Digital Communication
Professor Surendra Prasad
Department of Electrical Engineering
Indian Institute of Technology Delhi
Lecture No 32
Performance of M'ary Digital Modulations (Continued)

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We will continue with our discussion on the performance analysis of M-ary digital modulations based on constellations in two dimensions, right? Examples of which being Q A M, Q P S K, in general M-ary phase shift keying and many others, a variety of constellations. And you might remember

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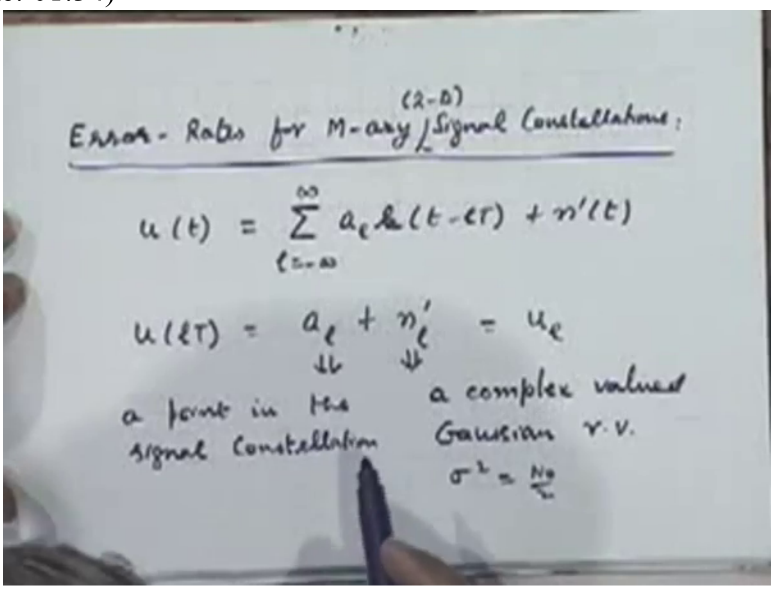

till now we have discussed the fact that in this kind of situation, the receiver uses a single matched filter whose output at the sampling instant, assuming that your outputs are really Nyquist pulse shapes, at the sampling instance these are essentially Gaussian random variables, this us u is are Gaussian random variables whose mean depends on the specific point that was transmitted as and whose variance depends on the, basically the characteristics of the noise output of the matched filter basically, it is a complex variable, so is this. In the two dimensions, both these points are in a complex plane and this is a random variable whose components are uncorrelated and since they are Gaussian, they are also independent, right? Each component having a variance of N zero by 2.

So this is the model with which we will work and this model corresponds to this kind of a picture

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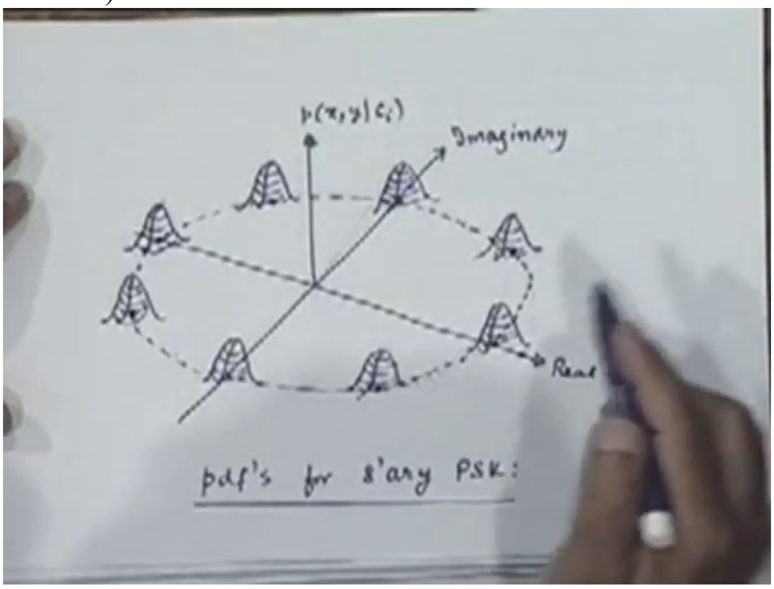

for the conditional density functions of the x and y components are the real and imaginary parts of the matched filter output given the particular symbol was transmitted. Even then the particular symbol c sub i was transmitted, Ok.

So suppose c sub i corresponding to this point, then the mean of u sub l

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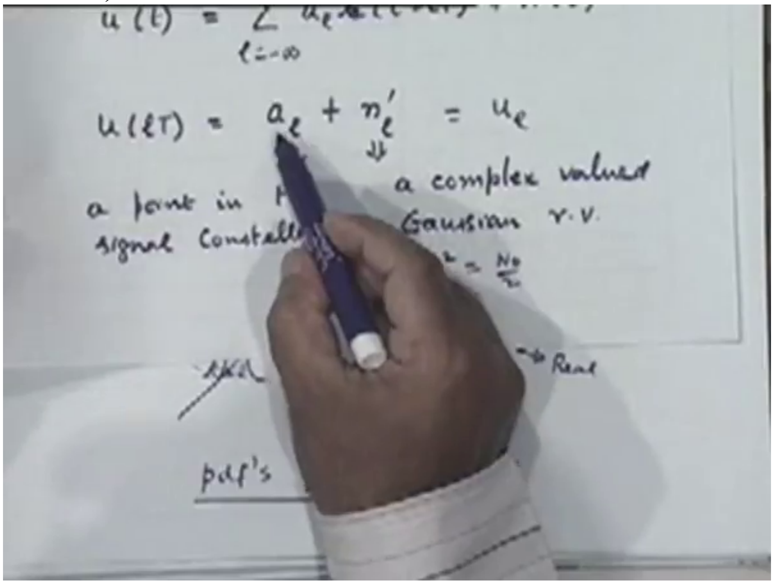

is this point

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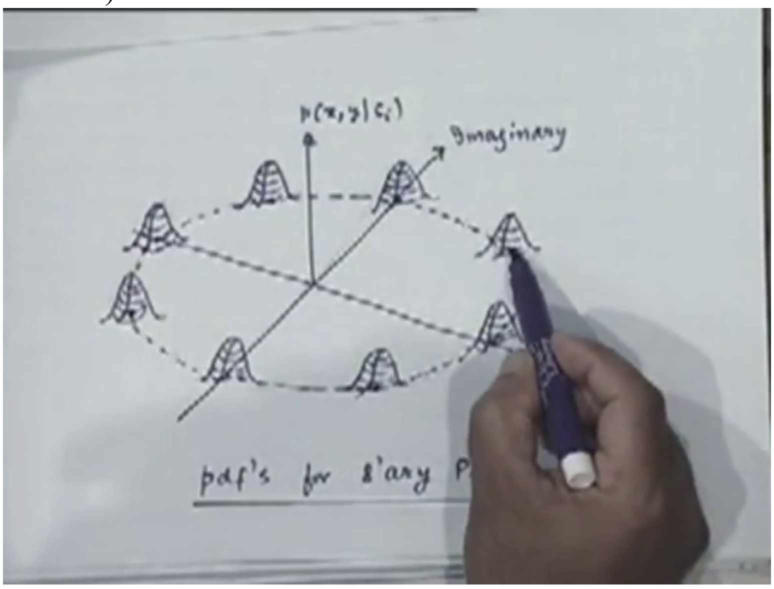

A sub l, alright and the density function that is described as this mean is essentially a normal density function in two dimensions, along x and y, real and imaginary right and therefore this kind of a density function, two dimensional Gaussian density function, right? Similarly for any other point in the constellation.

This is the picture depicting the case for 8-ary p s k, 8-phase phase shift keying, right? This is where we stopped yesterday. Do you have any questions about this model which is relevant here? Is it clear it is a relevant model to work with for analysis of 8-ary p s k or in general, two dimensional constellations, analysis of two dimensional constellations? Ok. So basically what do we have to look at? We have to define of course the error event in this case also and as you can well see, essentially how do you, how does the receiver decide which point is the most likely transmitted point by looking at whether there received u sub l lies in the corresponding decision or not.

Suppose you transmitted this and the corresponding decision, let say is this. Then if you received, u sub l lies within this sector, you are considering the case of 8-phase p s k here, then we will of course make the correct decision or else if it lies in some other sector that describes the error event for this; we have actually transmitted this but we have received u sub l, this value because of the noise

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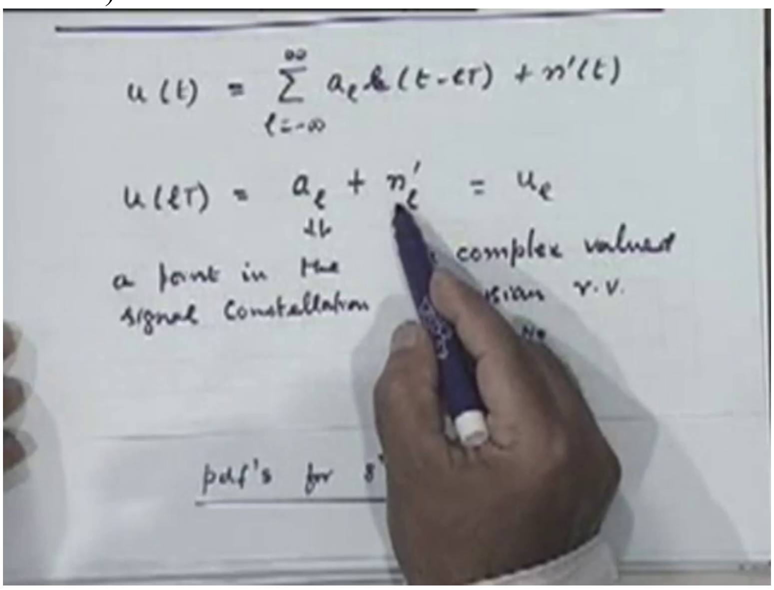

here, gets modified

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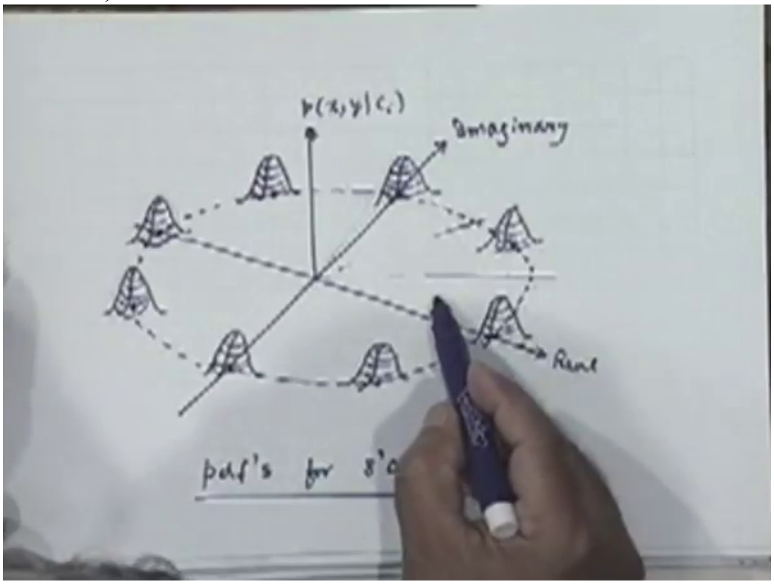

vectorally and takes this vector a sub l and modifies it to some other vector u sub l in some other sector, right?

So this is your, let us say the true vector, the noise, take it here, bring it here and now you wrongly

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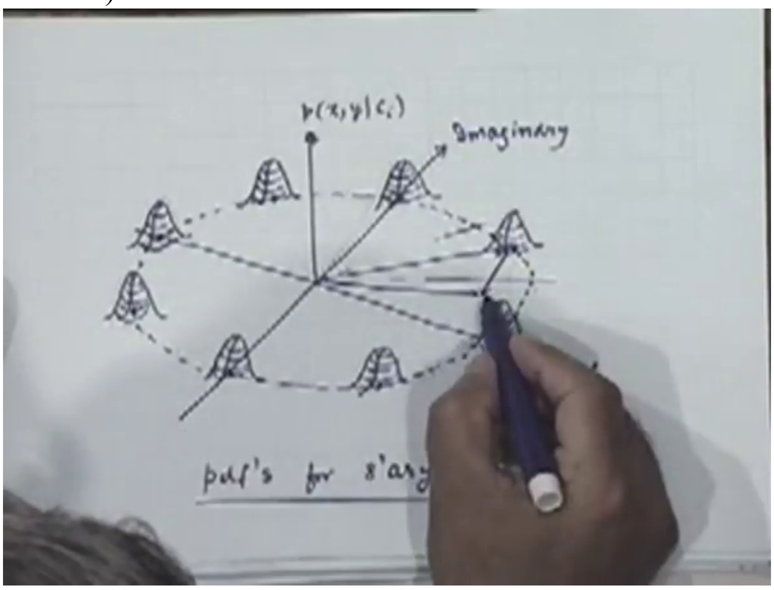

decide that this is a point that was transmitted, right and so on. It could, depending on how the noise two components behaved; you could have gone into any sector with different probabilities. So if a picture is clear, then the analysis is fairly simple and straight-forward to carry out. So how do we define our p sub e?

