L C and s square + 1 by L C so you are left with only capital R, this becomes a degenerate case. Let us not worry about it, what I am saying is that the numerator roots can be either real or complex alright, the denominator roots are purely imaginary alright. Suppose this quantity, the quantity under square root sign suppose this is greater than 0 then obviously both the roots shall be z 1 2 shall be real and negative alright.

Now these are indicated on the complex plane or the s plane like this. Sigma j Omega and the 2 roots of the numerator are denoted by circles, supposed they are negative there will be one will be here and one will be here, they are indicated by small circles and the roots of the denominator are indicated by crosses, they shall be on the imaginary axis purely imaginary. This is the s plane on which the roots of the numerator and denominator are indicated are potted. You notice that the original expression Z of s can now be written in terms of its roots like this, s – z 1 times s – z 2 and s – p 1 multiplied by s – p 2 alright, the original expression can be written like this and this is the reason why the highest power coefficient was reduced to unity alright.

And it is obvious that when s takes the value either z 1 or z 2, the expression becomes 0 Z of s becomes 0. When s takes the value z 1 s = z 1, Z of s becomes 0, when s takes the value z 2, Z of s becomes 0 and z 1 and z 2 are called the zeros of the impedance function, z 1 z 2 are called zeros of the impedance function and that is why we used to symbol small z small z for zeros. On the other hand when s takes the value p 1 or p 2, the impedance function becomes infinite it goes up alright, such values of s for which the impedance functions becomes infinity are called the Poles of the impedance function, p 1 and p 2 are poles and therefore this diagram that we have drawn here is called Pole-Zero sketch or Pole-Zero diagram alright, let us take another case.

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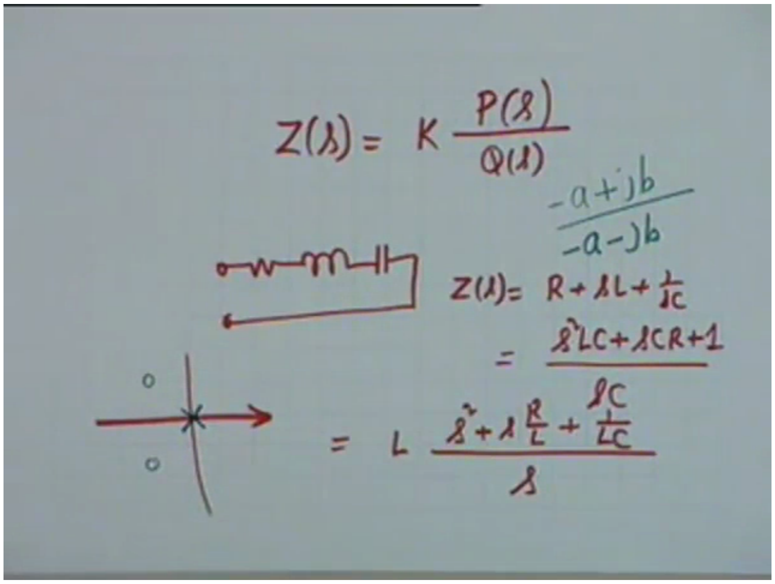


Is there any question on this? Definition of poles and zeros are let me repeat, if you write Z of s as equal to K times in general, it is not necessary the degree is 2 they could be higher than 2, it is not necessary that the degree of numerator and denominator is equal, it is not necessary alright. For example, if you take a simple RLC network alright 2 terminal network then you know the impedance is is R + s L + 1 over s C and this can be written as 1 over s c s square L C + s C R + 1, which I can write as L; L C taken out and C taken out from here s then you have s square + C R by L C s R by L + 1 over L C and you notice that the roots that the numerator is of degree 2 and the denominator is of degree 1 alright so the 2 degrees can be different can be different.

Now the numerator has therefore 2 roots, the denominator has only 1 root, where are these roots? The denominator root where is it? It is at the origin so there is a single pole at the original and if the roots of the numerator are let us say complex then you shall have 2 zeros like this. You might have noticed that if the roots are complex then they are complex conjugate that means if I have – A + j B as one of the roots then the other root must be – a – j B, now why is that why is this constraint?

Student: Because the coefficients are real.

