Digital Communication Using GNU Radio

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Week-04

Lecture-18

Hello, so welcome back to this lecture on Digital Communication Using GNU Radio. My name is Kumar Appiah and I belong to the Department of Electrical Engineering at IIT Bombay. In this lecture, we are going to continue our look at demodulation with a signal space picture in mind. So if you remember in the last lecture, we were talking about how you can obtain a set of orthonormal basis signals from a set of

s_1 , s_2 up through s_{M-1}

and you need to obtain a set of orthonormal basis signals. To do this, we need to use the Gram-Schmidt orthogonalization process which takes the first signal, finds the component of the second signal that is not along the first signal to get a new signal, finds the component of the third signal that is not along the first two and repeats. At the end, you get a basis set.

So let's actually do this for a specific example. So in this particular example, you have four signals. I have not used the s_0 , s_1 notation. In this case,

$s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$.

$s_1(t)$ is essentially 1 from 0 to 2 seconds. $s_2(t)$ is 1 from -1 to 1 second. $s_3(t)$ is 2 from 0 to 1 second and $s_4(t)$ is -1 from -1 to 0, 1 from 0 to 1 and -1 from 1 to 2. So let us now see how you can perform the Gram-Schmidt orthogonalization for this. I will make a remark.

The resulting orthonormal basis depends on which signals you choose in which order. We are going to choose them in this order 1, 2, 3, 4. If you choose another order, you will get a different basis set that is also equally correct and there is no problem. The basis set that you get from Gram-Schmidt orthogonalization is not unique. So let us actually perform Gram-Schmidt orthogonalization on these signals.

For reference, I have put them over here,

$s_1(t)$, $s_2(t)$, $s_3(t)$, $s_4(t)$.

All we need to know is where they are non-zero. So our basically times of interest are between -1 second and 2 seconds. The first signal is non-zero between 0 and 2. The second is non-zero between -1 and 1.

Third is 2 between 0 and 1. The fourth is like -1 in the first interval, 1 in the second interval, -1 in the third interval. Let us now carefully perform Gram-Schmidt orthogonalization on this signal set. So now what we are going to do is, we are going to first take our ψ_1 . Let us make it blue so that you know it is nicer.

Let me rewrite it over here. So I am going to write

$\psi_1(t)$ directly and $\psi_1(t)$, let me just get rid of this because it is confusing. So

$\psi_1(t)=s_1(t)/\|s_1\|$.

This is very easy because I am just going to choose my first signal as my first orthogonal vector or first basis element but just scale it to make it unit energy. So if you now look at inner product like $\|s_1\|$, that is very easy.

If you just calculate

$\int s_1^2(t)dt$,

I am dealing with real signals minus infinity to infinity, it is very evident that you can see that it is over here, it is 1 from 0 to 2, so this integral is going to be 2. Therefore, my

$\psi_1(t)$,

if I draw it, is very simple between 0 to 2, it is like this is actually $\|s_1\|^2$. So if you substitute $\|s_1\|$ over here, it is $1/\sqrt{2}$, so this amplitude will be $1/\sqrt{2}$. So we have our ψ_1 . The ψ_1 is just a scaled version of s_1 , it is just s_1 except that it is being scaled to have unit energy.

That is your ψ_1 . So now the next task is to obtain ψ_2 . To obtain ψ_2 , we will first go for an intermediate function called ϕ_1 that essentially find you a signal that is orthogonal to ψ_1 , but has the rest of s_2 in it, so let us do that. So I am going to define

$\phi_2(t)=s_2(t)-\langle s_2,\psi_1\rangle\psi_1(t)$.

So this is actually very interesting.

The first part says $\phi_2(t)$ is $s_2(t)$, but I want to take out that part of $s_2(t)$ that is along ψ_1 . So now needless to say, if you find

$\langle\phi_2,\psi_1\rangle=\langle s_2,\psi_1\rangle-\langle s_2,\psi_1\rangle\langle\psi_1,\psi_1\rangle$.

But what is $\langle\psi_1,\psi_1\rangle$? $\langle\psi_1,\psi_1\rangle$ is 1 because we chose it in that manner. It is

$s_1(t)/\|s_1\|$,

so this goes away. So if you now subtract, you get this equal to 0.

