Digital Signal Processing and its Applications

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Lecture 26 A

Introduction to Frequency Transformation, Bandstop Filter & Transformation & Critical Point Mapping

A warm welcome to the 26^{th} lecture on the subject of Digital Signal Processing and Its Applications. We have been discussing frequency transformations in the previous lecture. By analog frequency transformations, we mean transformations from an analog filter to an analog filter. In fact, more specifically, a transformation that takes us from a low pass analog filter to an analog filter of appropriate nature, whether it be high pass or band pass or band stop.

In the previous lecture, we had looked at the transformation that takes us from the low pass to a high pass filter and from the low pass filter to the band pass filter. We now need to look at a transformation that takes us from a low pass filter to a band stop filter. And that is going to be a little more difficult than the other two. In fact, in order of difficulty, the high pass transformation is the easiest, the band pass is slightly more difficult and the band stop, even more difficult. So, today we now proceed to look at the band stop transformation.

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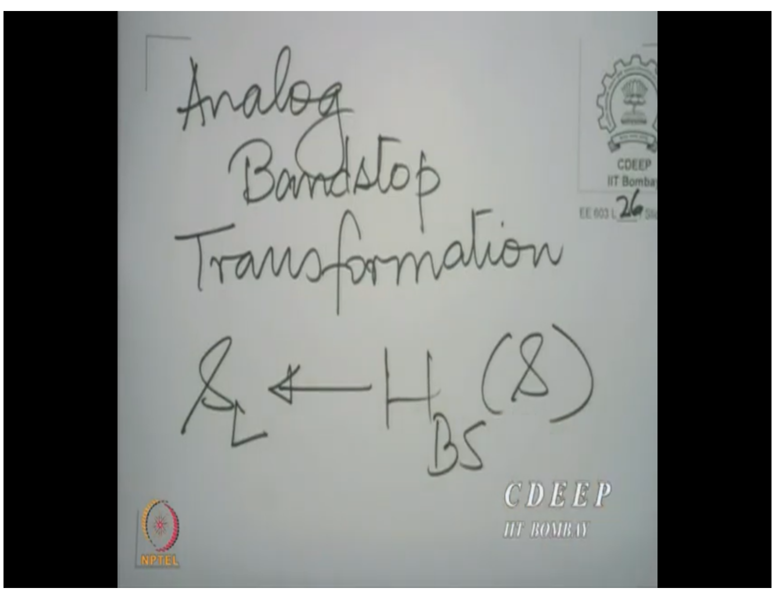

So, what we are trying to do is to get a transformation, let us call it $H_{BS}(s)$, which would replace the low pass complex frequency variable s_L. So that, when we replace s_L by this transformation, we would get a bandstop filter with specified characteristics, what we mean by specified characteristics.

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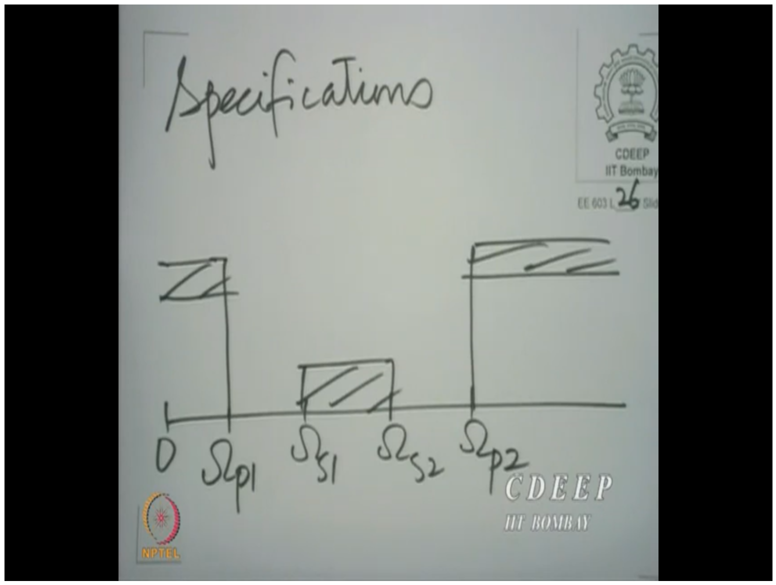

For a bandstop filter, the specifications look like this. Of course, we always show it in the positive frequency side. We have a pass band and another pass band, and the clear stop band. So, unlike the band pass filter, I am sorry, this should be the other way round, I am sorry. Let me redraw this. As I said, you have a one pass band, you have another pass band and the clear stop band.

So, unlike the band pass filter, we have just one stop band and two pass band, so maybe in a sense it is complementary or dual to the band pass filter. In the band pass filter, you have one pass band and two pass bands. Here, you have one stop band and two pass bands.

So, we want a transformation again which satisfies the three characteristics that we mentioned the last time. The first is that it must maintain rationality, the second is, it must preserve stability and finally it must take the imaginary axis to the imaginary axis in a manner that carries with it the kind of behavior that you desire. That is the frequency axis must map to the frequency axis in a way that a low pass character transforms to a band stop character.

Now, again we see here, there are two pass bands. And therefore, it is intuitively clear that we will require a second order transformation, we cannot make do with, it is not going to be a monotonic transformation. There is going to be some point in the transformation where you have a movement from one behavior to another. And that is going to happen if the transformation is at least second order, it cannot be just a first order transformation.