Let me, before writing the expression, I have plotted in more detail here the

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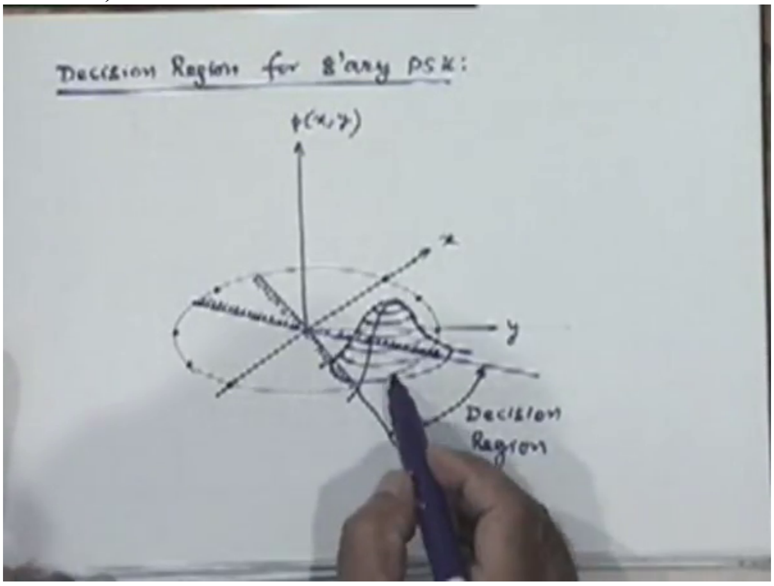

density function around a specific point. Let us say this was a point that was transmitted, right? This is the same 8-ary p s k situation; density function is shown over here. It is a two dimensional density function and therefore it is a three dimensional picture here. And these two shaded lines define the decision region around this point, right for the 8-ary p s k case. And as you can see the density function has space over outside the sector, right and it is the spilling over outside the sector which gives a finite probability to the error events, right?

This tent-like shape does not span or its base

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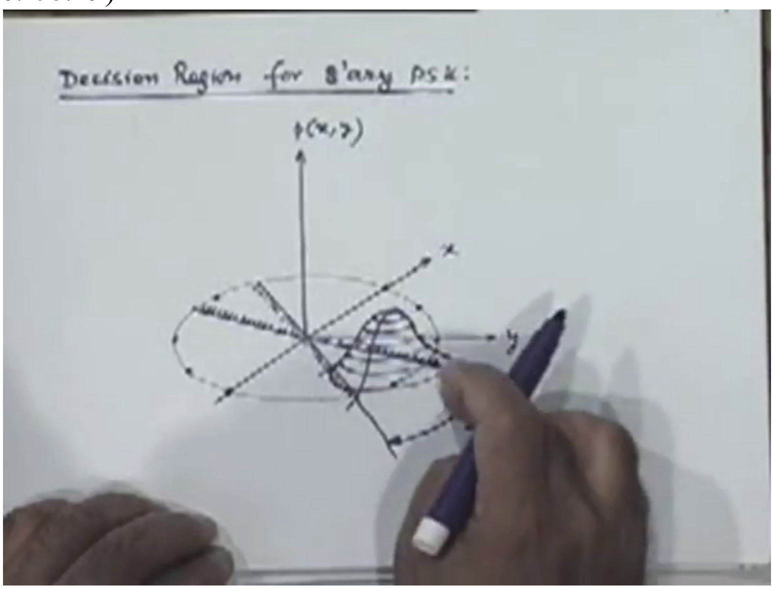

is not restricted only to its sector. It extends outside the sector, right and the volume under this density function which lies outside the correct decision region sector contributes to the error probability. This is what you must really appreciate in this picture, right. And therefore you must really evaluate what is the volume under the tails of this distribution function such that it lies; I have shown two planes here, either above this plane or to the left of this plane. I have shown 2 planes. Let me call this plane B,

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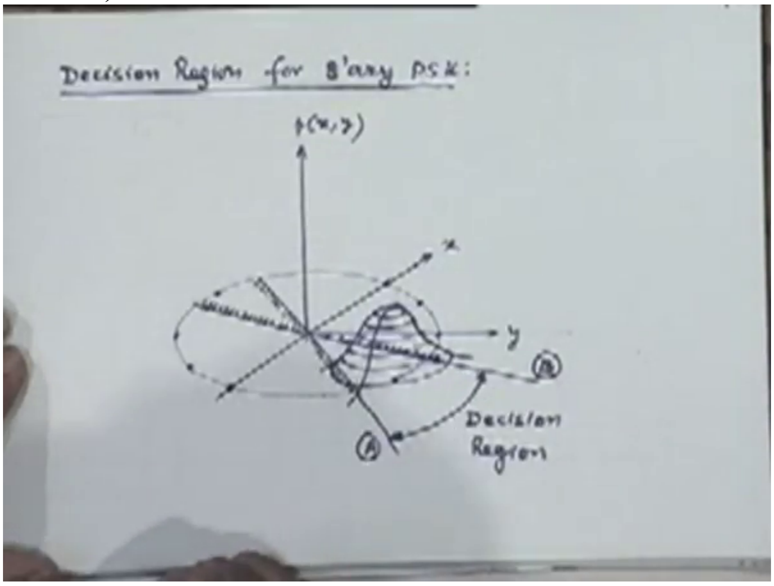

this plane A, sorry this line B divides the plane into two half parts to the left of this line.

That is why I have shown this shaded region to the right of this line, right? This is the top half, and that is the bottom half. So the volume of the, volume of the density function which lies outside this sector can be thought of, as consisting of two sets of such tails, one integrate out, integrate the density function to the right of this plane, right and the other to the left of this plane A and then if you add these 2 results you will get the total volume except the fact that the volume under this sector, this wedge like sector will be computed twice and added. We should have only done it once.

(Professor – student conversation starts)

Student: Can you repeat it?

Professor: What we are interested in doing is in computing the volume of the tail of this density function which lies to the right of this line B on this part of the plane, right, because that is outside your decision region. Similarly to the left of this plane which also lies outside

the domain of your decision region, right. You could have directly added this two results and obtained your total error probability except for the fact that the volume of the same density function under this sector will be added, will be computed twice because it will be computed as part of this plane, this half plane as well as part of this half plane, right? Where as you should have really computed it only once.
(Professor – student conversation ends)

So strictly speaking you could compute the volume on this right half plane, on this half plane, the volume on this half plane and subtract out the volume under this once. That would give you the exact result, right? However if again, we will assume that signal to noise ratio is sufficiently high so that the error that is encountered by counting this, this probability twice is negligible. As you can see, from typical density function that I have plotted here, the volume, you know the tail is really going to 0:10:11.8 out quite, quite a small value by the time it reaches this kind of a sector which is quite away from the basic point, right?

So the are/area, volume under this sector is going to be anyway negligibly small. So whether you count it once or you count it twice is not going to make a terrible amount of difference to your result. And therefore we will take recourse again to the union bound argument and say we can think of the error event as the, as the event corresponding to the received point lying to the right of this line or to the left of this line without worrying about this area being counted twice. Ok

So this is a basic approach for performance analysis for this kind of situation. Is it clear? Now we will proceed with the analysis? So your p sub b is the integral of the 2 D density function over the portion of a complex plane which does not include the decision region. That is the basic definition,

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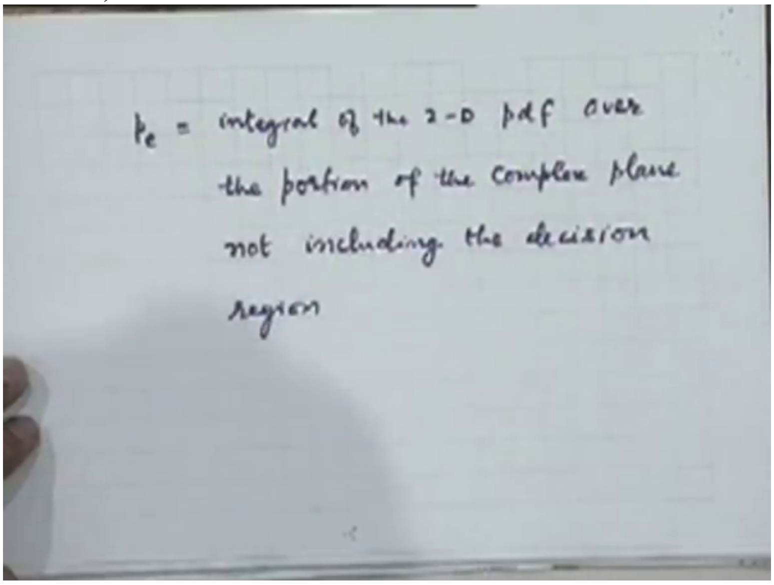

right? This is how we define error probability in this case. That is completely integral of that two dimensional p d f. On that portion of the complex plane which does not lie in the decision region. And one can evaluate this either numerically in the way that I just suggested or we can use that union bound argument that we just went through.

So obtain a union bound. Fortunately this, in this particular case, this bound is quite tight, right?

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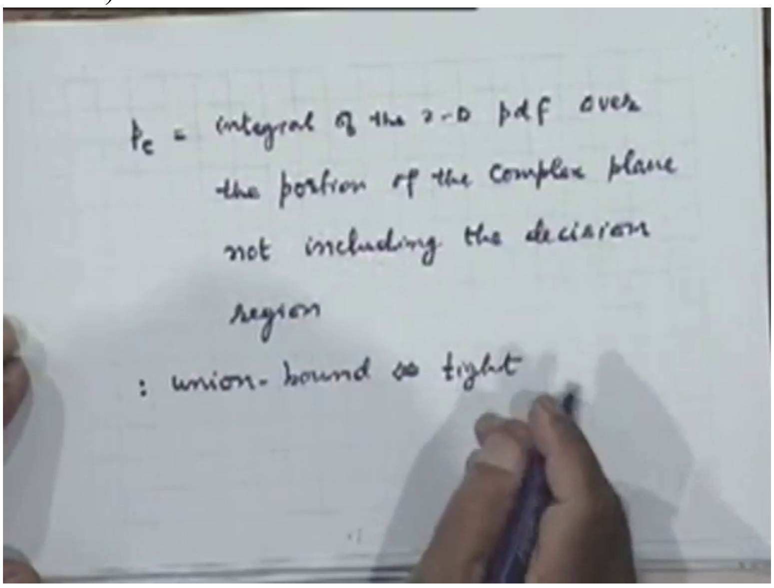

It is reasonably a very good approximation to the actual value because we expect the volume over that wedge to be very, very small. So counting it once or twice hardly makes any difference. But strictly speaking, p sub e will be equal to the integral over the half plane B,

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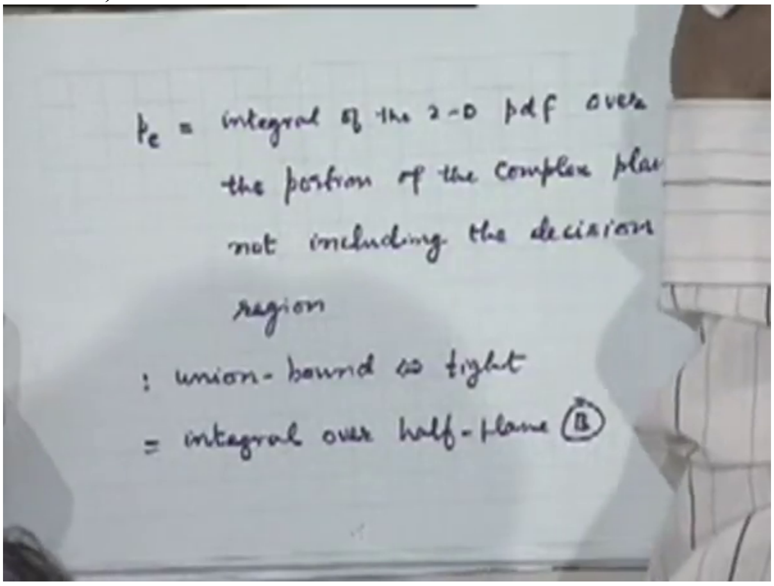

remember this is the half plane

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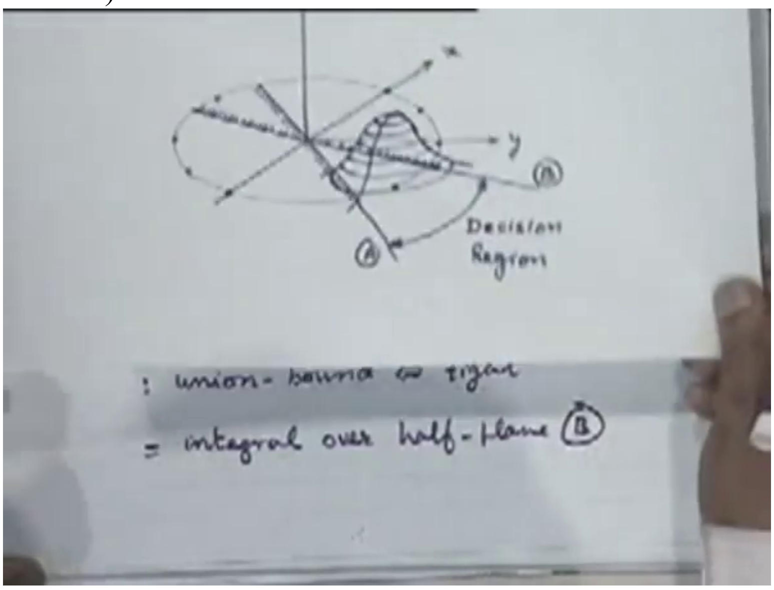

B that is to the right of this line B. That is the half plane B of the x y plane.