In other words, $\phi_2(t)$ is a signal that is specifically designed to be orthogonal to ψ_1 and have the remaining part of $s_2(t)$ in it. Let me issue a warning over here. If $s_2(t)$ is actually just a scaled version of $s_1(t)$, then you will get $\phi_2(t)$ to be 0, which is completely fine in which case you can just ignore it and move on to the next signal because $\phi_2(t)$ is just a scaled version of $s_1(t)$. So if your $s_2(t)$ is just a scaled version of $s_1(t)$, everything is going to be captured over here and you are essentially going to get 0. So that is something you have to keep in mind.

I am going to keep this part over here. So let us now move further. I want to find $\phi_2(t)$. So

$\phi_2(t)=s_2(t)-\langle s_2,\psi_1\rangle\psi_1(t)$.

So what is $\langle s_2,\psi_1\rangle$? So ψ_1 is this, ψ_2 of, you know, $s_2(t)$ is this.

You can see that ψ_1 is non-zero only from 0 to 2, while over here, this is between -1 to 1 and the only overlapping area is the part between 0 and 1. So if you now perform the

$\int s_2(t)\psi_1(t)\,dt$,

only the part between 0 and 1 is active and that part, if you integrate, you are going to get just $1/\sqrt{2}$. So you are just going to get $1/\sqrt{2}$. That is because if you multiply ψ_1 and s_2 , you are only going to get this particular part scaled by $1/\sqrt{2}$ and if you integrate, you get $1/\sqrt{2}$. So this means that your

$\phi_2(t)=s_2(t)-\psi_1(t)/\sqrt{2}$

and let us actually use a pictorial representation and draw it.

So it is

$s_2(t) - \psi_1(t)/\sqrt{2}$.

So $s_2(t)$ looks like this. I am not drawing the x axis marks. So it is all the time and this is actually $s_2(t)$. Let us subtract $\psi_1/\sqrt{2}$.

ψ_1 is over here. It is $1/\sqrt{2}$ everywhere. If I write it in this way, it is 0, 1, 2, amplitude is half. So if I subtract, what do I get? I am just going to draw the resulting signal. It is 1 over here and over here, the value is 1 and the value is here. I am going to subtract it.

I will get half and finally in the third part, the value over here is 0, value over here is half. If I subtract, I get minus half. So minus 1, half, minus half. So this is my $\phi_2(t)$. And remember, $\phi_2(t)$ is just a signal that has the part of $s_2(t)$ that is orthogonal to ψ_1 .

But it is not unit norm. So to get my ψ_2 , which is the actual basis vector element, I need to just normalize this. I need to just do $\psi_2(t) = \phi_2(t)/\|\phi_2\|$. So if you want to do this

$\int \phi_2^2(t)\,dt$,

it is very easy to quickly do it. Amplitude here is 1 and this is 1 in length, so it is 1 square. Similarly $(1/2)^2 + (1/2)^2$.

So it is

$1^2 + (1/2)^2 + (1/2)^2$.

So let me just write it down.

$\int \phi_2^2(t)\,dt = 1 + 1/4 + 1/4 = 3/2$.

So $\|\phi_2\| = \sqrt{3/2}$.

So I am just going to get this as $\sqrt{2/3}$, this to be the root times $\phi_2(t)$. So let us now draw our ψ_2 in a very neat way. $\psi_2(t)$ is essentially $\phi_2(t)$, which is over here, except that it is scaled by $\sqrt{2/3}$. So let us draw this and then it is half of this, half of this. I am deliberately not drawing the x axis points.

It is minus 1, 0, 1, 2. This is minus, sorry this is not minus, this is $\sqrt{2/3}$. This part is $\sqrt{1/6}$, which you can evaluate. It will be $\sqrt{1/6}$. Why? It is half times $\sqrt{2/3}$. That is half, it is like half times $\sqrt{2/3}$ is $\sqrt{2/12}$.

That is $\sqrt{1/6}$. This is $-\sqrt{1/6}$. If you want to now just do a sanity check, this should, the square of ψ_2 should integrate to 1. If you square it over here, you get two-thirds plus one-sixth plus one-sixth. Add those, you get 1.