Because the coefficients are all real alright and therefore it is a property of real polynomial real coefficient polynomials that if there is complex roots, its conjugate must also be a root. Now you notice that the point that I am making is that the numerator and denominator degrees P and Q degrees need not necessarily be the same they may be different and therefore the number of poles and the number of zeros may be different. Here for example, at s equal to 0 the impedance function goes up it becomes it becomes infinity that is it allows no current to flow, s = 0 means what e to the s t, s = 0 means a constant which means a DC and obviously this is corroborated by physical consideration that the capacitor does not allow DC to pass alright.

Okay then the 2 zeros here the 2 zeros here may be complex may be real and negative alright then at a 0 the function becomes 0 which means that the total circuit acts as a short-circuit, for that particular value of s not for any other value of s. We were taking we are going to take another example let us take that example.

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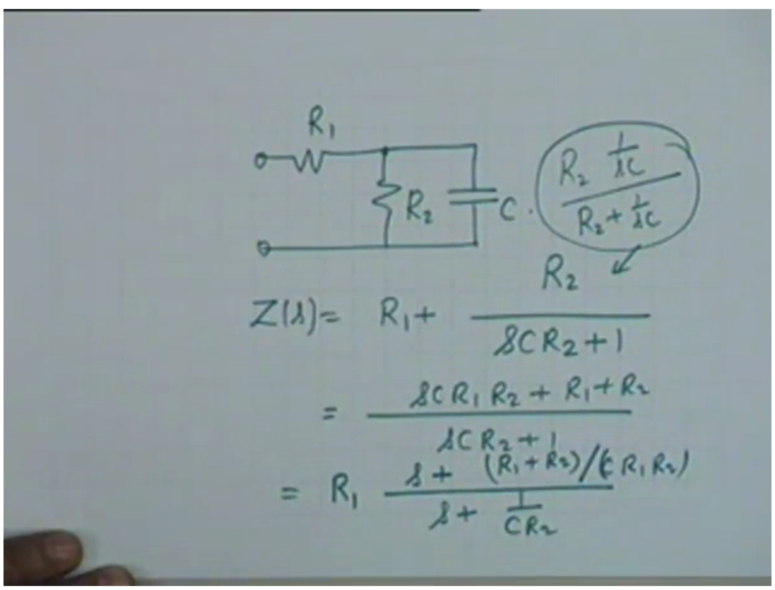


Before we generalise this let us take an example, let us say we have a resistance R 1, a resistance R 2, and let us say a capacitor C, the impedance function between these 2 points obviously is given by R 1 + R 2 divided by s C R 2 + 1, does it sounds magical? No, what I have done is R 2 1 over s C divided by R 2 + 1 over s C and this is what I have simplified to this okay by that algebra. So I can write this as s C R 2 + 1, s C R 1 R 2 + R1 + R 2 and now I must reduce it to that form that is the coefficients of the highest power should be 1 and so I take C R 1 R 2 out and C R 2 out therefore I get R 1 and the denominator I get s + 1 over C R 2 and in the numerator I get s + R 1 + R 2 divided by C R 1 R 2 is that okay? Alright. We can write this in a more elegant form…

Student: Sir I cannot see it.

Professor: You could not see the previous one okay, s + R 1 + R 2 divided by C R 1 R 2 and as + 1 by C R 2, you notice that R 1 R 2 divided by R 1 + R 2 is the equivalent resistance of R 1 and R 2 in parallel.

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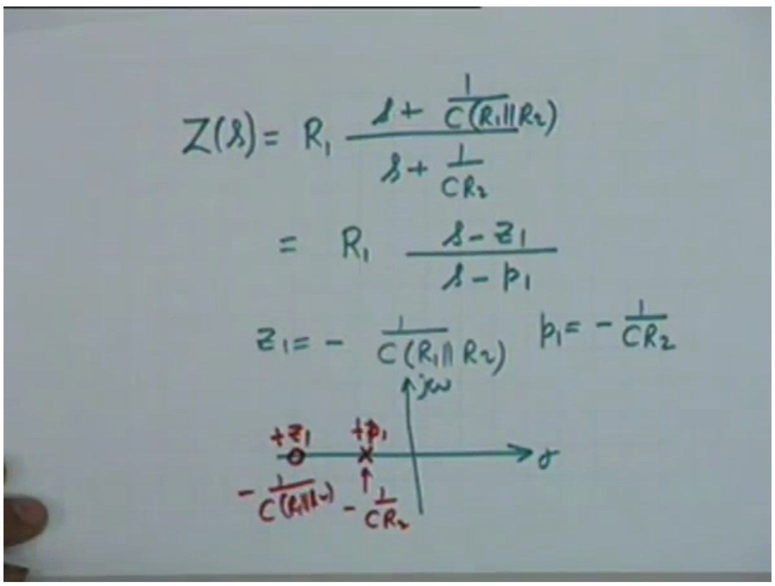