The x y plane has been divided into 2 halves by this line B, right. So this is equal to the integral over the half plane B plus the integral to the left of half plane A,

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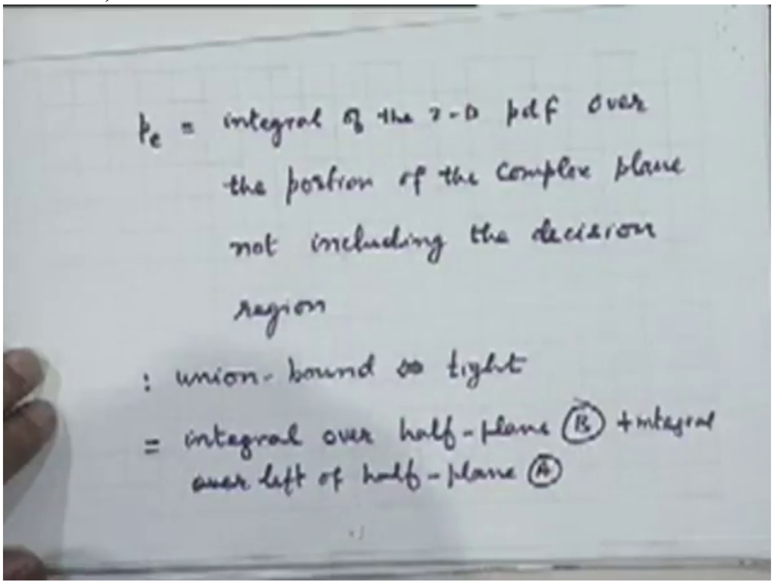

Ok minus integral over the wedge which is

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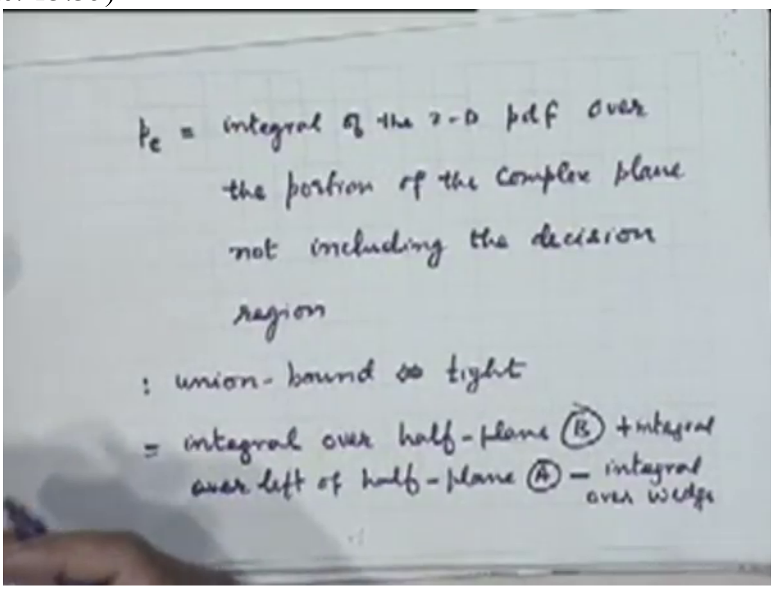

the region of intersection of these two planes on the other side, right. It is this which we are going to ignore in the region of union bound, right? So this is the true value

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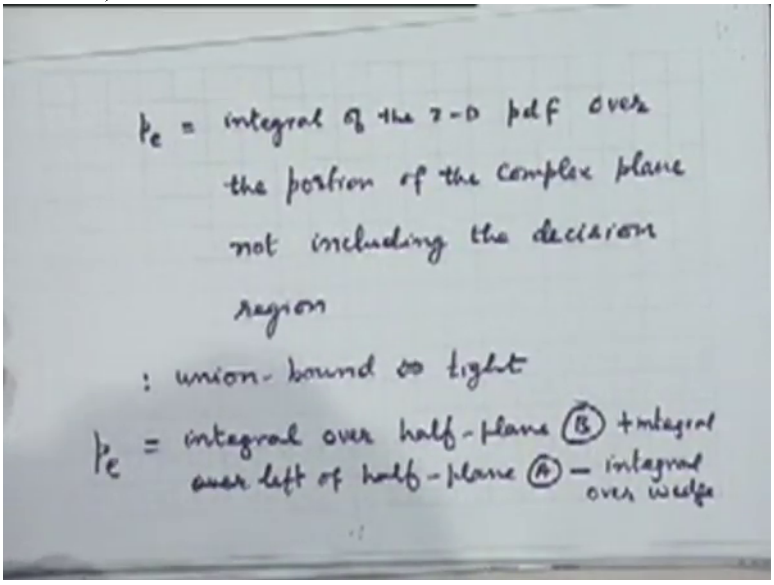

of p sub e and the union bound basically will ignore the third term because it is rather small. Is it Ok?

I have just re-explained what I had explained earlier purely in words. Ok, so the important thing then is to try to evaluate the volume of,

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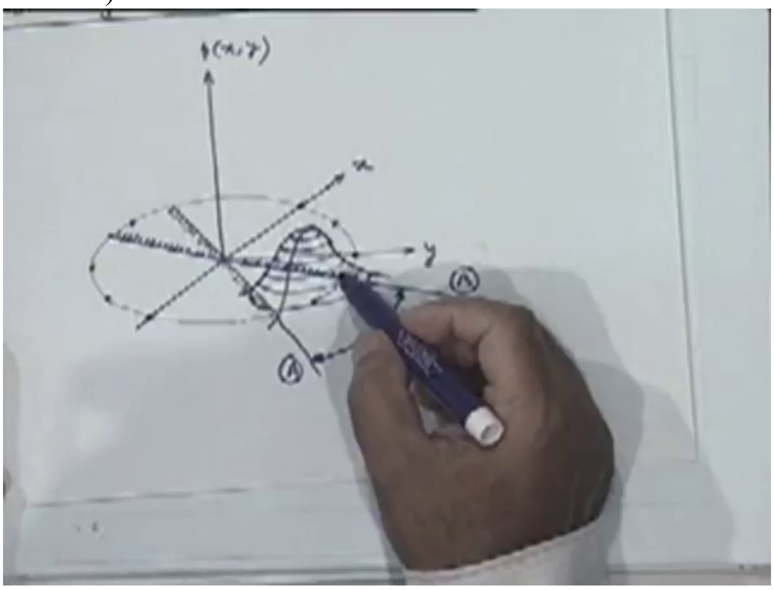

under this, volume under this density function in one of the half planes because in the other half plane it would be similar, right? We must find out how to evaluate this volume in one of these two half planes that we have been talking about. Is there any doubt there? Speak out if there is some problem there? We can discuss it. Ok?

So the question is how do you compute this volume under this half plane, any of these two half planes? And for that I have a following result. Suppose we have a density function, what is the density function we are concerned with? It is a density function in two variables x and y, Gaussian density function. Both the variables x and y have a variance sigma square so it will be simply, oh there will be no square root, right? Because the product of the, so please remove this square root, it is 1 by 2 pi sigma square exponential, now where is this density function located, what is its mean value? Let us say some mean value x bar y bar corresponding to a sub l, right?

Let us call this some mean value x bar y bar. So x minus x bar whole square plus, both of them have identical variance, plus y minus y bar whole square, this whole thing divided by 2 sigma square. Mind you this is not

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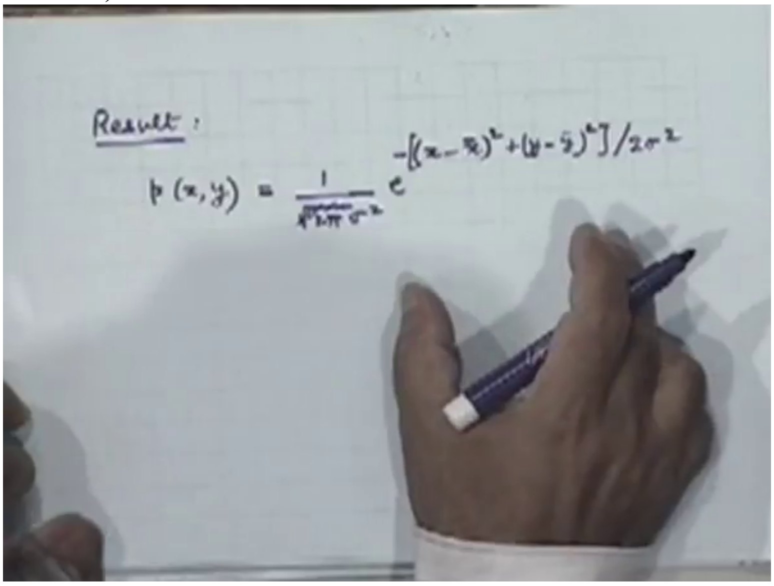

the most general form of the Gaussian density function. I hope you are familiar with the most general form. What is the most general form? There is also a cross correlation term between x and y, right?

In general if x and y are not independent, then the joint Gaussian density function of two random variables involves the third term which requires you to use the correlation

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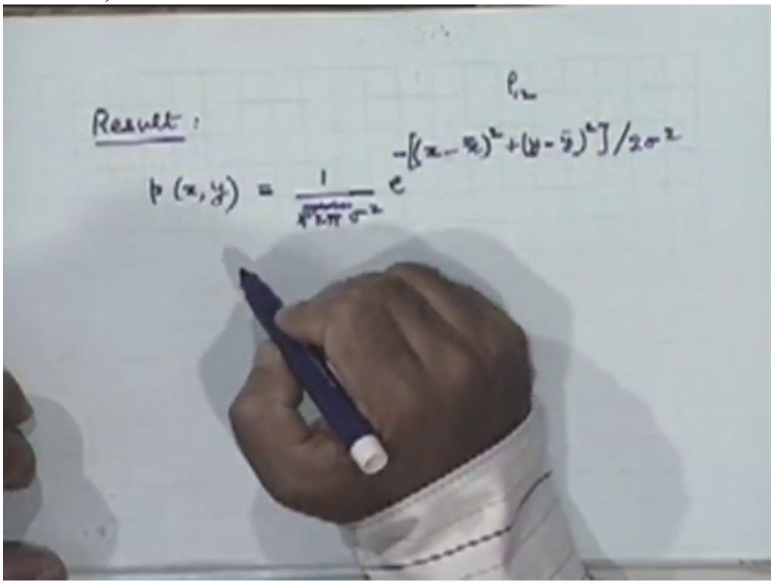

coefficient rho 1 2 between x and y which in our case happens to be

(Professor – student conversation starts)

Student: Zero

Professor: Zero.

(Professor – student conversation ends)

So that term does not come into the picture. This is from your basic knowledge of Gaussian density functions which I have not done in this course but which I am presuming you all know, Ok. But in any case we do not need it here, right? All we need is this form of the density function, joint density function of two independent Gaussian random variables which is simply obtained by the product of the corresponding individual Gaussian density functions, right, which is very simple.

Whereas if they are not independent, they are not in product. We cannot factor them like this. These can be factored out, the p x and the p y can be factored out here, right? But when they are not independent, they cannot be factored out. You know that from your basic knowledge of probability theory. Anyway that was a digression.

So this is a kind of density function we have to work with and we are interested in computing an expression which is going to be a double integration over x and y, right of this density function where

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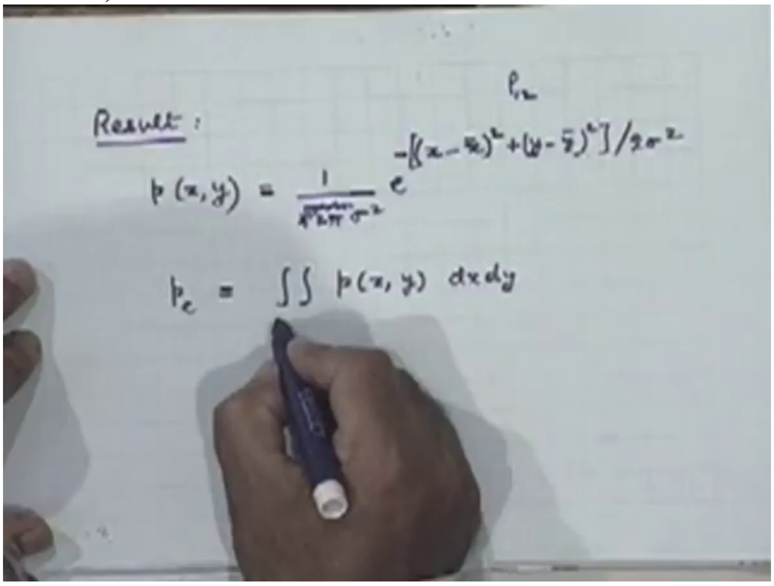

the region of integration, I have not specified the limits here at the moment, the region of integration is going to be one of the half planes that we discussed, either A or B,

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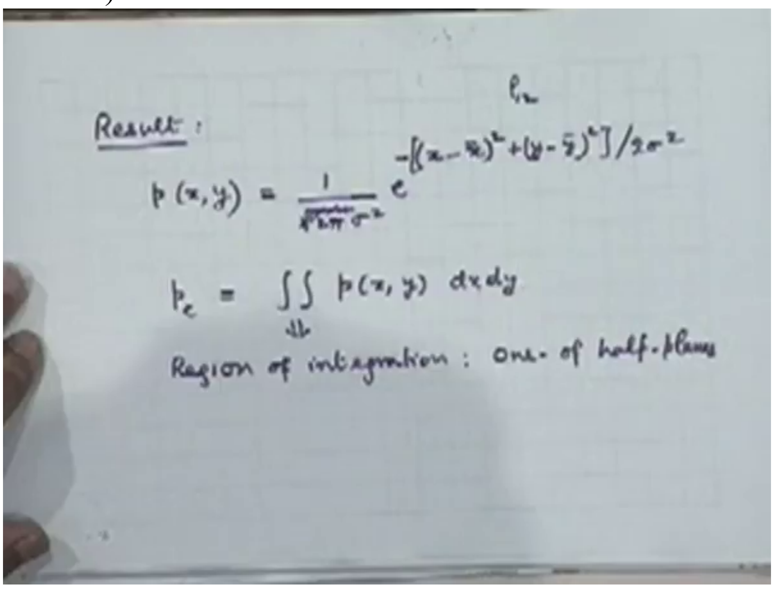

Ok?