That is the first point. Second, ψ_2 should be orthogonal to ψ_1 . ψ_1 is essentially the same thing between 0 and 2. ψ_2 is actually antipodal between 0 and 2. So the overlap will be 0. So ψ_2 is an orthogonal signal, orthonormal signal, ψ_2 is a unit energy signal and orthogonal to ψ_1 .

So ψ_2 is the second element of our basis vector. Fine. So we have done ψ_1 and ψ_2 . Let us look at ψ_3 .

So for ψ_3 , we need to find ϕ_3 . s_3 is 2 between 0 and 1. So 2 between 0 and 1. Fine. So how do we do this now? 2 between 0 and 1.

Let me just draw this. This is my $s_3(t)$. The value is 2. My ψ_2 is here and maybe I will just try to get ψ_1 . So I am just going to copy this and let us paste it over here. So this is my ψ_1 , this is my ψ_2 and we now have this s_3 .

So I am going to write ϕ_3 first. $\phi_3(t)$ is again for Gram-Schmidt, remember it is s_3 without the components along ψ_1 and ψ_2 . So it is

$$s_3(t)-\langle s_3, \psi_1\rangle \psi_1(t)-\langle s_3, \psi_2\rangle \psi_2(t) .$$

So now I am not going to do this, but if you take inner product of

ϕ_3 with ψ_1 ,

that is going to be

$$\langle s_3, \psi_1\rangle-\langle s_3, \psi_1\rangle$$

and it will become 1 and anyway $\langle \psi_2, \psi_1\rangle$ is 0. So ϕ_3 is orthogonal to ψ_1 and similarly ϕ_3 will be orthogonal to ψ_2 which is what you want because you want a signal which has that part of s_3 that is not in ψ_1 and not captured in ψ_2 as well.

So I am not going to do that. I am now just going to find ϕ_3 and then make it unit energy so that I get the third component. So now let us do this. So I have these signals over here. Let us now carefully just find

$\langle s_3, \psi_1 \rangle$, $\langle s_3, \psi_2 \rangle$.

So ψ_1 is here, $\langle s_3, \psi_1 \rangle$. I am just going to look at the pictures and do this for you. So if you look at $\langle s_3, \psi_1 \rangle$, this is 2 between 0 and 1. This is $1/\sqrt{2}$ between 0 and 1. So the common part is only 0 and 1 and you are just going to get 2 multiplied by $1/\sqrt{2}$. So that is going to be I think $\sqrt{2}$ because you are multiplying between 0 and 1.

The value is 2 here. The value is $1/\sqrt{2}$ here. So $\sqrt{2}$ is the result.

$\langle s_3, \psi_2 \rangle$

is also easy. So ψ_2 is over here and it is $\sqrt{1/6}$ between 0 and 1 which is the area that matters. If you now perform the multiplication and integration between 0 and 1, $2/\sqrt{6}$ will be the result which is $\sqrt{4/6}$ which is $\sqrt{2/3}$.

Now your task is to subtract these out with the appropriate scaling. That is

$\langle s_3, \psi_1 \rangle \psi_1$, $\langle s_3, \psi_2 \rangle \psi_2$.

So let us do this. So inner product $\langle s_3, \psi_1 \rangle \psi_1$ is going to be $\sqrt{2}\,\psi_1$ that is going to be you have ψ_1 over here.

If it is $\sqrt{2}$ times it is actually very easy. From 0 to 2 it is 1. Similarly

$\sqrt{2/3}\,\psi_2$, ψ_2

is also very easy. $\sqrt{2/3}\,\psi_2$ is going to be if you now multiply this part by $2/3$ you get $2/3$. If you multiply $\sqrt{1/6}$ by $2/3$ you are going to get you have to multiply $\sqrt{2/3}$,

$(1/\sqrt{6})\sqrt{2/3} = \sqrt{2/18} = \sqrt{1/9} = 1/3$.

So you get $1/3$ and then here you get $-1/3$.

Why? Because it is $-\sqrt{1/6}$ here. I am deliberately not writing the x axis but we really want this is -1, 0, 1, 2, -1, 0, 1, 2. Now you need to add these and subtract it from

s_3 . So let us actually do this. If I add these I am going to get, let us carefully add them. In this part between -1 and 0 there is no contribution from the first signal so I am just going to write $2/3$.