And therefore I can write it in a in a somewhat more decent form as R 1 s + 1 over C R 2 s + 1 over C R 1 parallel R 2, let us call this R 1 s – p 1 there is only one pole s – z 1 as only one zero. Then z 1 = – 1 over C R 1 parallel R 2 and p 1 is equal to – 1 over C R 2 so both the roots the roots of the numerator and denominator all of them are real and negative and therefore the Pole-Zero sketch shall be like this Sigma j mega. Now if I go from 0 to the left what should I need $1^{st}$, the Pole or the Zero? It has to be unique answer, the Pole so the cross and then the 0, this could be – 1 over C R 2 and this would be – 1 over C R 1 parallel R 2 or I say this is – – P1 and this is – z 1, this is the Pole-Zero sketch.

Student: (())(22:18)

That is what I do.

Student: You have put '–' sign there.

Thank you, this is p 1 and this is z 1 absolutely right. Now listen to me carefully, I am going to make what I defined here as poles and zeros are purely mathematical convenience. We said z of s is a rational function so I find out the roots of the numerator, roots of the denominator I called the roots of that numerator as zeros and the roots of the denominator as poles alright. So what? What is the physical interpretation of a pole or a 0? One of the interpretations is that at a pole the impedance becomes an open circuit, at a pole the impedance becomes infinity it becomes an open circuit so it does not allow current to flow agree. On the other hand at zeros the impedance becomes a short-circuit, it does not allow any voltage to be dropped agreed, they are completely duals of each other.

We can make them kind of interpretation with regard to an admittance after all admittance zeros are impedance poles is not it clear? Because one is the reciprocal of the other so I can make the same kind of interpretation, now another physical interpretation can be like this, it is the same thing but we will state in different terms and this will what will lead to application of Pole-Zero concept in finding the natural response of circuits.

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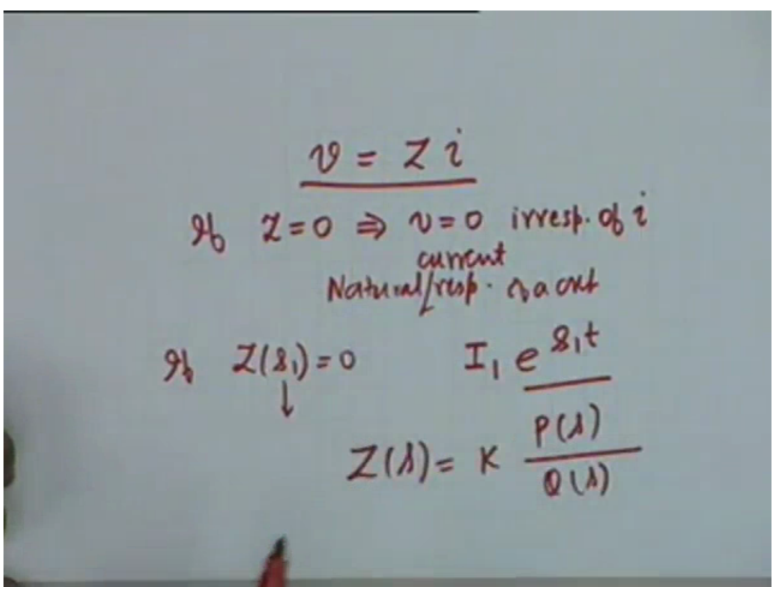


You see V = Z i provided V or i is exponential alright this is the definition. Now you notice that if Z = 0 at some value of s then V = 0 irrespective of i is not that right? Okay is that clear, this is simple mathematical statement that if Z = 0 whatever i is it does not matter the product shall be 0, so V = 0 irrespective of I, which means that a current can exist without a voltage agreed, a current can exist without an external voltage, at this value of s a current can exist without an external voltage which means for a current to flow there must be there must be charges available current after all is the flow of charge, which means that if the current has to flow.