And once again, we take recourse to our intuition of looking for functions that take you from the imaginary axis to the imaginary axis and there again, we have LC networks to come to our aid. And last time, we saw that the series LC network comes to our aid in the band pass filter and therefore, we expect that if we use a parallel LC network, it would come to our aid in a band stop filter.

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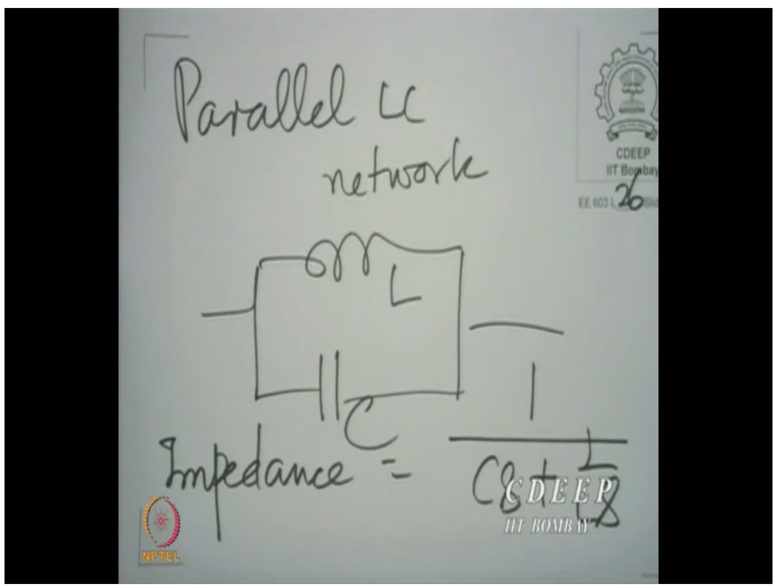

So, let us see how the impedance of a parallel LC network would look. So, the impedance would be of the form 1 by the sum of the admittances. And the admittances are $Cs + \frac{1}{Ls}$.

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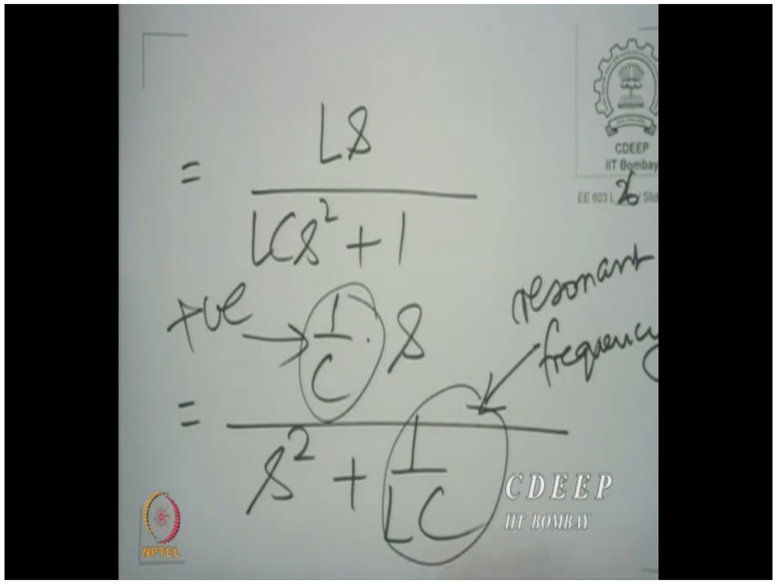

And this is easy to simplify, we can easily simplify this to get Ls in the numerator, $LCs^2 + 1$ and that can be simplified once again. So, $\frac{\frac{1}{C}s}{s^2+\frac{1}{LC}}$ and once again, we note that $\frac{1}{LC}$is the resonant frequency of the network. And this is a positive constant.

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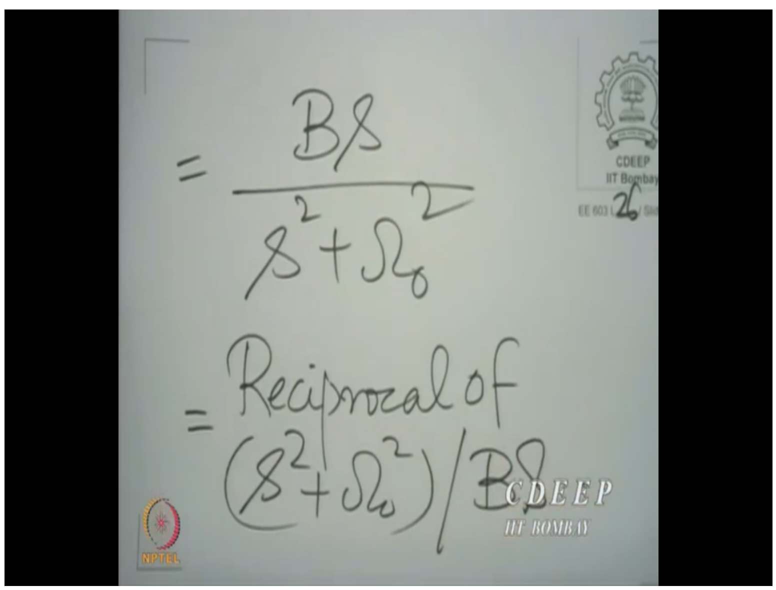

And therefore, we can rewrite this expression in the form $\frac{Bs}{s^2+\Omega_0^2}$. And we are not terribly worried about the specific meaning of B, Ω_0^2now, that will in fact now, was the case to the band pass filter, it will become obvious when we actually go through the transformation. Now, one thing we need to do, is to note that this is the reciprocal of $\frac{s^2+\Omega_0^2}{Bs}$. And of course, it is rational.