Now, then this will be of course twice, right?

(Professor – student conversation starts)

Student: We are having symmetry here 0:18:17.0

Professor: Yeah, we are taking a symmetric situation here and the result is, suppose this distance,

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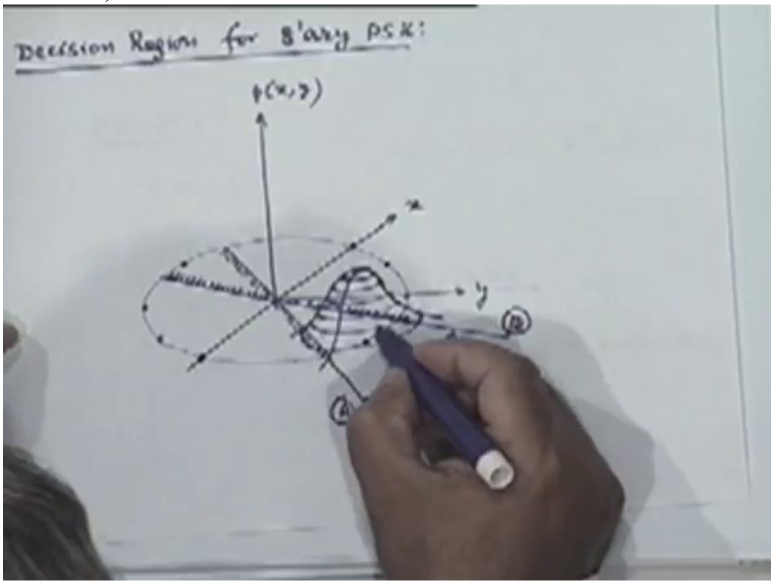

distance of this mean value, mean point to this half plane is d by 2, Ok. So if I draw a perpendicular from here to this, suppose this mean value is, as a distance of d by 2 from the corresponding half plane outside which you want to compute, on which the, you know, the particular half plane on which you want to compute the volume, then value of this P sub e is 2, 2 considering the fact that you are doing 2 such computations, Q function with an argument of d by

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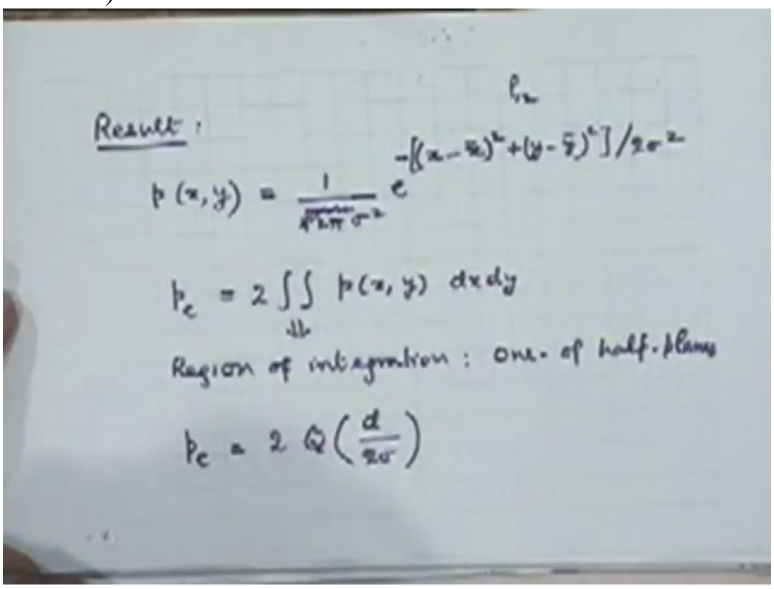

2 sigma squared, Ok. That is your result.

This factor 2 comes because you have to compute over

Student: Twice

Professor: two different half planes, identical situations but on any one half plane, the value of any of these integrals is this one.

Let us see how.

(Professor – student conversation ends)

Have you understood the result?

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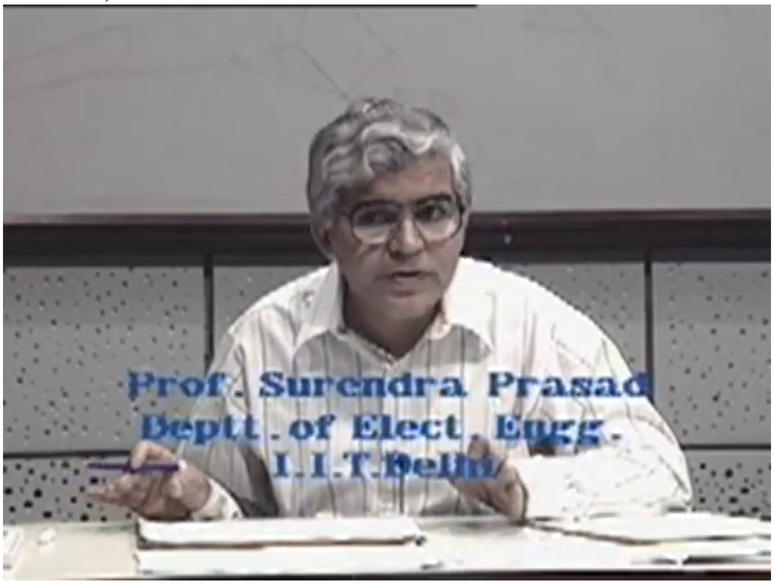

We are going to see how to obtain this but have you understood the statement of the result? Because that is the starting point. To understand how this comes, let us look at this picture here.

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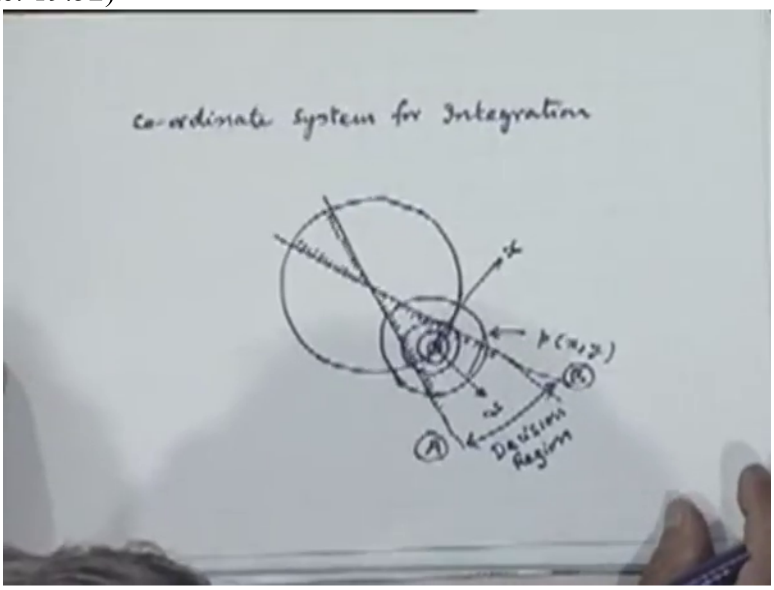

This is the same picture as this, this is really the same picture, there is not much difference except that I have not plotted it, three dimensional probability density function here but shown the density function in the form of contours, contours of constant value, right? So I have just shown it like that.

So that, these are the contours or the density functions p x y, this is your mean points corresponding to x bar y bar. And therefore your d by 2 corresponds to this. Actually this picture is not very nicely drawn; this y should have been drawn parallel to this up here, Ok. Suppose I want to evaluate the volume on the density function which lies in this half plane, Ok? This is the half plane we are considering.

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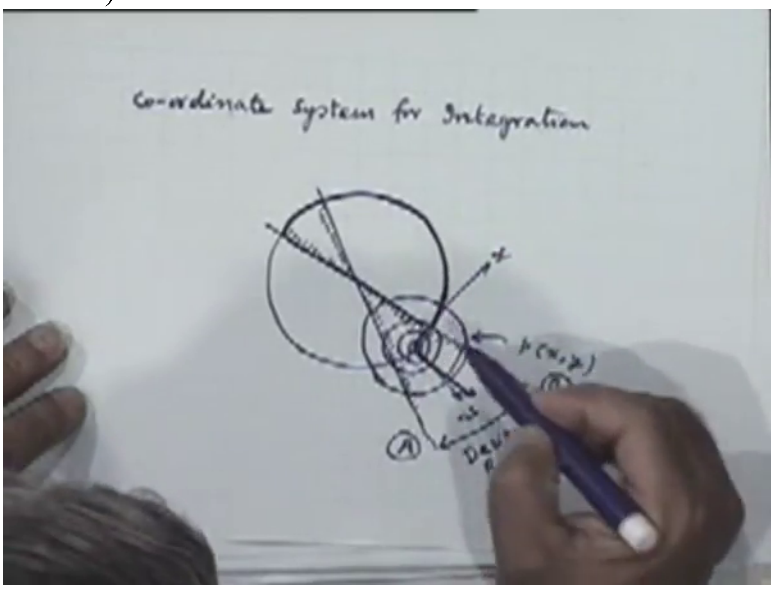

Actually, strictly speaking, this entire this, entire half plane. This circle was just a convenience to indicate the points of transmission, right?

The signal points were lying on this circle; right but actually speaking as far as error probability calculation is concerned you have to calculate the volume under this function over this entire half plane, right? What I have done is change the coordinate system for representation of the e x y. The earlier x y was some arbitrary x y

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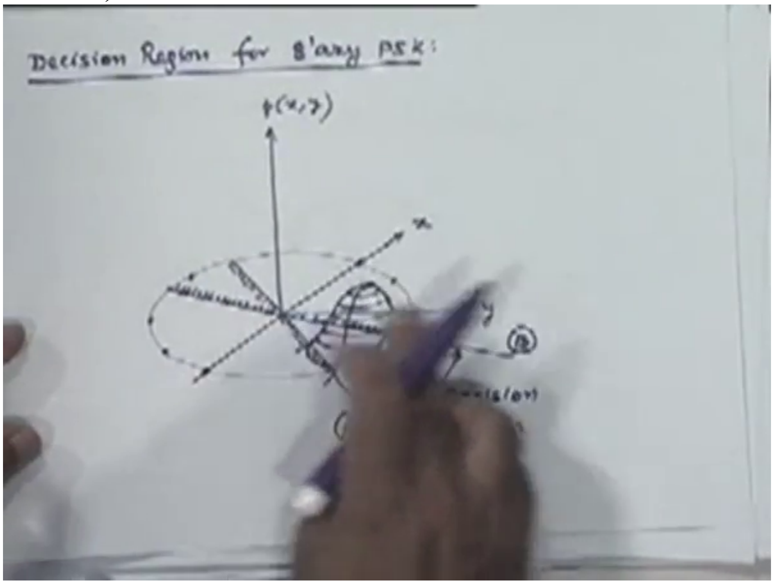

here. I am going to first of all shift it here. I am taking the coordinate system to this point so that the mean becomes zero zero both along x as well as along, so x bar and y bar both become

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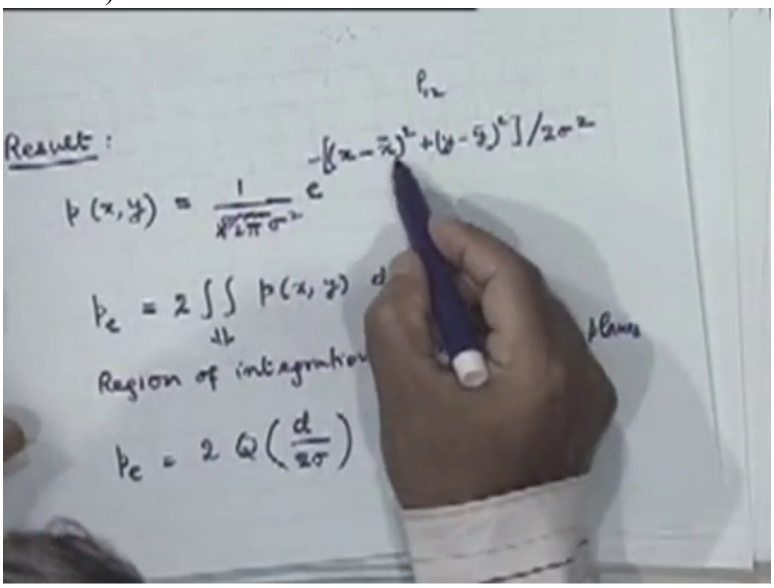

zero each.