Well it must be due to internally stored energy agreed and therefore a 0 in some way related to the natural response of a circuit, the 0 is in some way related to the natural natural response, what response? Current response natural current response of a circuit. And precisely if you sync if you ponder a little, you see that if Z of s 1 some value of s if Z of s 1 = 0 then s 1 is one of the roots or one of the functions occurring in the exponential of the natural response, which means that if Z of s 1 = 0 then the natural response natural current response of the circuit should include a term of the form I 1 e to the s 1 t, I shall illustrate this in a moment but let the concept come.

The concept is that you start from this relation and you say that if Z is 0 at some value of s then that s must have some relation to the natural response of a circuit. And you have seen in the examples that we have worked out that the natural response is of the form e to the s t natural response for example for a $2^{nd}$ order circuit it was A 1 e to the s 1 t + A 2 e to the s 2 t, how did that e to the s t come? Because we converted all equations into homogeneous differential equation and we sought the solution in the form of e to the s t agreed? And therefore the impedance concept therefore gives us the values of s, which shall occur in the natural current response of a circuit, let me illustrate this yes.

Student; Sir how Z = 0 is related to natural response of circuit.

Okay this is what I have explained. If Z = 0 then from this relation you see that V = 0 irrespective of i, in other words even without the presence of voltage the current can flow. Now without any external voltage current can flow if it is the natural response if it is if there is internally stored energy and therefore conceptually it appears that it must be related to the natural current response. And s the value of s at which the functions become 0 should occur in the natural current response of a circuit and s occur as a power of e alright, s and t the product occurs as power of e.

And what is the dimension of s 1, it is the frequency and therefore s 1 the value of s at which the impedance is 0 is called the natural frequency of current response natural frequency of current response and in that sense in that sense the zeros identify the natural frequencies of current response zeros of impedance function identify the natural current response of a circuit yes.

“Professor–student conversation starts”

Student: Sir, if Z is 0 and if we apply some voltage then will the circuit burnout?

Professor: Will the circuit burnout, yes the current shall be infinity the current shall be infinity but then you have to generate the voltage of the form e to the power s 1 t okay these questions we shall take up later. Conceptually a 0 at 0 of impedance function, the punch and become 0 yes.

Student: Sir you had said that the impedance concept we assume that inductor capacitor should be relaxing.

Professor: Yes in calculating the impedance, I understand your question.

“Professor–student conversation ends”

In calculating the impedance we say it is the ratio of voltage to current and this voltage or current is an external excitation, when you apply that all elements must be relaxed because otherwise the current response would not be purely e to the s t it might have a constant component also alright. So what we are doing is, in calculating the impedance let us say one has effect on the other, in calculating the impedance we assume the circuit to be relaxed. Once we calculate that then the parameters of the impedance are this constant K I am sorry let me write it again.

This constant K and the coefficients of P and Q these are the parameters of the impedance. Alternatively we say that P and Q let them be factored out, then these factors or the roots what is their physical interpretation, this is what we are looking into. The physical interpretation is that these the roots of P of s identify the natural frequencies of current response identify, it has nothing to do whether the charge is stored or not. If you wish to find natural response of a circuit s 1 I am sorry z 1, z 2 and z 3 and so on which are the roots of P of s shall show their teeth shall show their face. Let me take a very simple example which might which might convince you that it is so.

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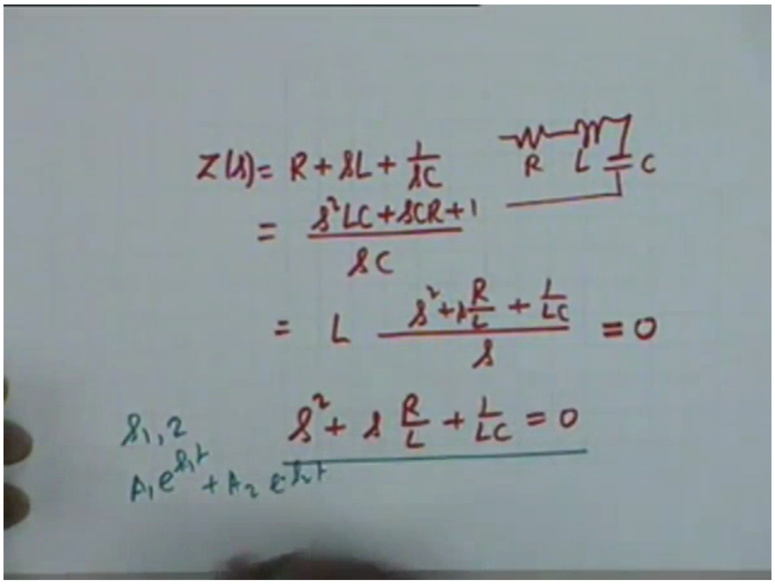