So, the movement from rational to rational is guaranteed, there is no problem. So, we do not need to worry about the fact that after transformation, the resultant filter would remain rational, there is no problem on that account. But of course, we need to worry about stability.

Now, stability actually follows from a very simple argument. In fact, we will show that if a particular transformation is stable, in other words if it takes the real part of s to, I mean if it takes, if it preserves the sign of the real part of s, if it takes the left half plane to left half plane and the right half to the right half plane, then the same thing happens to the reciprocal.

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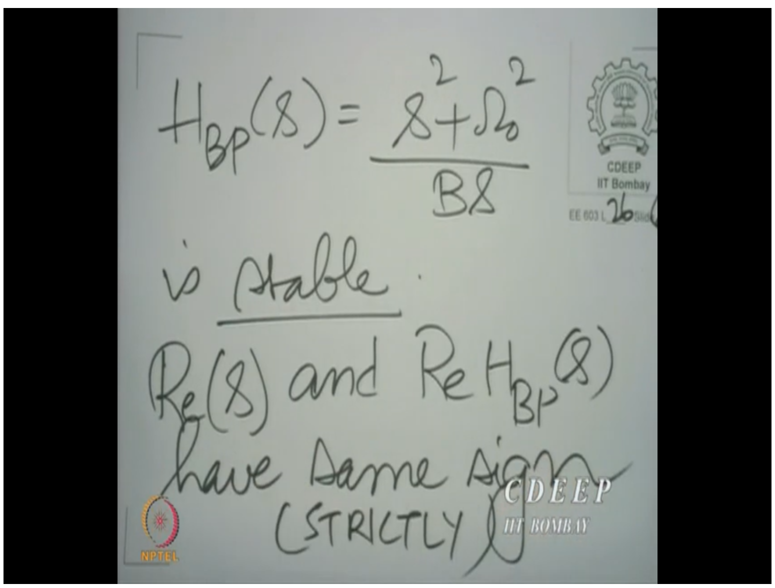

So, we will say, $H_{BP}(s)$ which is $\frac{s^2+\Omega_0^2}{Bs}$ is stable. In other words, the real part of s and real part of $H_{BP}(s)$ have the same sign. Of course, strictly the same sign. By strictly I mean, if the real part is *0*, the other real part is *0* as well. The imaginary axis goes to imaginary axis.

Now, of course the fact that the imaginary axis would go to the imaginary axis is obvious because we have taken an *LC* network. So, when you take its impedance, if you put $s = j\Omega$, it is definitely going to be imaginary. So, we do not need to worry about that. So, the imaginary axis going to the imaginary axis is not a problem. But what we need to check is this, whether the sign of the real part is present.

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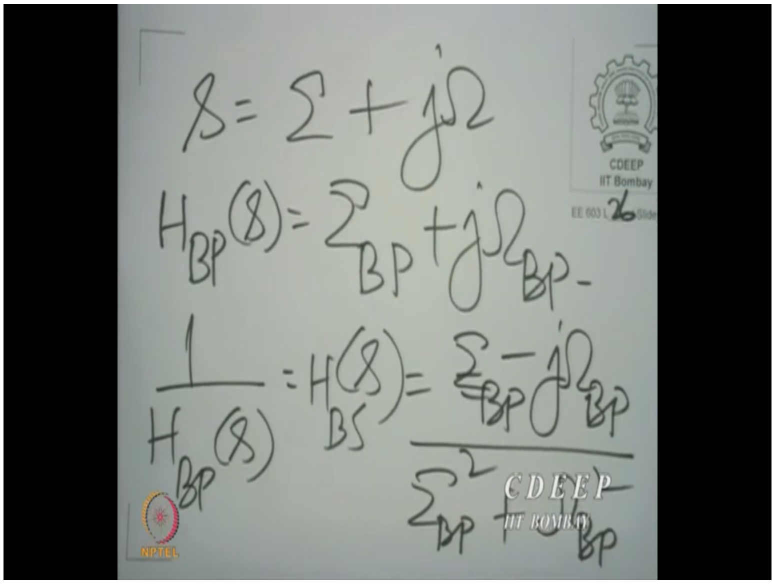

Now, let $H_{BP}(s)$, you see, let $s = \Sigma + j\Omega$, whereupon, let $H_{BP}(s)$ give $\Sigma_{BP} + j\Omega_{BP}$. Now, $\frac{1}{H_{BP}(s)}$ which is $H_{BS}(s)$ is the reciprocal of this. The reciprocal of this is, $\frac{\Sigma_{BP}-j\Omega_{BP}}{\Sigma_{BP}^{2}+\Omega_{BP}^{2}}$. The complex conjugate divided by the mod squared.

