Digital Signal Processing and its Applications
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Lecture – 22B
Introduction to Chebyshev Filter System Function and Design Parameters

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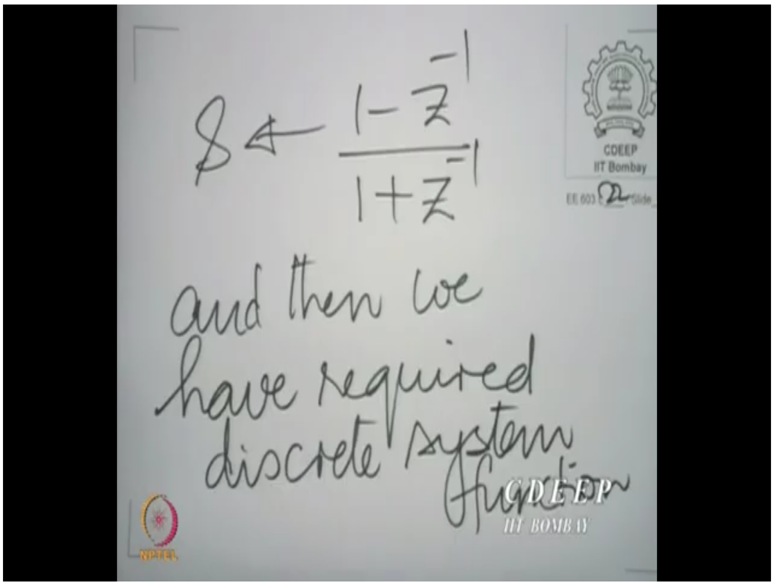

Now, we shall look at the possibility of an equiripple passband and a monotonic stopband. So, you see we move on to what is called the Chebyshev approximation of filter design.

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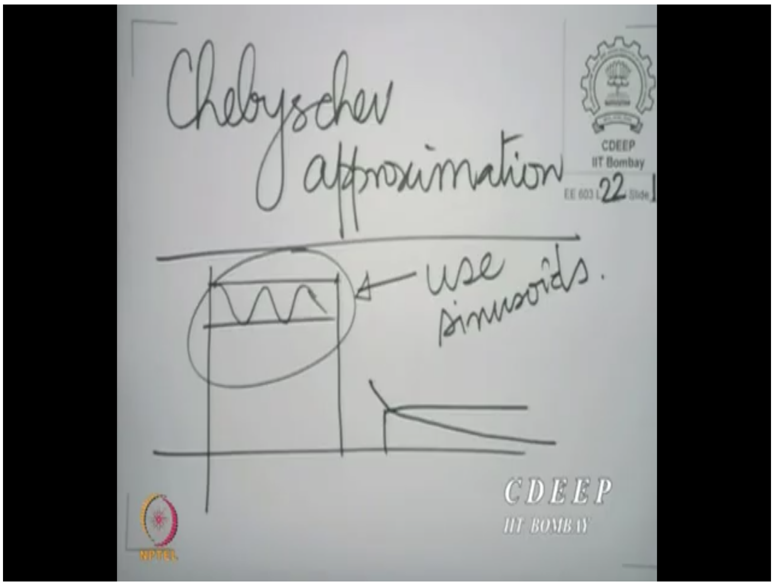

Chebyshev is a very well known mathematician; he has made some very important contributions to the theory of polynomials and rational approximations. So, his work has given a lot of insight into the behavior of polynomials, and into the choice and the use of polynomials and rational functions.

In particular, one class of polynomials that he proposed are called the Chebyshev polynomials, which are actually related to trigonometric functions; but are not trigonometric functions on their own. And we are going to employ those polynomials in our design here. What we want to do is to have an equiripple passband and a monotonic stopband. So, we expect the passband to have a magnitude something like this; and we expect of course the stopband to have a magnitude something like this.

Now, if you want an equiripple behavior in the passband, you want this essentially to correspond; where do you get equiripple behavior from? Equiripple essentially one way to get equiripple behavior is to use the sinusoidal function; because, the sinusoids oscillate in equiripple manner over cycles. And what is a function which is sinusoidal in a certain range, and then becomes monotonic later. Now, here we have to use a little beyond high school mathematics. We have to extend our ideas of trigonometry to complex functions; so in fact if we choose the function cos θ for example.