Let us take the family of RLC circuit, well the impedance is Z of s is R + s L + 1 over s C, which is equal to s C, s square L C + s C R + 1 alright. Now which is we wrote this as L multiplied by s square + s R by L + 1 over L C divided by s. Now you put this equal to 0, which means that the numerator that is s square + s R by L + 1 over L C = 0, do not you see that this is precisely the characteristic equation of a circuit. If you write differential equation and try a solution of e to the s t, this is what we had got earlier and it is the roots of this s 1 2 in terms of which we express the natural response as e to the s 1 t + A 2 e to the s 2 t agreed, so s 1 and s 2 are indeed the natural frequency is of the current response agreed? The simple example we want to stress that what we are saying is correct.

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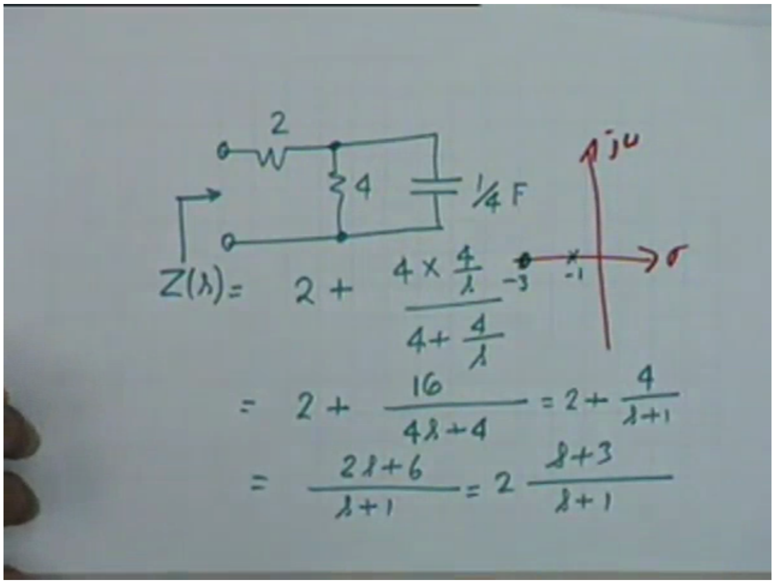


Let us take let us take an example, suppose I have a resistance let us say 2 ohms, another resistance 4 and a capacitance of value one quarter farad. We assume this circuit to be initially relaxed, we are calculate in the impedance Z of s. Obviously this is 2 + 4 multiplied by 4 by s that is correct divided by 4 + 4 by s, I simplify this 2 + 16 divided by 4 s + 4 is that okay? I multiply by s so 16 divided by 4 s + 4, which I can write as 2 + s + 1 and here 4, this I can write as s + 1 let me make it into a rational function, 2 s + 2 + 4 so 2 s + 6, and I write this as 2 s + 3 divided by s + 1 alright, therefore the Pole-Zero diagram is Sigma j Omega the pole is at – 1 and the zero is at – 3 no I am sorry this should be a circle, the 0 is at – 3 alright.

Now what we are saying is we have identified the zero and if you wish to find the current response natural current response of the circuit then e to the – 3 t shall appear in the natural current response, let us see now how can you obtain a current without having initially stored energy?