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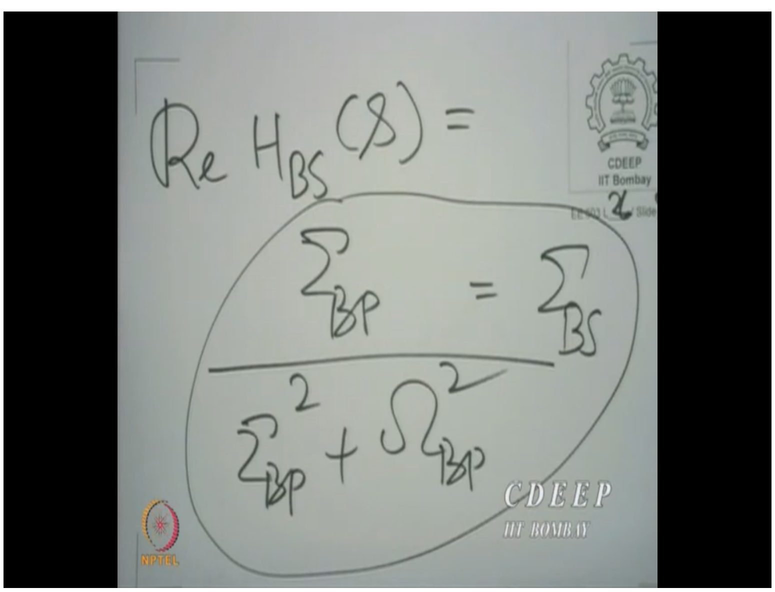

And now, you can see very clearly that the real part of $H_{BS}(s)$ is simply $\frac{\Sigma_{BP}}{\Sigma_{BP}^{\ 2}+\Omega_{BP}^{\ 2}}$, which is of course,$\Sigma_{BS}$ if you wish to call it that. And now, it is very obvious that when $\Sigma_{BP} \neq 0$, this denominator cannot possibly be 0. And if the denominator is not 0, it must be positive because it is a sum of two squares.

That means Σ_{BP} and Σ_{BS} must have the same sign, it is obvious now. So, we have proved stability for the bandstop transformation as well. So, now we have a very strong candidate, all that we need do is look at what it does to the sinusoidal frequencies. And let us indeed embark upon that question. So, what does it do when you substitute $s = j\Omega$?

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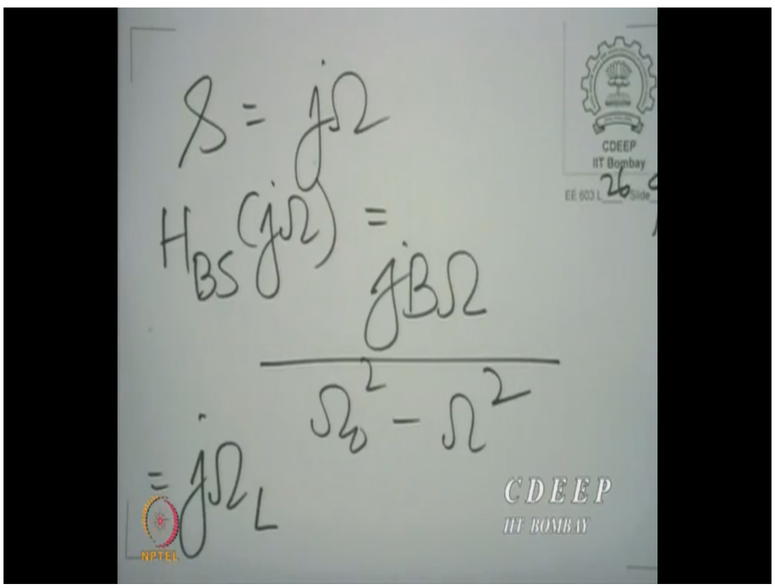

$H_{BS}(j\Omega)$ is $\frac{jB\Omega}{\Omega_0^{\ 2}-\Omega^2}$. And therefore, this is of the form $j\Omega_L$, it is of course imaginary as you can see.

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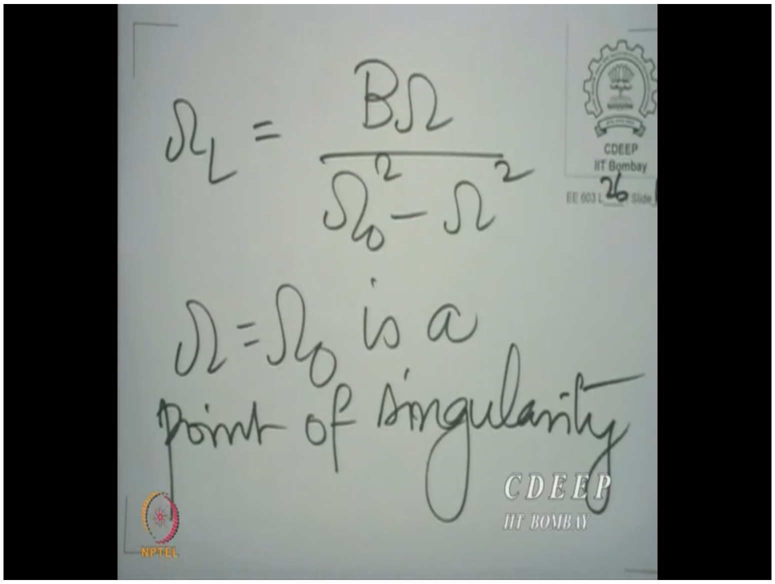

Where Ω_L is $\frac{B\Omega}{\Omega_0^2 - \Omega^2}$. And here we have a little more work to do. You see, it is very clear that we now have a point of singularity somewhere on the positive real axis. $\Omega = \Omega_0$ is a point of singularity. Singularity means it diverges; the function is undefined at that point.