So first I am doing is just the translation of the

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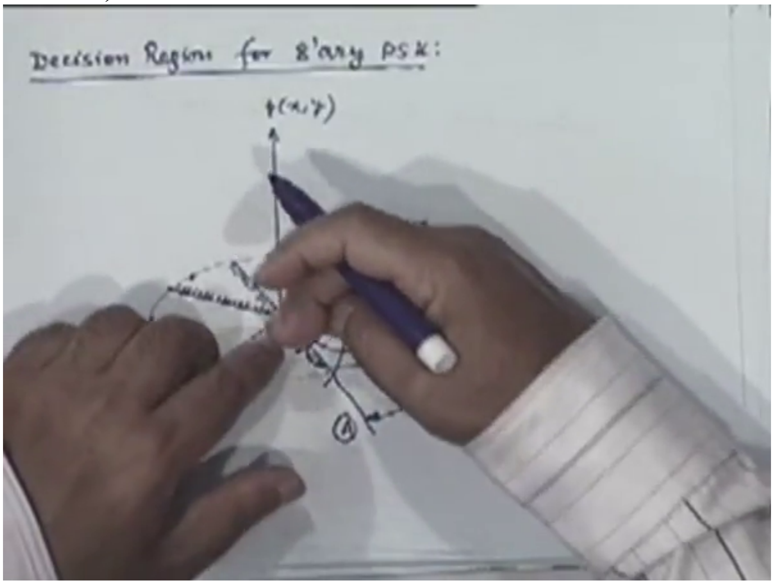

coordinate system to this point that makes the mean value zero, the second thing is to rotate the coordinate system so that,

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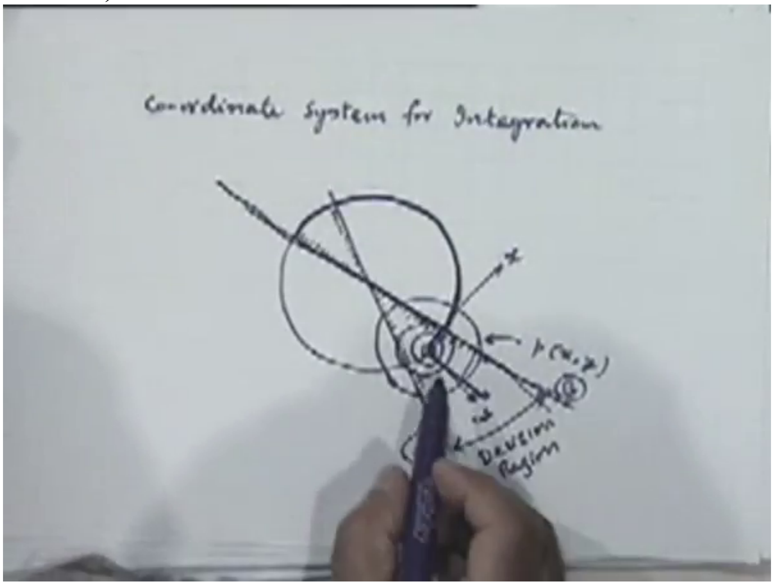

let us say the y axis becomes parallel to the half plane over which you want to compute the integral, Ok. So obviously the x axis will be perpendicular to this and then this value will be simply

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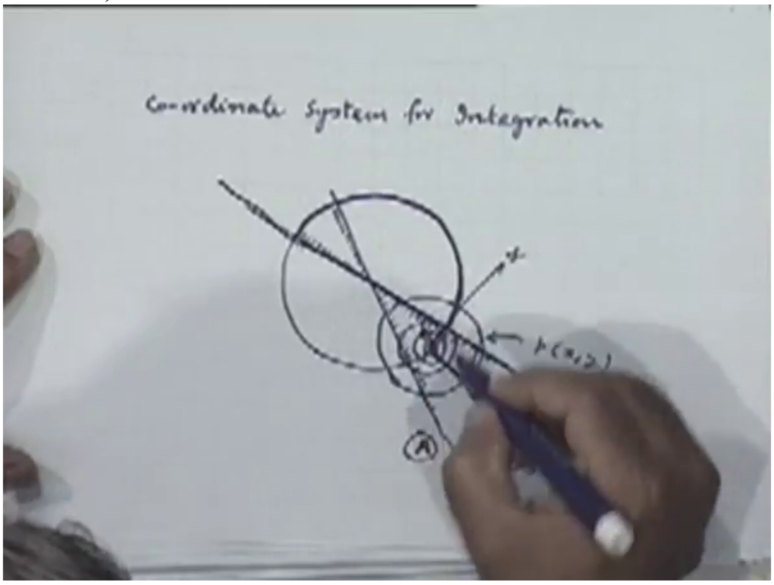

d by 2.

(Professor – student conversation starts)

Student: Rotation is such that

Professor: Rotation is such that the y axis becomes parallel to the line which defines the half plane over which you want to compute the integral, alright? So the line B here is the line which defines this half plane. y axis is made parallel to this. Actually I should have called them y prime and x prime to avoid any confusion with the previous x and y. But I hope it does not cause any confusion. Then I will continue to work with x and y, right?

(Professor – student conversation ends)

Because these contours are circular, it is obvious that if I rewrite the density functions in terms of these new coordinates, they will continue to be of the same form, Ok. Now this was pictorially quite obvious. Whether I write these circles along this coordinate system or along this coordinate system, equations of these circles are going to remain the same, x squared plus y square is equal to some constant, right? And therefore the form of this density function does not change as a result of this coordinate transformation because it is only a rotation.

And because, to start with this contour is not circular. If any of these conditions is not there, then we would have had trouble, right? Actually this is equivalent to saying, just to, for those who have some better idea of probability theory or who can appreciate this kind of statement better; see to start with we have x and y, right? By coordinate transformation, what are you doing? You are generating some x prime y prime

(Professor – student conversation starts)

Student: Through shifting

Professor: Through a matrix transformation, right.

What we are doing is this matrix is such that it only rotates

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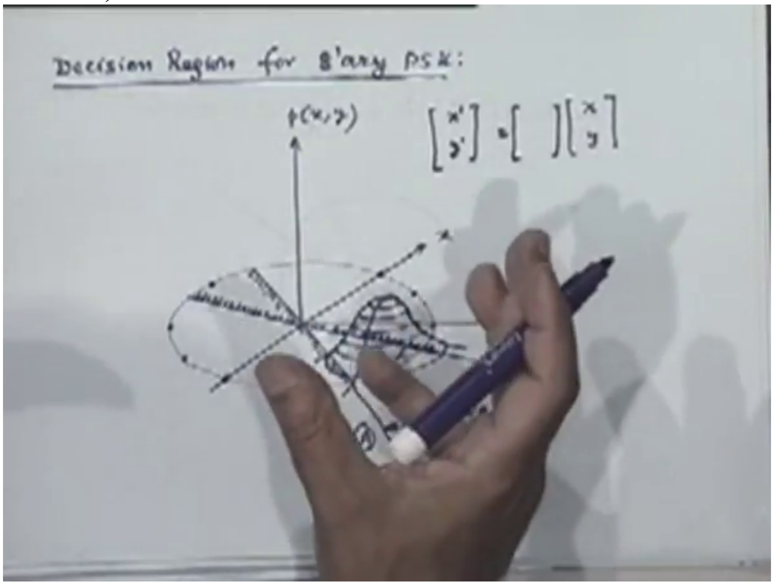

the axis. So what kind of matrices you are familiar with? Are you familiar with some matrix...?

Student: 0:24:19.2

Professor: The unitary matrices, the so-called unitary matrices are the ones which yes, cos theta sin theta is an example of that kind of unitary matrix.

(Professor – student conversation ends)

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Any unitary matrix such that U U conjugate, suppose this is U, I assume U U transpose is equal to identity matrix,

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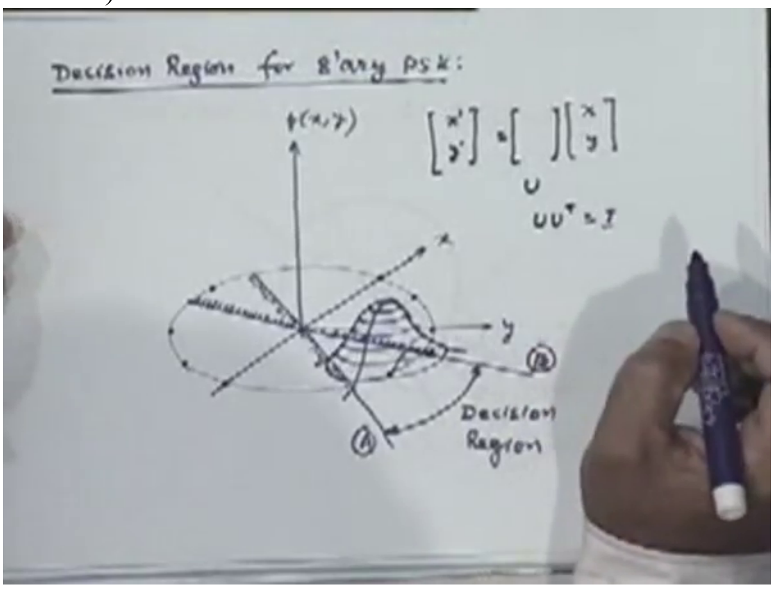

is unitary matrix. That carries out a rotation of the coordinates. Now it can be shown, it is very easy for you to verify that if x and y are independent to start with, and if you transform it by means of a unitary operation like that then x prime y prime will also be independent, right? So that is what you want to know. They will be described by the same Gaussian density function.

If the, unitary matrix has two, unitary transformation has two properties. The independence of these random variables will be maintained. And their norms will be maintained. The norm of the vector x y will be maintained. That is the length of the vector will not change. Because it will only do the coordinate transformation.

(Professor – student conversation starts)

Student: Sir, 0:25:21.4 the length of the vector

Professor: Ok, don't go into it. I did not, this was just a, you know, to give you an alternative argument to basically say that, to imply the fact that x prime y prime will be continued to be governed by the same density function that we started with. x prime y prime will continue to be independent Gaussian random variables with each having a variance of sigma square because we have only done a coordinate rotation.

Student: That is obvious from the symmetry of the whole thing.

Professor: That is obvious from the symmetry of the circular contours that we have. Because whether you take, whether you take sigma along this direction or along this

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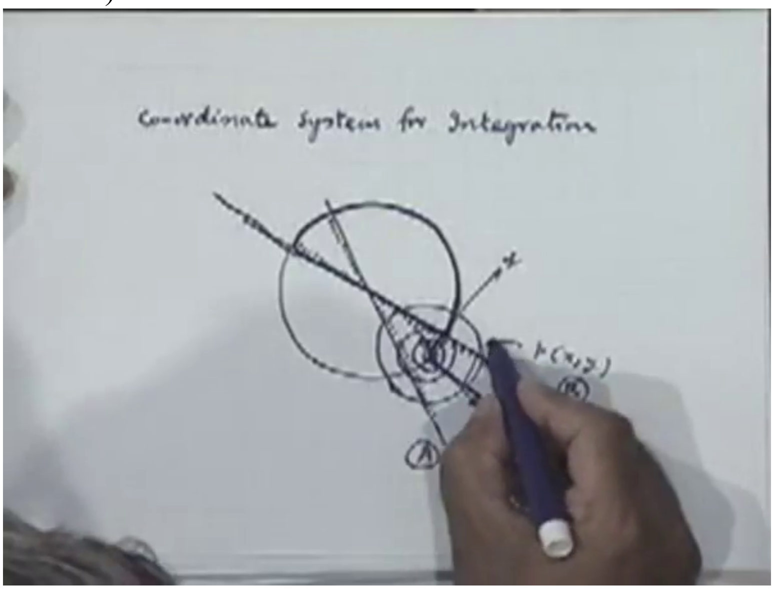

direction or along this direction, its value is same because these are circular contours. If they are elliptical, for example, that will be a situation when rho 1 2 is not zero or when both of them have different variances, right, then, then there will be problem.

But in this case, they are circular; no matter how you rotate them the variances along each axis will continue to be the same. And they will continue to be independent.

Student: We have shifted also...

Professor: Oh, the shifting only makes the mean zero. After shifting I am doing a coordinate rotation, right. So I have really gone into 2 coordinate transformations. Number 1, shifting, translation, after translation I am doing, translation does not do anything except making the mean zero. The second coordinate transformation we are doing is rotation. The rotation does not alter the variances along either axis, the new axis or the fact that they are independent, because it is a pure rotation, Ok.

(Professor – student conversation ends)

This is something if you are not very comfortable with, I will just suggest you pick up any elementary text on probability theory or even the text that you have with you, Lathi's book. These things are dealt with in the review chapter in that book. So just make yourself comfortable. Or you come to me, I will explain to you in more detail because I do not want to spend too much time on this, alright. So if you have understood this, it is very easy to write down

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now the expression we are looking for. Your p sub b is twice,

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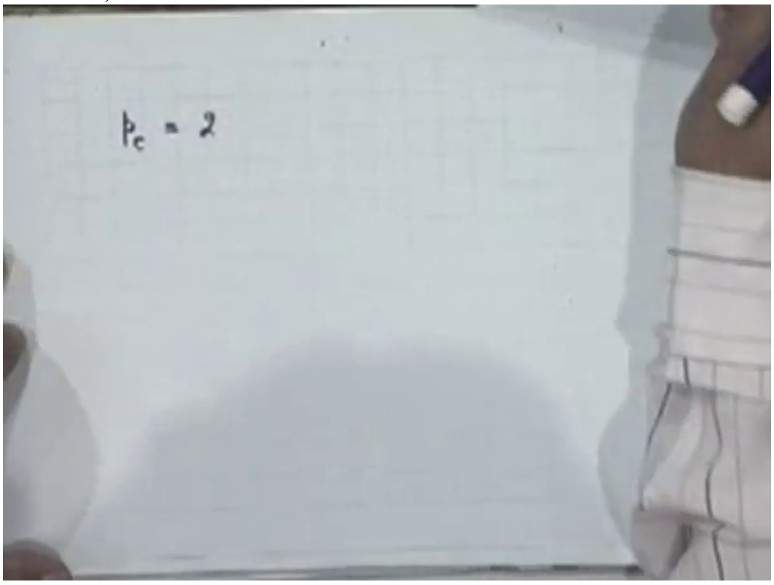

I want to write the integral

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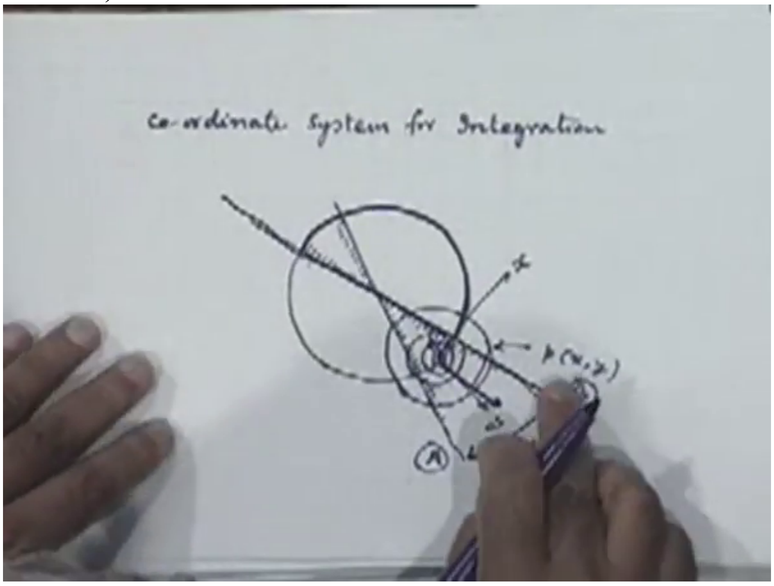

over p x y to the right of this plane where y axis is like that and x axis is like that. Is it obvious? What the integral will be?