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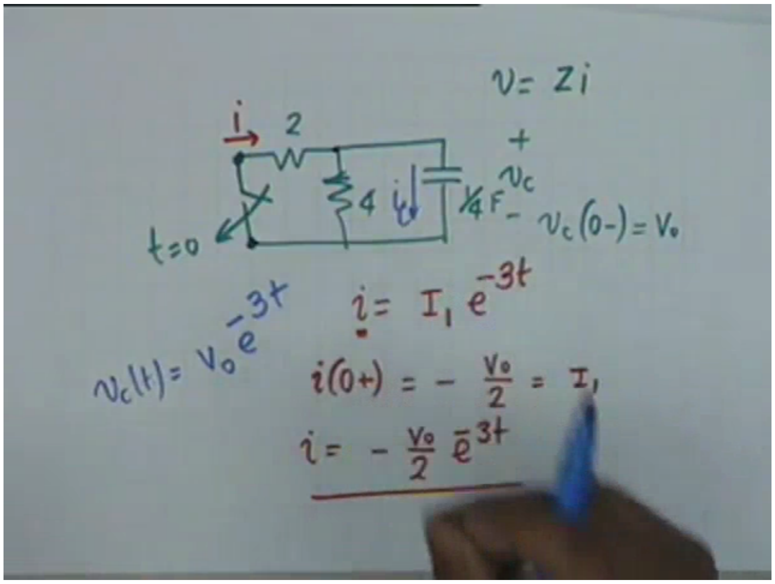


Therefore let us say that in this circuit 2, 4, let us say that in this circuit one quarter farad let us say that V sub c 0 – let us say is equal to V 0. Suppose V sub C 0 – = V 0 and then what we do is, between these 2 points we have measured the impedance we put a switch ON at t equal to 0 and I used to find current through this short-circuit. You see you remember that V = Z i what did we argue? We argued that if Z equal to 0 then i can exist without an applied voltage okay, it is between these 2 points that we considered V and i, we consider V between these 2 points so let me make V = 0, what does it mean? It means that I short-circuit this and let us say this is the current i then without doing any differential equation I can say that i must be of the form some I 1 e to the power – 3 t alright to find I 1 we have 2 apply an initial condition.

What is the initial condition? At t = 0 the current i now have made initial condition on I, I of 0 + after the switch is closed obviously is V 0, I did not say what is v 0… – V 0 by 2 that is correct because the voltage from here to here becomes V 0, current is in this direction assumed was the – V 0 by 2 which means that this equal to I 1 and therefore i = – V 0 by 2 e to the power – 3 t, I have not solved the differential equation I have simply found out the zeros of the impedance between these 2 points and then what I have done is I have shorted them, shorted the 2 points, found out the current through this and I I suspect that e to the – 3 t well I am convinced that e to the – 3 t shall occur in the natural response natural current response, we will come to voltage a little later, natural current response and therefore this must be the total solution.

For example, instead of the current suppose I have to find out V ct then I can write it immediately as the initial voltage is V 0 so it would be V 0 e to the – 3 t or I can find out the current through the capacitor i sub c well i sub c will be one quarter C times dVc dt is that clear? Okay, let us consider the other the other case in which we have to interpret the poles physical interpretation of the poles. Physical interpretation of zeros is let me repeat that you measure the impedance between 2 terminals alright, if you short-circuit these 2 terminals that is if you apply voltage equal to 0 and then initial stored over somewhere in the circuit then the current response shall contain the frequencies that are identified by the zeros of the impedance function alright, let us see the other case namely the poles.

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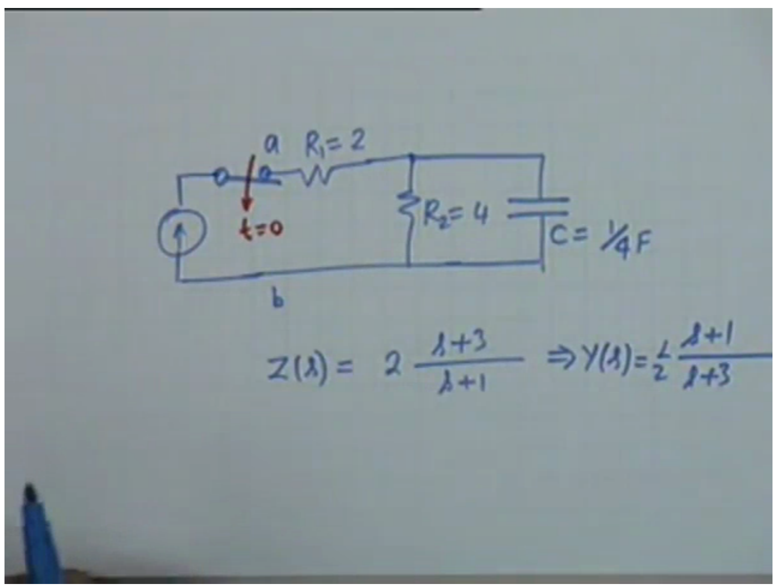