And therefore, we expect something unusual to happen at that point. We will see in a minute that that is exactly what the case is.

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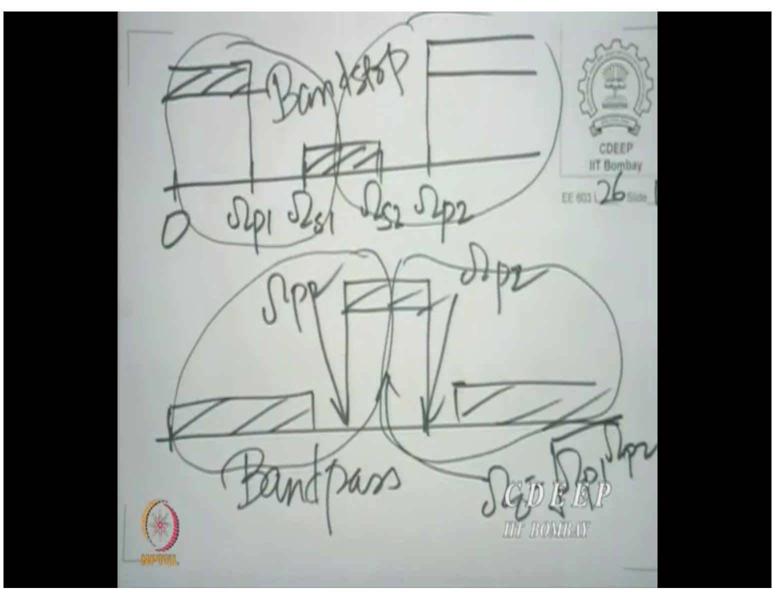

So, let us now consider the bandstop transformation. So, here you have, let us map the critical points. You know the critical points are going to be a little more difficult to map now. You see, let us try and visualize what it is that we desire from this transformation. Here, if you look at the bandstop, if you look at the bandpass transformation, what happened there, and let me draw it once again for you, this is bandstop.

In the bandpass transformation, what happened was, you had these two stopbands and a passband in between. And what effectively happened was, you have Ω_0 somewhere in between here where it is the geometric mean of Ω_{p1} and Ω_{p2}. And this was Ω_{p1} and this was Ω_{p2}. From the effect of this transformation was to bring Ω_0 to *0*. This to $-\infty$ and $+\infty$ went to $+\infty$.

So, in fact a pair, this entire thing went to one side of the frequency axis in the low pass factor and this entire thing, went to the other. Now, in a way, the bandstop transformation, we have exactly the reverse situation. So, we have two such pairs of structure, this pair and this pair if you wish to call it that. And therefore, what we expect is that one of these pairs would go on to the positive side of the frequency axis and the other would go on to the negative side. By a pair, I mean a segment or the entire stopband and a segment of the entire passband. So, a stopband passband pair.

In the bandpass filter, it was a segment of the passband and the entire stopband. Now, here we expect it to be a segment of the stopband and the entire passband that will go to one side and the other side of the frequency axis in the low pass filter. And therefore, the infinity is going to come somewhere in between and as expected, that infinity is going to come to $\Omega = \Omega_0$, that is why it is a point of singularity.

So, at $\Omega = \Omega_0$ there is going to be a jump, it is going to jump from $+\infty$ to $-\infty$. Let us see indeed what happens. So, let us take inspiration from the bandpass filter.

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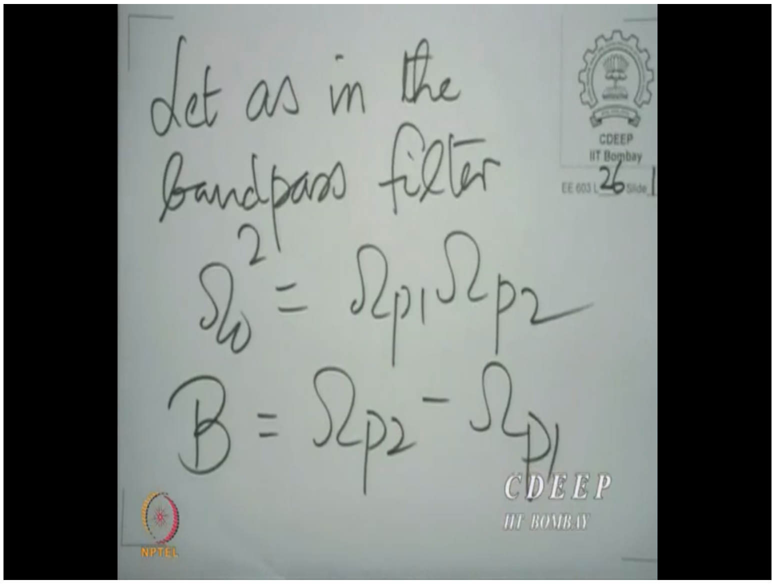

Let, as in the bandpass filter, Ω_0^2 be $\Omega_{p1}\Omega_{p2}$ and B be equal to $\Omega_{p2} - \Omega_{p1}$. So, here maybe it is not quite appropriate to call B the bandwidth, it is like a combination of a stopband and the transition bands. It is not exactly a bandwidth. It is something slightly different but it is indeed a segment of the real axis from the frequency axis, that is of course true.