The x limits will be from d by 2 to infinity and the y limits will be from minus infinity to infinity, right? The entire y axis span is being taken but the x axis span is going from d by 2 to infinity. That is all. That is all you need to appreciate. So you have this value, right? x goes from d by 2 to infinity, y goes from

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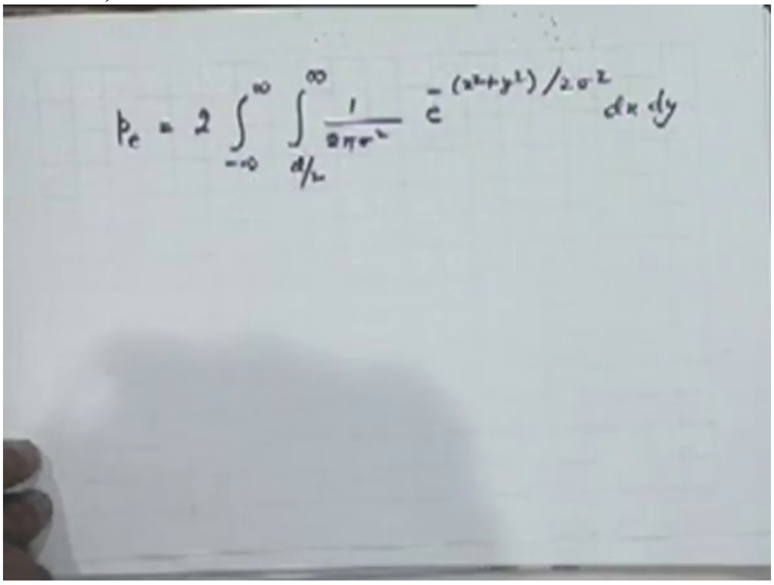

minus infinity to plus infinity, Ok? Now the rest is all very easy. As far as y is concerned, I can factor out the y portion of the density function, 1 by root 2 pi sigma e to the power minus y square by two sigma, two sigma square and integrate it separately and the value will be 1. So I totally ignore that.

From what I have left is 1 by root 2 pi sigma e to the power and that is precisely

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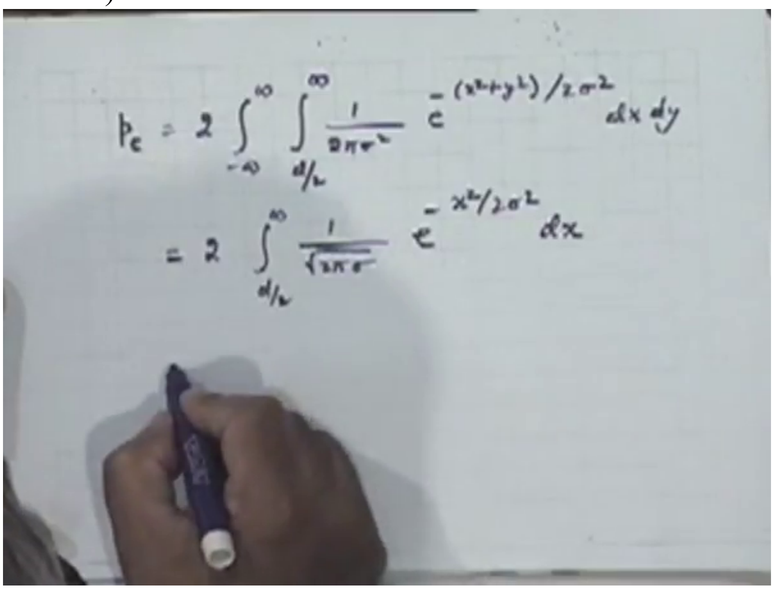

what you define the Q function, right? Your Q function is defined precisely in terms of this integral except that this is d by 2. So it is d by 2, by definition this is...

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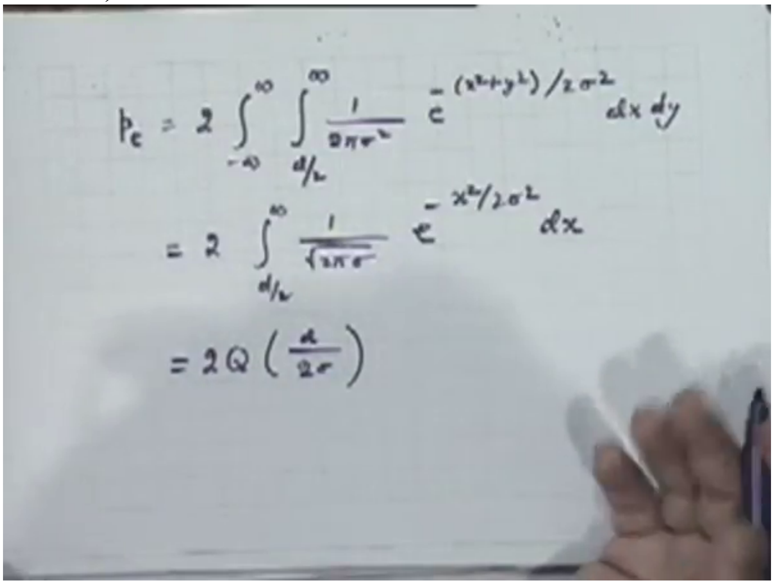

this is how this result comes, very easy to appreciate it. Just study it a little more for the special case that we are really discussing, namely the 8-ary p s k, 8 phase p s k, phase shift keying, Ok.

For the 8 phase p s k, any questions before I proceed further? Is this clear how we obtained the result? Mind you this is slightly less than the most correct result.

(Professor – student conversation starts)

Student: Greater

Professor: less than, why, Ok it is greater. What I meant was it is not really the exact result, right. It is the union bound, yes. The actual probability will be less than this quantity. So this value is slightly greater than the actual probability. But it is for, for reasonable signal to noise ratio, it is very, very accurate.

(Refer Slide Time: 30:35)

It is good enough to; you might as well take it as exact result, Ok. So the bound is quite tight for large values of e b by N zero. So let us now specialize this result for the case of

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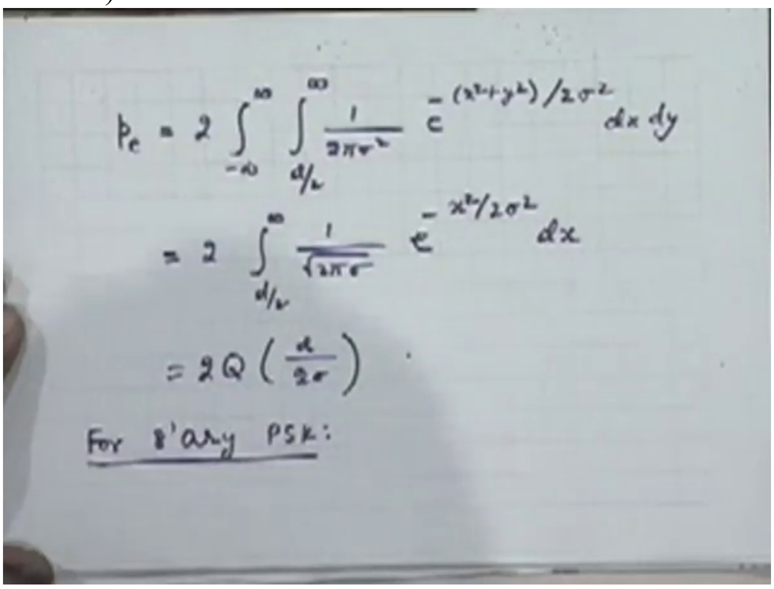

8 phase p s k.

It is obvious that the overall symbol error probability will be the same as, that is average error probability will be the same as the error probability associated with any particular symbol because we are assuming all symbols to be equally like, right, equally likely. What matters here is the equal prob/probability, a priori probability of all the symbols.

Student: Also the energy is equal.

Professor: Energy is obviously equal, right.

(Professor – student conversation ends)

But even if energies are equal, if various symbols occur with different probabilities then we cannot say this, right. The average error probability is the same as error probability associated with any specific symbol given that any specific signal, symbol was transmitted, at any specific signal point. That is if I compute the error probability

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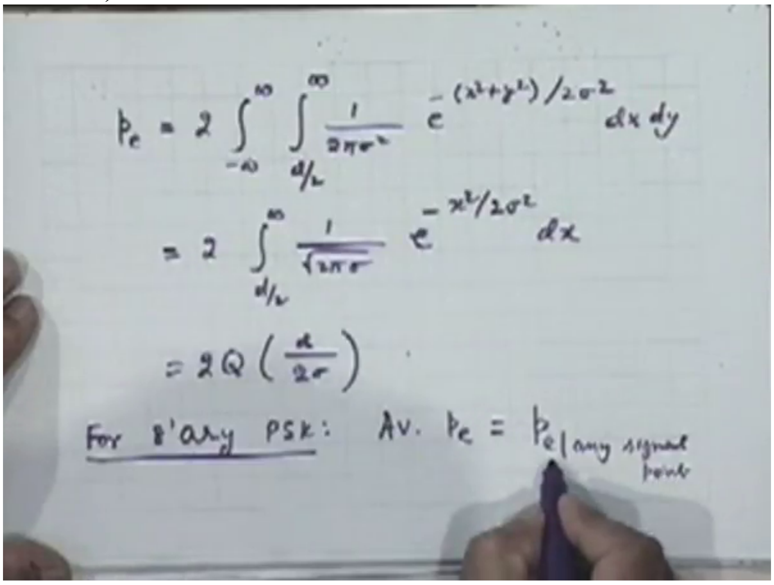

given at a specific signal point was transmitted and if all the specific signal points have the same a priori probability, then the overall average probability will be, probability of error will also be the same as this. Because it will be really 1 by 8 times this plus1 by 8 times this again and so on, if all these are same. So final results is the same.

Ok, now your p s k is described by a constellation in which the points are distributed on a circle. Let us say the circle is of radius A. So let A be the radius of the constellation which essentially says that your

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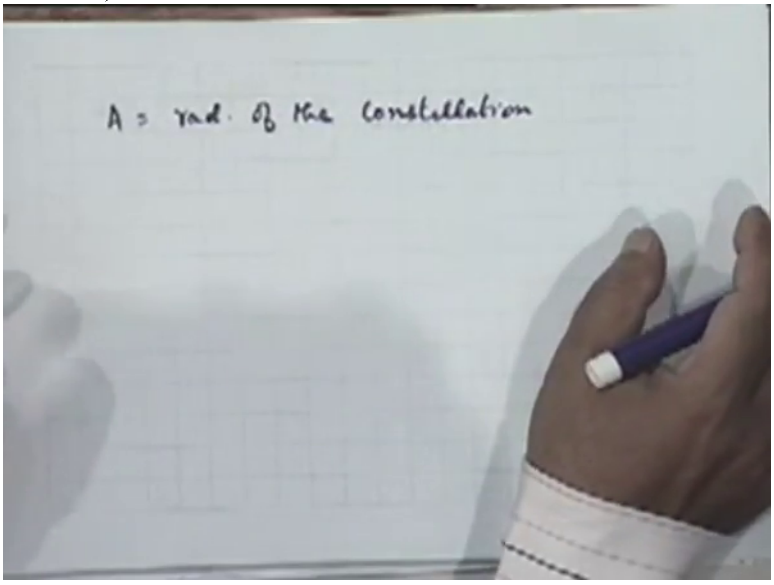

received signal has an amplitude A right? What will be d by 2 in this case? Because d by 2 is a crucial parameter here as you see.

(Professor – student conversation starts)

Student: A sin pi by 8, Sir.

Professor: That is right. It is quite obvious so I don't have to explain this at all.

(Professor – student conversation ends)

It is A sin pi

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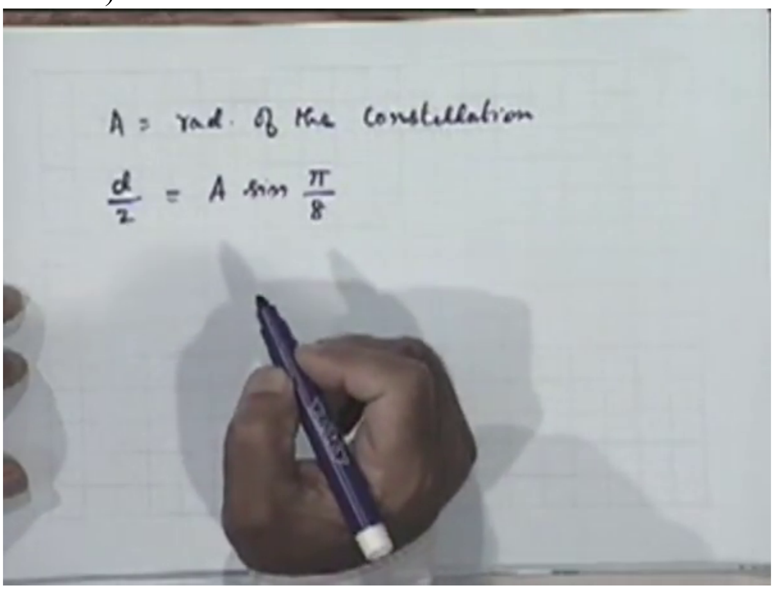

by 8, right? Because your sector around each point is described by an angle of pi by 8 in either direction. Total angle is pi by 4, 2 pi by 8 is pi by 4, and pi by 8 is the half angle which turns out to be point 3 8 2 7 A.