For the poles of an impedance function, poles of Z of s we work in terms of admittance instead of impedance, these are zeros of capital Y of s capital Y conventionally is the symbol for admittance alright. And V = Z i can be written as i = Y V and we argue the same thing that is if Y s 1 = 0 some value of s at which at which the admittance is 0 then s 1 must be the pole of Z alright, a 0 of Y is a pole of Z because one is the reciprocal of the other, if Y s 1 = 0 then i = 0 and v is arbitrary, any V would do there is no restriction on V, which means without the current excitation voltage can exist alright. Without current excitation voltage can exist so this value of s 1 must be related to the natural voltage response of the circuit.

In other words, the natural voltage response must be of the form must have a component V 1 e to the power s 1 t, the voltage response V of t must have this particular component and alright, let us take another example to illustrate this point that is interpretation of poles. I repeat, zeros are associated with the natural current response and poles are associated with natural voltage response, these are new concepts and I would go very slow and I want these concepts to soak in because in later classes we will simply assume, we will not explain why we are doing this, we will simply find out the poles zeros and say this is the voltage response, this is the current response and so on. Let us take an example to illustrate this point that is interpretation of poles, the same example we take.

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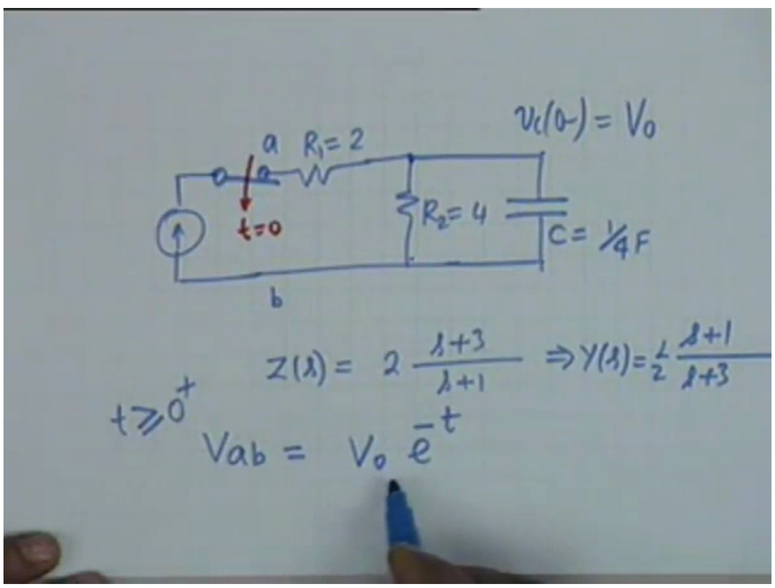


R 1 = 2 ohms please follow this carefully, R 2 is 4 Ohms and C is let us say one quarter farad and we have a situation like this let us say we have a a switch. Now what we have to do is without the current excitation, voltage can exist that means we will have to open the switch, the current source has internal resistance of infinity and therefore you will have to open it okay so this is open at t = 0 and there is a source that is a source let us say a current source here okay. Between these 2 points a and b the impedance we have already found out Z of s is 2 times s + 3 divided by s + 1 or Y of s is half s + 1 divided by s + 3 so the 0 of Y is that s = – 1.

Now when this current source is here , the capacitor accumulates a voltage alright because the current passes like this and it charges the capacitor alright. Suppose at t = 0 – V c 0 – suppose the capacitor accumulates the voltage of V 0, now the switch is thrown open, what we have to find out if V ab for t greater than equal to 0 +. Obviously this would be it must contain a term e to the – t and some constant, and that constant is obviously by inspection is V 0 alright, because when you open it there is no drop in this so it must be voltage across R 2 which is same as the voltage across C, is this point clear?