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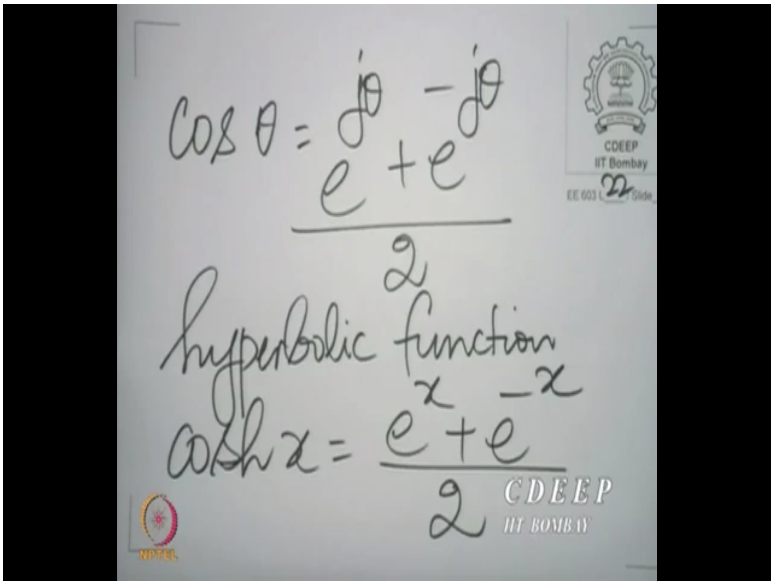

Cos θ can be written in complex form as, $\frac{e^{j\theta} + e^{-j\theta}}{2}$. And at the same time the hyperbolic cosine or the hyperbolic function cosh; cosh x is defined as $\frac{e^{x} + e^{-x}}{2}$, is a close relationship between them cos hyperbolic. Now, θ in principle can be complex and so can x; so, one does not have to restrict this definition only to real's. And when we do that, we get this whole class of functions; that is equiripple in a certain region and then becomes monotonic.

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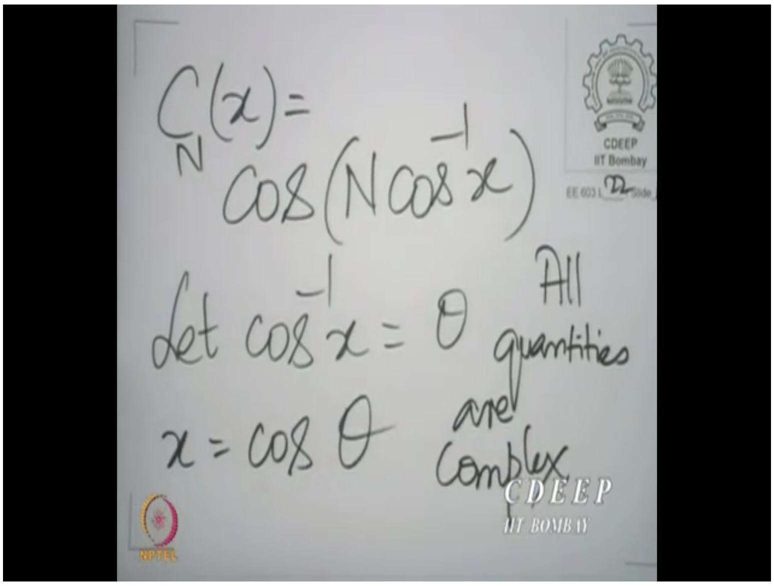

In fact, let us define the function $C_N(x) = \cos(N \cos^{-1} x)$. Now, interestingly we will show that it can also be written as well in terms of the hyperbolic cosine; so, let $\cos^{-1} x = \theta$. Remember we are now not restricting ourselves to real x at all; although we will later on take the special cases of real x in different regions.

But, we are allowing x to be complex, and therefore let cos x, $\cos^{-1} x = \theta$; and we are agreeing that all quantities are complex. And therefore, x is of course $\cos\theta$ and therefore we could expand x in terms of θ.

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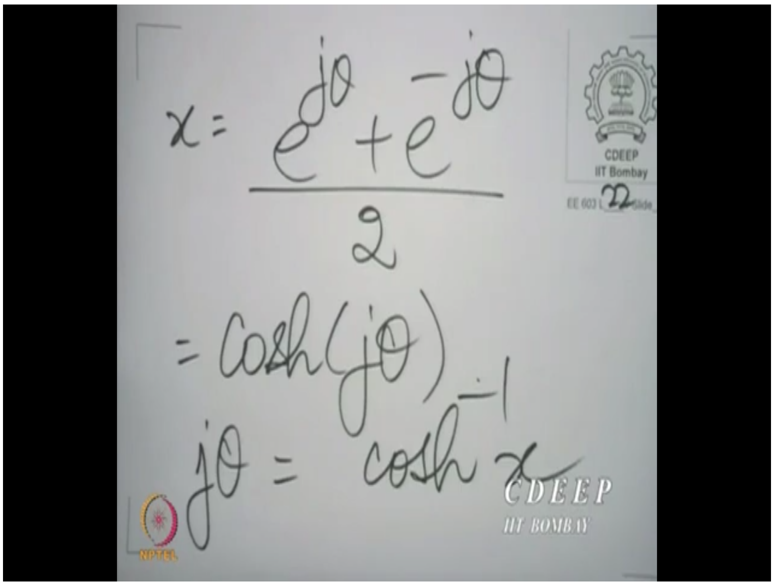

It is $\frac{e^{j\theta} + e^{-j\theta}}{2}$. But, this is also cosh $(j\theta)$ and therefore $j\theta = \cosh^{-1} x$. Now, go back to $C_N(x)$.