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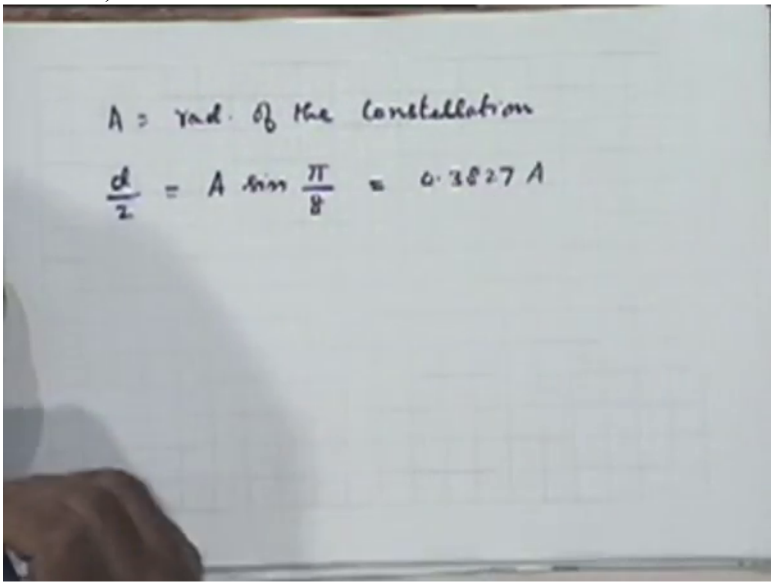

Also from your matched filter theory that we have already discussed the output A by sigma square is given in terms of the pulse energy and the noise pass spectral density function, 2 E p by N zero,

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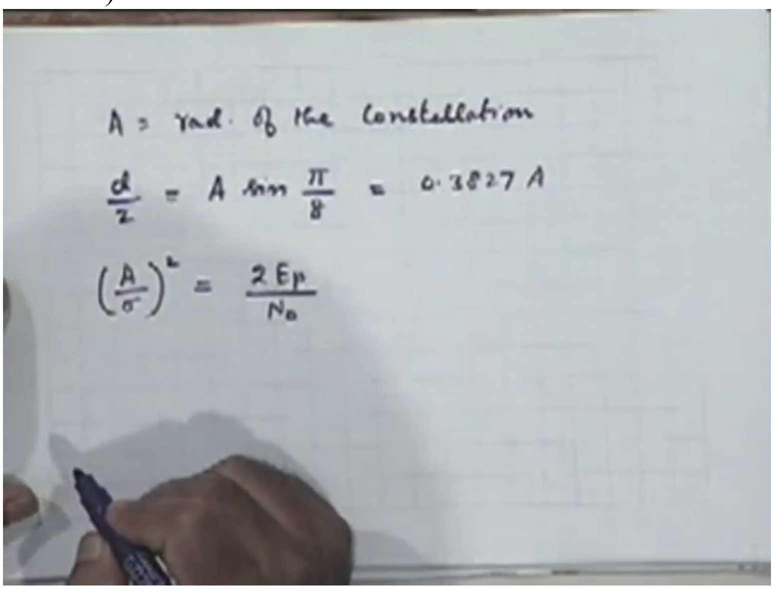

right? So you are union bound, then if you substitute these results, so from there you can get d by 2 sigma as point 3 8 2 7 A by sigma, right and A by sigma you substitute from here, and you will get this result.

2 Q right, it is d by 2 sigma, now A by sigma is square root of this, so I will put this also into the square root sign and write, I will ignore this 2 7, just point 3 8 square 2 E sub b by N sub zero,

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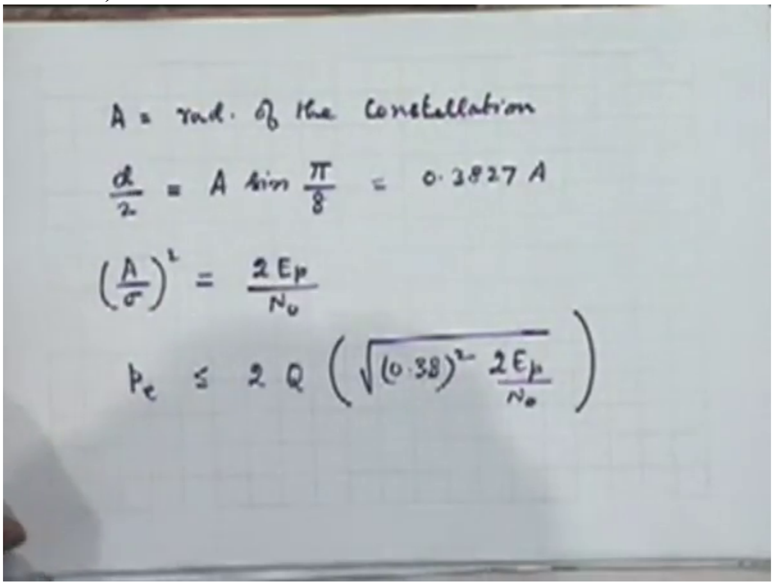

Ok. And this is going to be really, nearly equal. One does not even have to write less than equal to provided this is sufficiently large. The last point to consider is, this is what? What kind of probability is this? Symbol error probability, right? In practice like we did it yesterday, we will be interested also in bit error probability. So how do we go from symbol error probability to bit error probability in this case?

(Professor – student conversation starts)

Student: 1 by 0:35:10.7

Professor: No, in this case one has to keep in mind that in the two dimensional signal constellation one will

(Refer Slide Time: 35:19)

usually do a Gray coding so that a single symbol error causes at most a single bit error in the, at least in the most common situation where a symbol is mistaken for its adjacent symbol. Of course if it is mistaken for some other symbol then we cannot guarantee.

Student: The probability of 0:35:40.3, is not for just adjacent symbols.

Professor: That is true, you are quite right. The probability you have calculated corresponds to the particular symbol mistaken for any other symbol, right?

(Professor – student conversation ends)

But the major component of this probability comes from contribution of that event, right? Because the values, the contribution by other events, probabilities of other events would be very, very small. It is obvious because, you know, for a Gaussian density function that area under the density function keeps going smaller and smaller as you keep going away from the mean value. So the major portion comes from there. So one does not really worry too much about that second order of error that is taken, that is effected by this computation.

So if you keep that in mind, if you are doing the Gray coding, let us consider the 8-ary case. Suppose you mistake a particular symbol for its adjacent symbol, how many bits will go in error?

(Professor – student conversation starts)

Student: 1

Professor: 1 out of 3. So for every 3 bits, you will encounter only single bit error so essentially it implies that your bit error probability will be this probability divided by 3, right? Also what will be your E sub p then?

Student: 0:37:12.4

Student: Sir, I will 0:37:13.1only one...

Professor: Because we have gone from one symbol to its adjacent symbol, right,the Gray coding ensures

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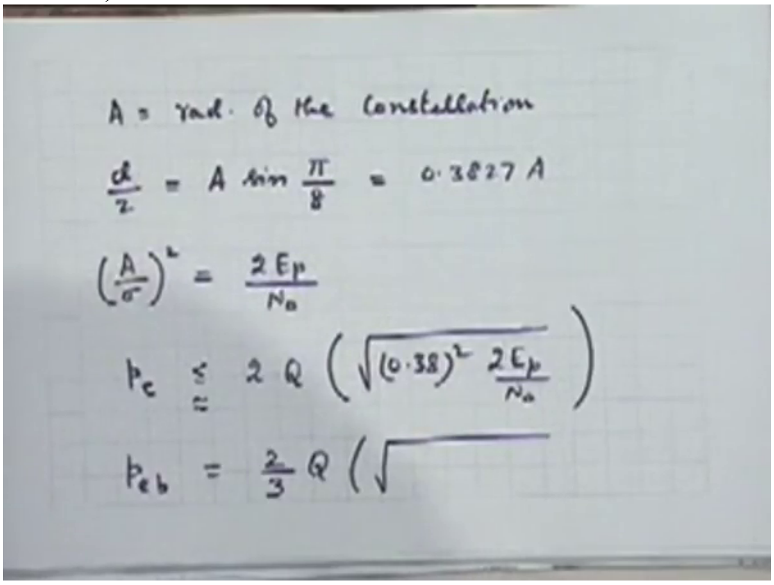

that only one of the bits will go in error, right

Student: Ok

Professor: That means, out of every 3 bits, we are making one bit wrong on an average, right? So every time an error occurs, such error occurs, we are assuming that only single, this implies only single bit error will occur. Of course assumption involved is that errors of the kind in which you go to a non-adjacent symbol have very, very small probability. Therefore this error probability is largely comprised of errors of going from a symbol to its adjacent symbol. That is the assumption implied here. So there is a further approximation implied here to be sure, right but that approximation is also

Student: Upper bound

Professor: Yeah

Student: It does not remain upper bound.

Student: It is still upper bound. It is still...

(Refer Slide Time: 38:22)

Professor: It is difficult to say that but since it is a second order effect, one does not worry about it too much. It is really a second order effect. In fact a very, very accurate.

Student: Probability is more than the accurate.

Professor: You are quite right. It does not remain the upper bound because you are neglecting certain probabilities.

Student: Sir but we are including all that in this.

(Refer Slide Time: 38:47)

Professor: The two will be compensating for each other but we do not know how. So we will not enter into that. You are quite right.

(Professor – student conversation ends)

Ok, I also, before I wrote this expression I also wanted to substitute for E sub p in terms of E sub b. E sub p will be 3 times E sub b because

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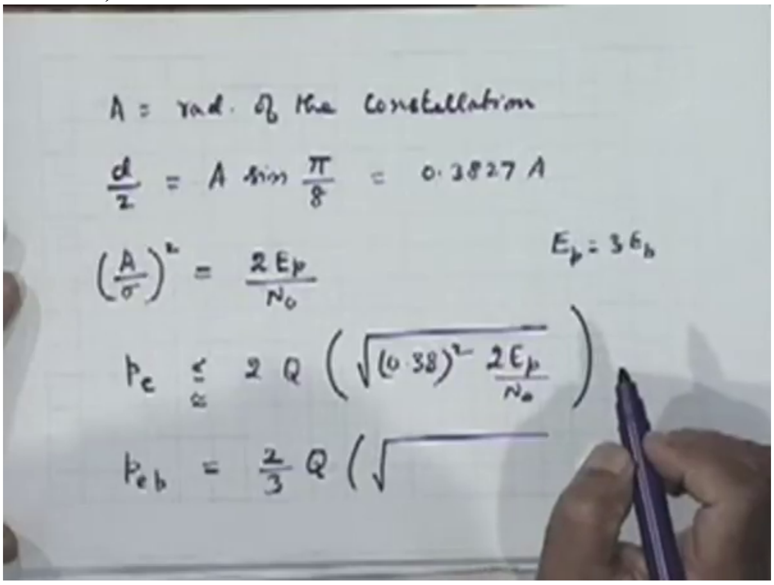

bit energy is defined as the pulse energy divided by log 2 m, which in this case is 3. So if you do all that substitution, you will get this as point 8 8 E sub b upon N zero.

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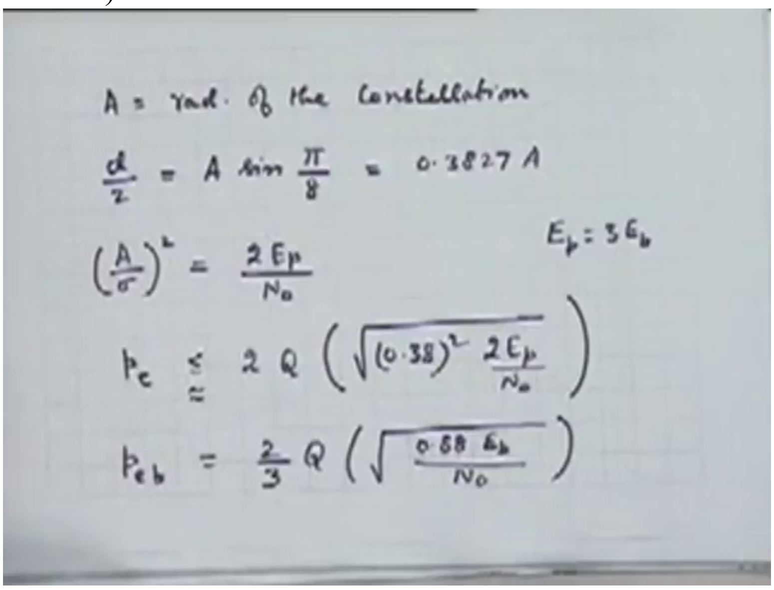

What was the result for binary p s k? Do you remember? That is a very standard result, you should remember it almost. There is no factor of 2 by 3; it is Q of 2 E b by N zero, right?

This factor of 2 by 3 is not very significant. More important is what is inside the argument Q, right? Because this will not affect orders of magnitude where as that will affect the order of magnitude of the probability. Because this is inside that Q function. So roughly if you ignore this factor of 2 by 3 in one case and 1 in the other case, which one is better, binary p s k or 8-phase p s k?

(Professor – student conversation starts)

Student: 8-phase p s k.