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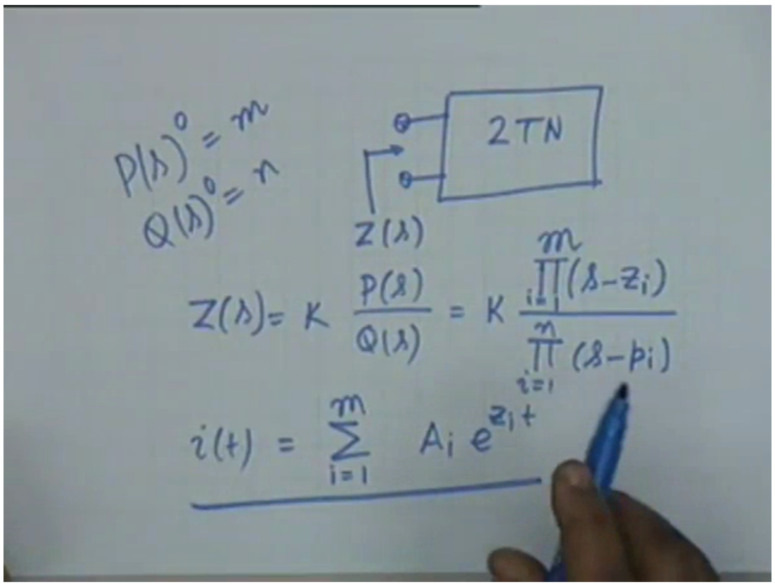


Now let me generalise, what you do is you take a 2 terminal network and between these 2 points you measure or you calculate the impedance or admittance, let the impedance be of the form K P of s divided by Q of S, which I can write as K, I can factor this out, you remember how K was taken, K is taken out such that P of s has the highest power coefficient as unity Q of s has the highest power coefficient as unity alright, then suppose the degree of P = m this is how I write the degree of the polynomial just like temperature okay. Suppose the degree of Q = n, so m and n need not necessarily be the same they can be different.

If it is so then the polynomial of degree n can always be written as the product, are you acquainted with the symbol? Okay, so the product of factors like this s – z i where i goes from 1 to n. Similarly in the denominator we shall have product of factors like s – p i, where i goes from 1 to n okay. Next, if you wish to find out the current flowing through the short-circuit that is if you short-circuit these 2 terminals and if you want to find out the natural current response then the natural current response i of t shall be of the form A i e to the power z i t where i = 1 to m is that clear?

Due to internal stored energy if you wish to find out the current when these 2 terminals are short-circuited then the current response shall be of the form.

Student: (())(46:17)

Yes that we shall come back a little later, if there are repeated roots then we shall have to modify in the same manner that we did for differential equations that is instead of let us say z 1 and z 2 are equal then instead of taking A 1 e to the z 1 t and A 2 e to the z 2 t, we take A 1 + A 2 t multiplied by e to the power z 1 it is the same kind of modification.

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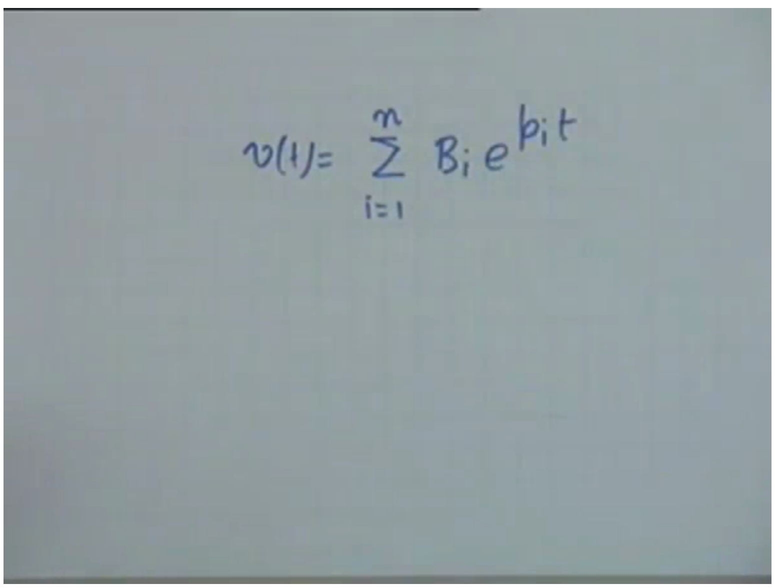


On the other hand, if these terminals are left open then the natural voltage response across the terminal shall be given by V t = submission B i e to the power p i t that is the poles shall now should obtain, the poles determine the natural frequency of voltage response where i = 1 to n alright, and then these constants have to be evaluated from initial conditions, we shall consider several examples next time to illustrate this point and that is where we conclude today.