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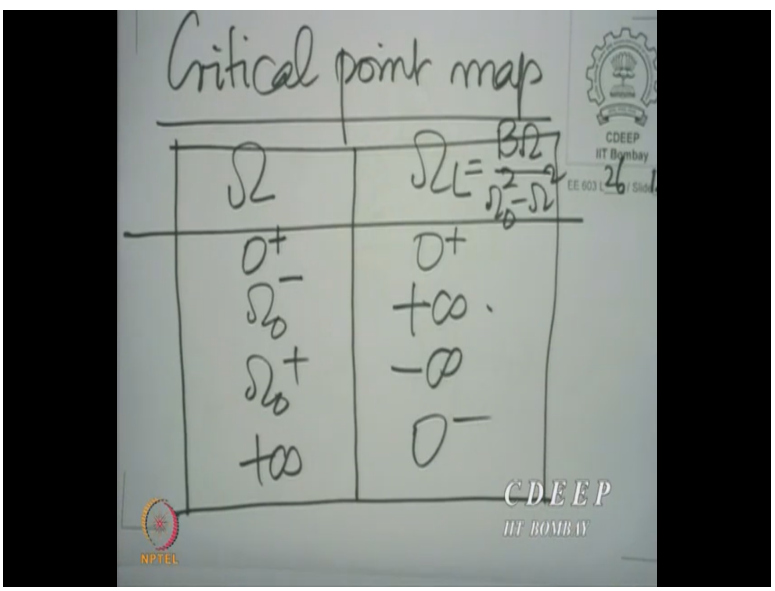

If we get this, then we can now map the critical points. *0*, it is very clear map to *0*. In fact, let us take 0^+, by 0^+ we mean a very small positive frequency which goes towards. So, 0^+ of course goes to 0^+. Now, as we move from 0^+, the first point that you encounter of importance is Ω_0. Now, Ω_0 is a point of singularity.

So, we cannot simply use $\Omega_0$itself and $\Omega_0$itself, the expression diverges, we cannot use it. But we can use Ω_0^- and Ω_0^+. So, let us use Ω_0^-, by Ω_0^-, we mean a value just before Ω_0. Now, Ω_L as you know, is $\frac{B\Omega}{\Omega_0^2-\Omega^2}$.

This is the frequency transformation. So, Ω_0^-, the denominator is slightly positive. And of course, the numerator is finite and positive. And therefore, this is going to go to $+\infty$. In contrast, when we go to Ω_0^+, the denominator is slightly negative and of course the numerator is clearly positive.

And therefore, this goes to $-\infty$. So, now we know what we mean by $\Omega = \Omega_0$being a point of singularity. Not only is the function discontinuous at that point, there is a huge jump. So, in fact we have seen its derivative is also discontinuous and that is why there is a jump. A function could be discontinuous but its derivative might turn out to be continuous.

In that case, you do not have an infinite jump. But here, we have an infinite jump. So, it is more than just discontinuity. Its discontinuity in the derivative, the *CES* kind of singularity. And of course, when you go to $+\infty$ and, you see now $+\infty$ is tricky. We must divide the numerator and denominator by Ω.

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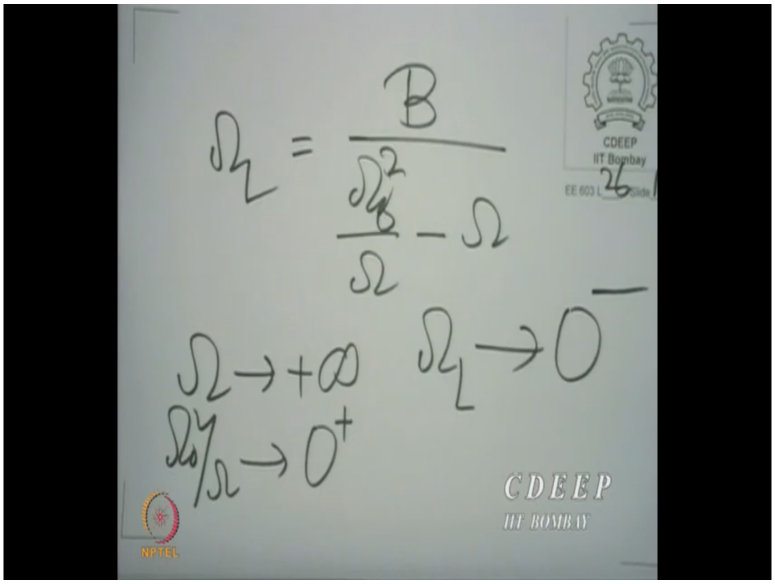

So, Ω_L is $\frac{B}{\frac{\Omega_0^2}{\Omega}-\Omega}$, that is how we write it. And therefore, as $\Omega \rightarrow +\infty$, $\frac{\Omega_0^2}{\Omega} \rightarrow 0, 0^+$ if you like and of course, $\Omega \rightarrow +\infty$, so this whole thing, $\Omega_L \rightarrow 0$.