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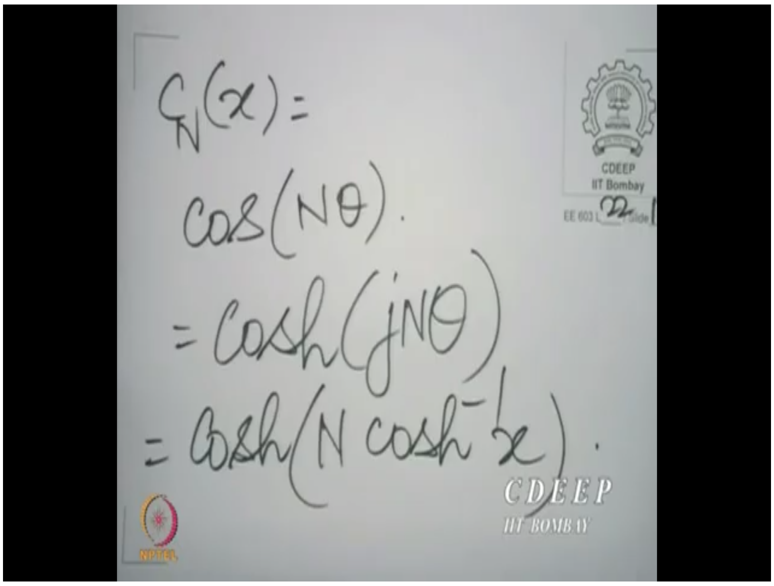

It is $\cos(N\theta)$, which is also cosh $(jN\theta)$; and therefore that is also = cosh (N $\cosh^{-1}x$). Because $j\theta$ is $\cosh^{-1}$ x; so therefore it is a very interesting observation that we have. That cosh (N $\cosh^{-1}x$) is also equal to cosh (N $\cosh^{-1}x$). And now we will write down this expression for a few values of the integer N. So, let us take N = 0 and N = 1; now those are easy.

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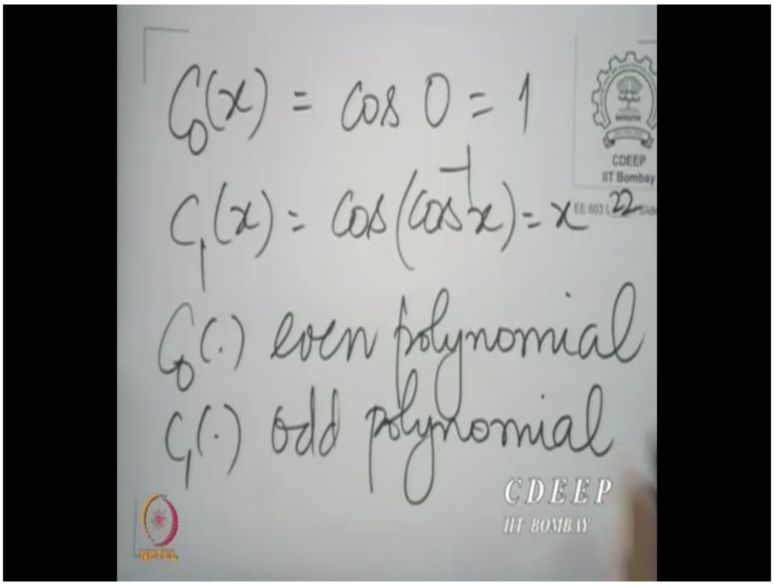

So, C_0 x = cos 0 = 1. C_1 x = cos($\cos^{-1}$x) = x. So, as you see both C_0 and C_1 are polynomials; you see C_0 is an even polynomial. By an even polynomial, I mean a polynomial with only even powers of the argument, and C_1 is an odd polynomial. So, it is a polynomial with only odd powers of the argument; now we shall generalize this by using a simple step of induction. We shall write down C_{N+2} (x) and C_N (x); so let us write down.