In the second part there is a contribution from here that is 1 and here there is a contribution of $1/3$. So you get $1+1/3=4/3$. In the third part there is $1-1/3=2/3$. This is going to be the combination of these two. Now you have to subtract this from your $s_2(t)$ which is this.

Now it is very easy to subtract because $s_2(t)$ affects only the parts between 0 and 1 and if you now subtract you have this $2/3$ becomes negative, this $2/3$ becomes negative and this $4/3$ is going to get subtracted from 2. So you are going to get

$\phi_3(t)$ to be $-2/3$ and over here you will get $-2/3$ and over here the amplitude is going to be $2-4/3$, $2-4/3$ is just $(6-4)/3$, you are just going to get $2/3$. This is your $\phi_3(t)$. So now if this is your $\phi_3(t)$ which is $-2/3$, $-2/3$, it is like $-2/3$, $2/3$, $-2/3$, have a look at your s_4 . This is actually -1, 1, -1 and your ϕ_3 looks a lot like s_4 and now if you want to scale this to make it unit energy, this part is if you do ϕ_3^2 , this part is going to give you an integration contribution of I think $4/9$, this will be $4/9$.

So you are essentially going to get $12/9$. $12/9$ is going to be I think $4/3$, $4/3$, so you have to multiply by $\sqrt{3}/2$. So you are going to get

$\psi_3(t)$ as, so $4/9$, $4/9$, $4/9$, $12/9$, so 3 by, $12/9$, so if you take the square root, this is actually going to be $4/3$, you will have to multiply by $\sqrt{3}/2$. So let us multiply by $\sqrt{3}/2$. So this part is going to be $-2/3$ multiplied by $\sqrt{3}/2$. So $2/3$ multiplied by $\sqrt{3}/2$, this is going to give you $\sqrt{3}$, this is going to give you, it is $1/\sqrt{3}$.

So let us now use our eraser and cancel this rough area out. So you are going to get $-1/\sqrt{3}$, $\sqrt{3}$, $-1/\sqrt{3}$. That makes complete sense because if you now integrate ψ_3^2 , you get one third plus one third plus one third that works out to 1. Now the question is do we have to worry about ψ_4 . Now in the case of ψ_4 , what happens is something which is very interesting.

It is actually just a scaled version of ψ_3 . So if you take

$\langle s_4, \psi_3 \rangle$

and if you subtract that out from s_4, you will get 0, which means s_4 is essentially a linear combination of these three waveforms. So in this case, you do not need another basis signal. ψ_1, ψ_2, ψ_3 capture this four signal system. So that is something which is very very obvious and interesting. So this is a full Gram-Schmidt orthogonalization process that we have performed for these signals.

But there is just some extension which I want to talk about. The one thing is in this particular case, we were dealing with waveforms, but we could have got away with just dealing with vectors. Why? The reason is because you have these signals, but they have a nice property in that they vary only at let's say 0, they vary only between, they don't vary between -1 and 0, they don't vary between 0 and 1, they don't vary between 1 and 2. Their values vary only at 0 and at 1. So you can actually treat these as three dimensional vectors.

So for example, I can as well write this as $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$. Why? Because this is a 0, 1, 1. This I can write as $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. This I can write as here as $\begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$. I am going to write this as 0, 2, 0 and I am going to write this as $\begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$.

So if I now look at these vectors and if I perform Gram-Schmidt orthogonalization on these vectors as opposed to signals, I will get the same set of vectors. That is, if you now look at ψ_1, I will actually get $\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}$. If you use ψ_2, if you look at ψ_2, you are going to get $\begin{bmatrix} \sqrt{2/3} \\ 1/\sqrt{6} \\ -1/\sqrt{6} \end{bmatrix}$ and if you look at ψ_3, you are going to get $\begin{bmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$. You will get the same result if you use vectors also.

There is one final remark that I wish to make. First of all, Gram-Schmidt orthogonalization is necessary when you have these kinds of signals but of course, in this case these were rectangular signals. The same trick of vectorizing would work even if you had sinc or some other waveforms if you can identify it carefully. But there is an interesting observation that you can make in order to not have to do all this work. What is it? The signal varies only between -1, 0, 0, 1, 1, 2. So why do I need any other special basis vectors? Let me give you a set of basis vectors without having to do any such work.