Now, again you may ask whether its 0^+ or 0^-. It should go to 0^- and therefore, we need to write here 0^-. Now, we have the critical point mapping very clearly before us. Now, this critical point mapping makes it clear that Ω_0 is a point where we have trouble. So, it will be difficult for us to show both the segments all in one figure, so let us show them in two figures, that will be easier to show.

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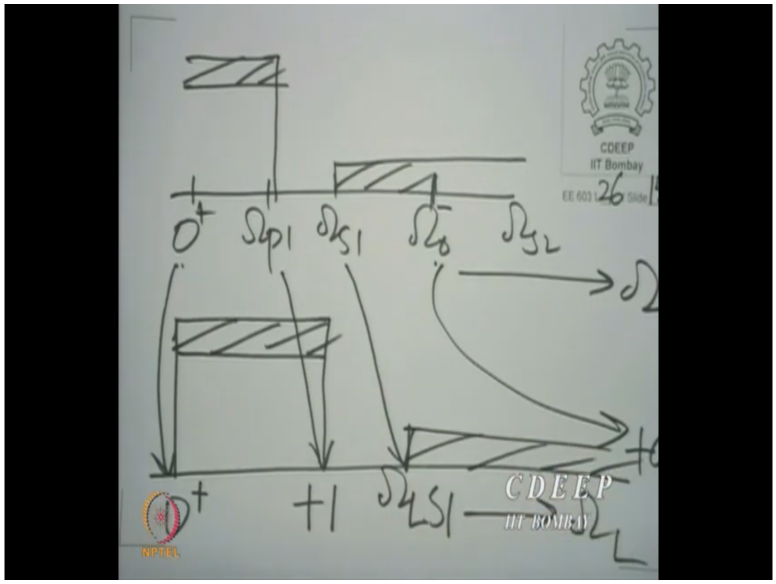

So, we will take the first segment of the bandpass, the bandstop transformation in Ω_0, somewhere here in between. Now, we know Ω_0 is a geometric mean of Ω_{p1} and Ω_{p2}. So, it will be somewhere in between. And Ω_{s2} is going to be somewhere beyond there. So, when we say Ω_0^{+} or rather Ω_0^{-}, we will go from *0* to Ω_0^{-}.

And *0* to Ω_0^{-} is carried, so, this is Ω, this is carried 0^{+}, 0^{+} is carried to 0^{+} on Ω_L. As we know, you know the expression, you have already worked this out. Ω_{p1} is going to go to $+\ 1$. That we know from the band pass transformation, we have worked it out. The reciprocal of *1* is *1*. So you see, Ω_{p1} will go to *1*.

Ω_{s1} is going to come to another point beyond $+\ 1$. So, let us call that Ω_{Ls1}. And Ω_0^{-} would go to $+\ \infty$. And therefore, we have this mapping for the segment between 0^{+} and Ω_0^{-}, this is the frequency mapping. And of course, here too we must not forget that the dependent variable is carried as is from the bandstop filter to the lowpass filter. The tolerances are carried as they are, the nature of the passband and stopband are carried as they are.

Here too, we must not forget that we have two passbands and therefore, in principle you could have had two different tolerances for the two passbands. But then, when you put down the specifications for the lowpass filter, you need to choose the more stringent of the two, whichever demands more, that must be put and the other will automatically be satisfied, if at all they happen to be different. If they are the same, there is no issue at all.

As far as nature goes, you do not have the luxury of having two different natures in the passband. The two passbands must be of the same nature, either equiripple or monotonic. And of course, the stopband gets carried as it is. So, that is for the segment from 0^+ to Ω_0^-. Now, that happens for a segment from Ω_0^+ to $+\infty$.

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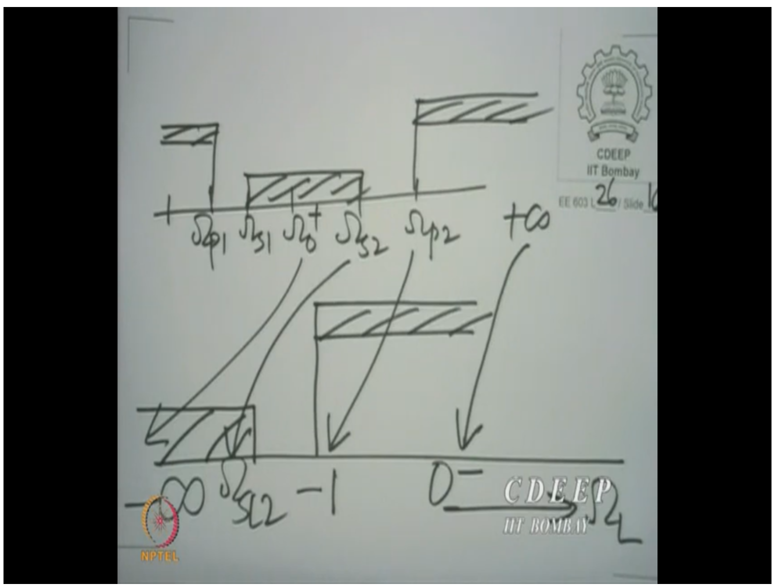