Professor: No, for a given bit energy this argument is smaller. That means the area under the tail will be larger, right? This is monotonically decreasing function of its argument. The larger the argument, the smaller the value of this probability, right? Because you are testing the area under the tail of the density function, Gaussian density function. The further you go away from the mean, the smaller the value of the area under the tail, Ok? In fact, roughly it is about 3 point 6 d B poorer than binary p s k. That is to get the same error probability, 8-phase p s k will require 3 point 6 d B more signal to noise ratio as compared to the corresponding value required for the binary phase shift keying.

(Professor – student conversation ends)

And this is quite understandable, isn't it? Because the noise now has an easy access to adjacent symbols as compared with binary p s k for the same bit energy, right? As you can quite see from this picture, which picture? Any of the one which we discussed. For example this one, right?

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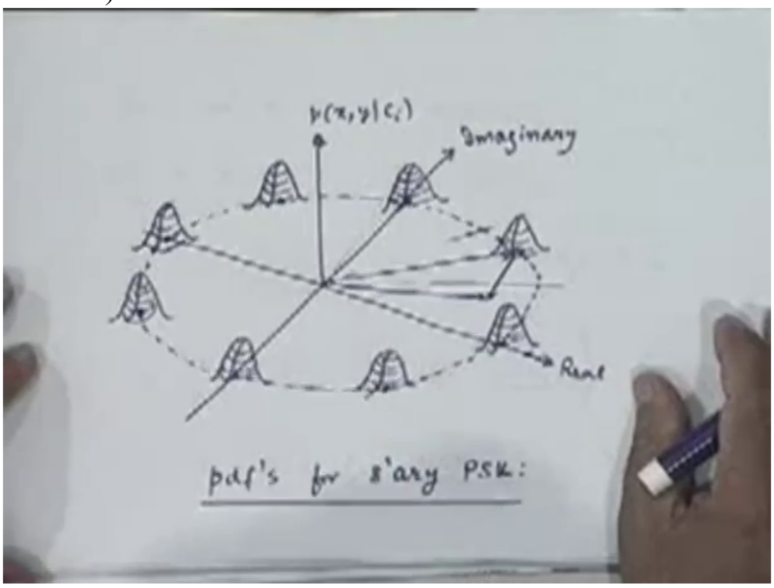

Because for the same, Ok, this is pulse M-ary, your pulse are becoming closer and closer together, and for the same noise variance, there is a larger probability that one symbol will be mistaken for another symbol. This is therefore quite unlike the M-ary orthogonal signals that we discussed yesterday.

For the M-ary orthogonal signals, as we increase the value

(Refer Slide Time: 42:10)

of m, and if your signal to noise ratio is above certain basic threshold value which is very low, you are guaranteed to get better and better performance with increasing value of m where as in the case of two dimensional signal constellations, as you increase the value of m, your performance is going to degrade further and further, right?

So there is a trade-off involved. To choose between orthogonal signals or two dimensional signals. The behavior is, the reason for the behavior is quite understandable because in the, in one case you are allowing the dimensionality to explode for the orthogonal case, there is no lim/limit, as more and more signals are being added, your signal space dimension is increasing, right, more and more? In the other case signal space dimension remains fixed at 2 and it is within that dimension that you have to distribute the various signals, right?

So in one case, we are working with strictly the bandwidth itself namely the two dimensional set. Whereas in the other case we have no bandwidth limitation, right? Because as we are increasing the value of m, we are expanding bandwidth, right which is...we have discussed the situations, power limited situation and bandwidth situation, the orthogonal signal set corresponds to

(Professor – student conversation starts)

Student: Power

Professor: power limited situation not bandwidth limited situation; we are assuming that bandwidth is infinite, right? So therefore they have the potential of improving your performance as you increase the value of m. whereas s in this case the application domain is different. Your bandwidth is a major constraint and this is what you can, you can trade off let us say, signaling rate with performance.

(Professor – student conversation ends)

As you go for higher value of m, you can improve on your signaling rate, right? Because every symbol can contain multiple bits depending on what is the number of points in the constellation but the price you are going to pay for is in terms of performance, error probability, bit error probability. It is going to become poorer and poorer in bit error probability, right?

So this was major thing coming out of this which was intuitively quite obvious. How much time is there? Ok, there is very little time.

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There is a further, one little result which I would just like to mention here, and the rest of it, a few paragraphs or so I would like you to read yourself, I will give you the handouts tomorrow.

This is a result for more general constellation, because this result that we derived for really pertained to

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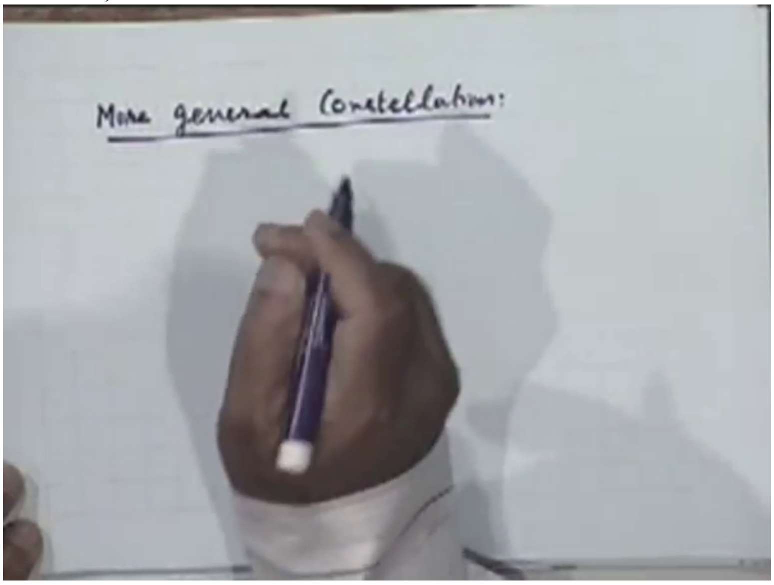

8-ary p s k. It was derived in the context of 8-ary p s k. But we could have a general constellation, not in the form of points distributed in a circle. It could be Q A M, on a grid or concentric circles or any such arbitrary shape.

Now the general result is motivated from this special result

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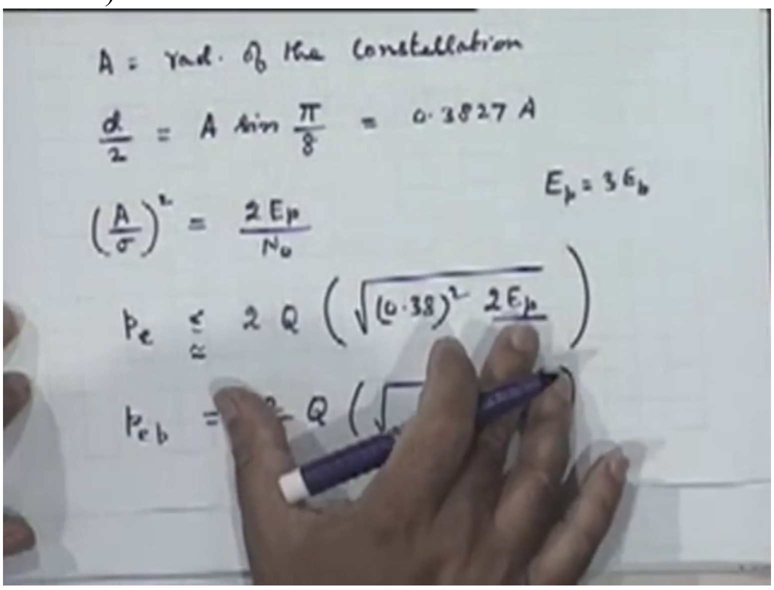

very easily. Let me first quote the general result. The general result is p sub e is given by N sub m into Q evaluated at d min upon 2 sigma,

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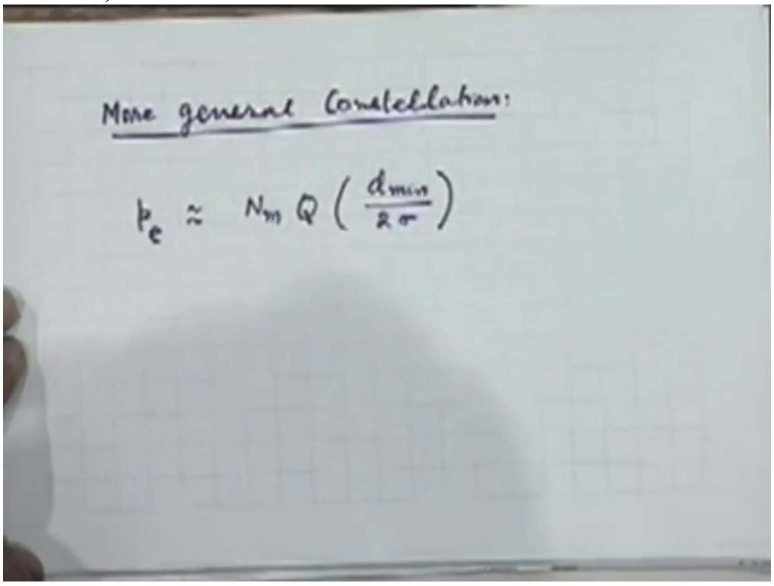

Ok where N sub m is the average number of nearest neighbors

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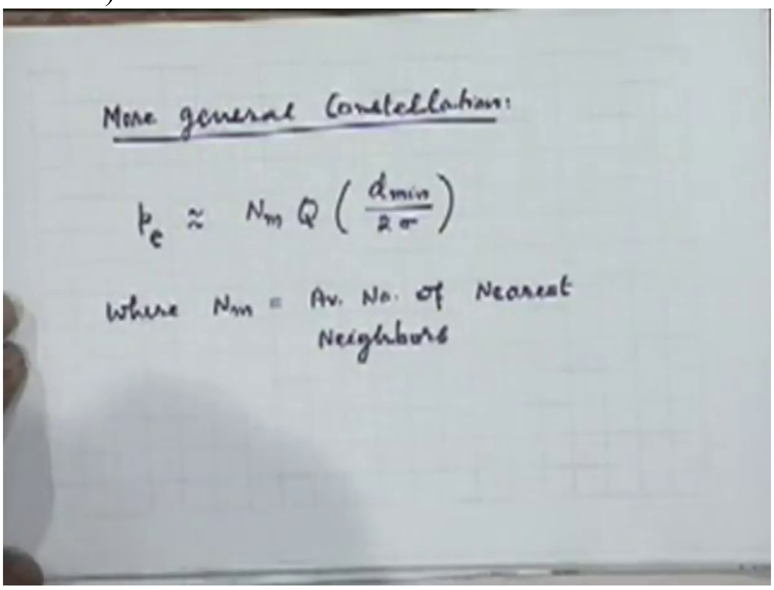

in the constellation.

(Professor – student conversation starts)

Student: d min this and

Professor: d min of course, you already know, it is the minimum distance between any two points on the constellation, right? d min is the minimum distance between any two points on the signal constellation. Alright let us see how this, this result can be motivated directly from what we have obtained here.

(Professor – student conversation ends)

So in this case, in the 8-ary p s k case,

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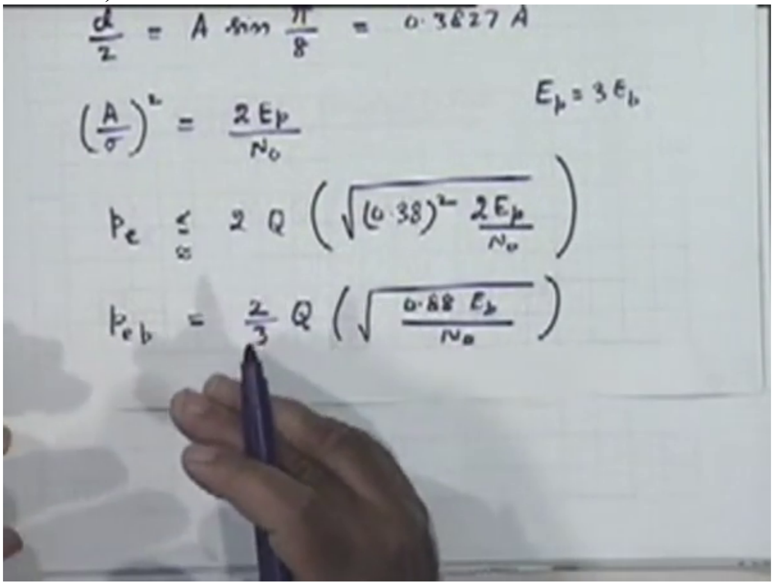

what was the number of nearest neighbors? 2, one on either side. That defined 2 half planes for which you had to evaluate the volume under the density function, volume under the tail of the density function, right? Now if you have more than two nearest neighbors, we will have to, your decision region will get modified accordingly and your tails definition will also get modified accordingly, right?

Suppose you have a grid there will be 4 planes which will be bounding the, which will be defining the decision region, so the area which will contribute to the error probability will have to be calculated on 4 different kind of half planes, right? Each of them will give you the similar kind of result because of the symmetry of the situation, right and therefore the result does make a lot of intuitive sense. We are not really going to the exact derivation of this but intuitively it is reasonable to expect this kind of a behavior.

(Professor – student conversation starts)

Student: Sir

Professor: Yes

Student: In a more general case the 0:47:40.0 of signal would be the same.

Professor: Therefore this is

Student: Generalized

Professor: More 0:47:46.7, kind of result than an exact result. The result which you can almost apply independent of what constellation you are working with. Otherwise for each constellation you have to derive an exact result of same thing, right? So therefore this is a nice designed result to work with, which does not specifically depend on which specific constellation you are working, right? So it is a rule of thumb kind of thing.

(Professor – student conversation ends)

I think the rest of it I will leave you to study yourself and as I promised I could not start information theory today. We will start that tomorrow.