By the way, let us just try to copy this. So let me just take this particular part. So I am just going to… Okay, so let me just talk about it below. Okay, so now if you now look at the same set of signals that is 0, 2, sorry this is 1, 0, 0, 2, -1, 1, the third one is 2, 0 and the fourth one is -1, 1. So here is a trick which we will play.

We will just write three basis signals directly. What is the observation? These signals do not vary between -1 and 0. They do not vary between 0 and 1, between 0 and 2. We just need to construct three such signals.

Here are three such signals. So this is another basis vector.

$$\psi_1(t)=1$$

between -1 and 0.

$$\psi_2(t)=1$$

between 0 and 1.

$$\psi_3(t)=1$$

between 1 and 2. So this was a construction where you essentially just did not have to do Gram-Schmidt orthogonalization and you were able to infer ψ_1, ψ_2, ψ_3 almost by observation because if you look at this particular signal construction, these are time orthogonal meaning they have no overlap in time which means that they are orthogonal and they are one in amplitude for one second duration which automatically means that they are going to be unit energy.

So in this particular basis we have the same vectors which we talked about. In this case this will be $\begin{bmatrix}0\\1\\1\end{bmatrix}$. This will be $\begin{bmatrix}1\\1\\0\end{bmatrix}$. Let me actually write them in different colors just so that you know things will be much easier to infer. So let us choose something like a red color.

This will be $\begin{bmatrix}0\\1\\1\end{bmatrix}$ because it is $\psi_2(t)+\psi_3(t)$. This will be $\begin{bmatrix}1\\1\\0\end{bmatrix}$ because this signal is $\psi_1(t)+\psi_2(t)$. This will be $\begin{bmatrix}0\\2\\0\end{bmatrix}$ because this signal is just $2\psi_2(t)$ and finally this signal let me just point an arrow here and this signal is minus 1, 0, sorry it will be $\begin{bmatrix}-1\\1\\-1\end{bmatrix}$. Why? It is essentially take ψ_1 , flip it, take ψ_2 , keep it, take ψ_3 , flip it. So sometimes by observation you are able to infer a basis without having to do Gram-Schmidt as well or you can do a hybrid.

You can guess some of them and not get others and things like that. So while the Gram-Schmidt orthogonalization process is a very systematic way of finding orthonormal basis vectors, there are some situations where you can find basis vectors using just observation or other intuition. The basis vectors you obtain in this manner are going to be equivalent in every other way in the sense that your representation of these vectors is going to be different. For example, if you look at this particular basis vector, in this basis vector, for example if you take ψ_1 , your original vector is going to be, original vector had 1, so it is going to be $\sqrt{2}$ times $\sqrt{2},0,0$. That is going to be the representation of the original s_1 .

Notice that the energy or the sum square of this is 2. Over here also the sum square is 2. Similarly, let us look at the second vector. Where is your ψ_2 ? It is a little more complicated for the ψ_2 case. So your ψ_2 is $\sqrt{2/3},1/\sqrt{6},-1/\sqrt{6}$. Now in this case, this part has to cancel and you have to essentially scale it appropriately, but if you now represent the vector in the same manner, you will again find that the energy is the same.

So even though you are using a different basis, all your computations and results are the same and this leads to a very important conclusion. The basis vectors themselves do not really matter because you can actually use different basis vectors depending on your convenience as long as you keep the basis vectors orthonormal, your performance and all other design aspects are going to be the same. So the conclusion from this exercise where we did this gram-schmidt orthogonalization is that the first, the basis vectors are, the number of basis vector signals, the number of basis signals ψ_1 , ψ_2 is always going to be less than or equal to the number of signals. In this case, you had four signals but only three orthonormal signals that captured this.

The second thing is that these orthonormal basis are not unique. You can come up with any other one. For example, even if you did gram-schmidt orthonormalization using this signal as the first, then you will actually get 1 0 0. You will get the 1 between 0 and 1 as the first basis vector and then if you choose this one, you will get the second and in fact you will get the basis vector that we wrote over here without having to do much work using gram-schmidt as well. So just keep these things in mind when you perform gram-schmidt orthogonalization and wherever possible you can infer the orthogonal basis using your intuition or use a combination of gram-schmidt and your intuition whichever works best. Thank you. Thank you.