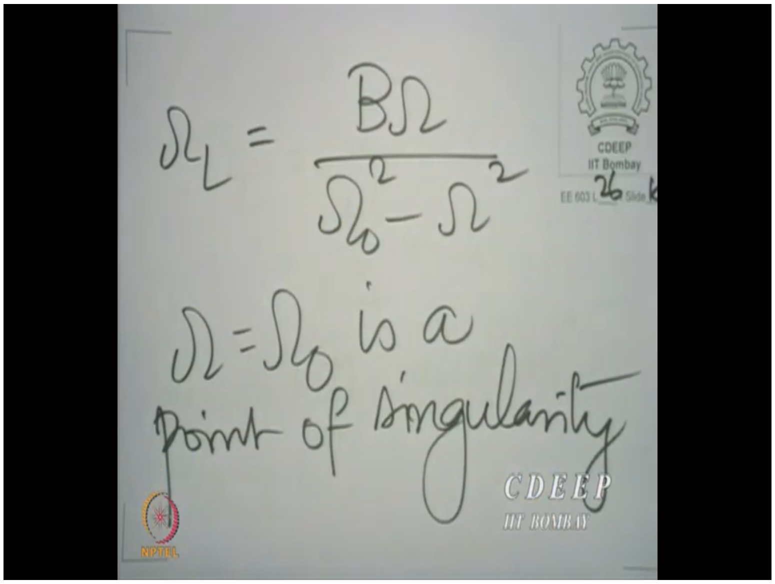

We will just squeeze this part. $\Omega_0^{\ +}$, as you know, goes to $-\ \infty$. We have already worked out for the bandstop transformation, that $\Omega_{p2} \rightarrow\ \ -\ 1$, this is easy to verify. And therefore, Ω_{s2} is going to come to a point, Ω_{sL2}. And of course, $+\ \infty$ is going to come to 0^{-}. And therefore, we have this specification being put down to the low pass filter.

There is one little word of caution. If you look at the frequency transformation here, this or if you like write this one, if you recall, in the bandpass transformation, it was $\frac{\Omega^2 - \Omega_0^{\ 2}}{B\Omega}$. Here, this is not exactly, the frequency transformation is a negative reciprocal and that is not surprising because the reciprocal of *1* by, or the reciprocal of an imaginary number has the, of the reciprocal magnitude and the opposite sign.

So, the sign has been reversed. So, $+\ 1$ and $-\ 1$ have got reversed here, that is the point of observation, the reciprocal of an imaginary number is reciprocated in magnitude and the sign is reversed. That should be noted, these are small, and that is why you should have to careful that the $+\ 1$ and $-\ 1$ roles have gotten reversed. So, Ω_{p2} now maps to $-\ 1$, not to $+\ 1$ and Ω_{p1} maps to $+\ 1$ and not $-\ 1$, that should be clearly noted.

Anyway, it is a minor point but we have now understood the transformation. So, now we have the totality of the transformation before us and we can see now that, we can put down the specifications of the low pass filter except for one important detail. We have got the specifications and the positive frequency side and the negative frequency side, but they have to be symmetric, magnitude symmetric and phase antisymmetric. Phase anyway we are not putting down anything at all.

So, it is only the magnitude which is of concern. Now, again, the symmetry is there anyway as far as the passband edge goes. The pass band edge is $+\ 1$ on the positive frequency side and $-\ 1$ on the negative frequency side. The problem is with the stop band edge. Ω_{sL2} and Ω_{sL1} or Ω_{Ls1}, whatever you want to call it, sL or Ls, now you see these mapped stopband edges may not be the negative of one another.

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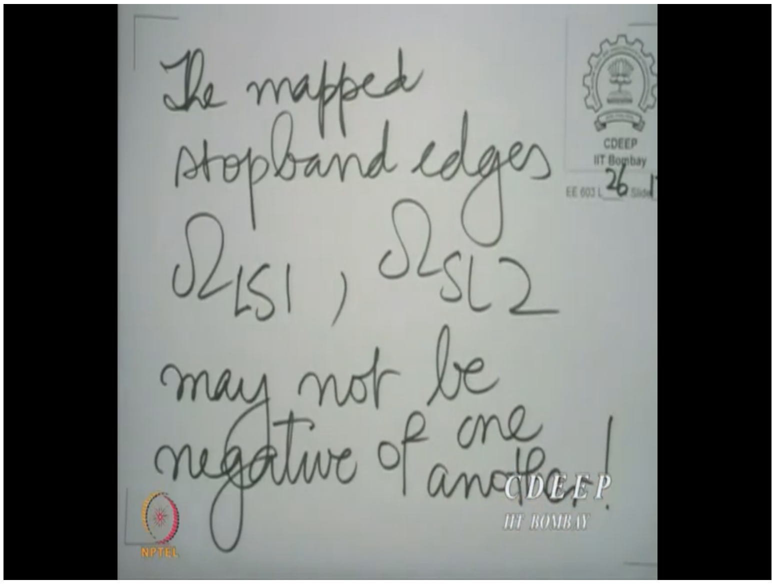

The mapped stopband edges, Ω_{Ls1} and Ω_{sL2}, will not be negative of one another. So, we have to do what is harder to do here. We take the minimum of $\left|\Omega_{Ls1}\right|$ and $\left|\Omega_{sL2}\right|$.