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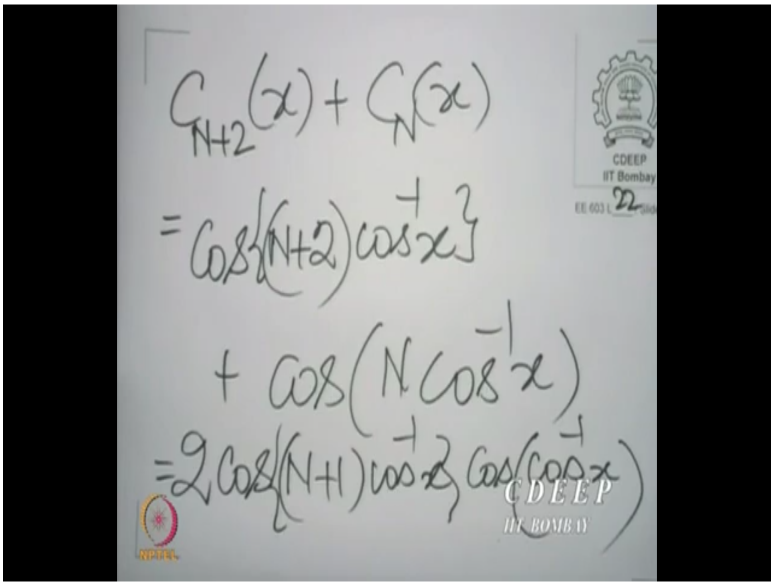

C_{N+2} (x) and C_N (x) = cos(N+2) $\cos^{-1}x$ + cos(N $\cos^{-1}x$). And we can make use of the trigonometric identity, when you add two cosines. So, this is 2 times quote the product of 2 cosines; the first one is the average of these two, and the second is this minus this by 2. So, you have this is equal from trigonometric identities to 2 cos{(N+1)$\cos^{-1}x$} cos($\cos^{-1}$ x). And therefore, we have a very interesting recursive relation between the C_Ns.

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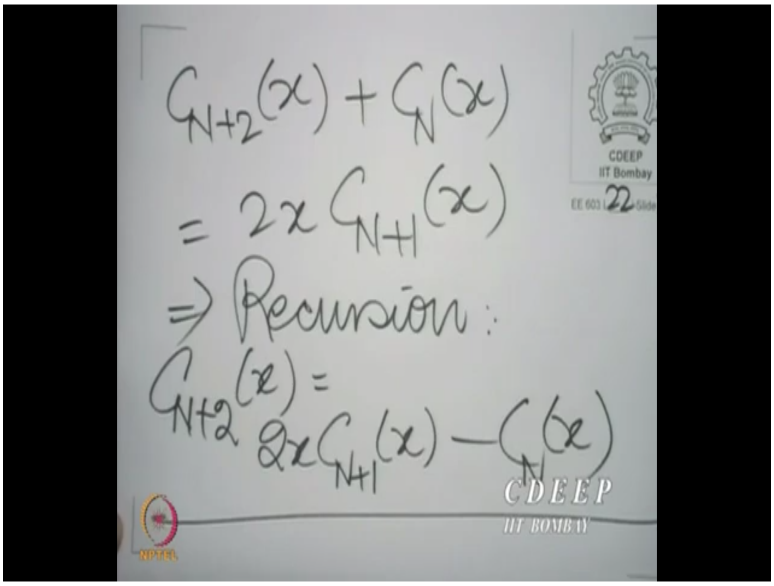

$C_{N+2}(x) + C_N(x) = 2x\, C_{N+1}(x)$ and therefore we have a recursion. $C_{N+2}(x) = C_{N+1}(x) - C_N(x)$; this is a recursion which takes you to successive C_N x, starting from 0 and 1. So, by induction we can now construct the C_N x beginning with the 0 and 1 case.

So, if I know C_0 and if I know C_1 ; I can construct C_2. If I know C_1-C_2, I can construct C_3; I can keep doing this. And this also makes it very clear that the C_Ns are all polynomials; because by induction if C_N is a polynomial in x. If C_N plus 1 is a polynomial in x, then this expression must be a polynomial in x.

Because you are multiplying 2x by polynomial in x, and subtracting another polynomial of x; and therefore when you subtract two polynomials, you must get a polynomial. In fact, you can also say something about the order of the polynomial. We have already seen by induction that C_0 is of degree 0 and C_1 is of degree 1.

So, in general if we assume C_N to be of degree N, then you have a degree N polynomial here; and you have a degree N+1 polynomial here. So, multiplying by x increases the degree by 1; so it would become a degree N + 2 polynomial coming from here and a degree N polynomial coming from there.

And therefore the overall polynomial is of degree N + 2. So, if we have proved in the inductive step that C_N is of degree N and C_{N+1} is of degree N + 1. Then, it follows by mathematical induction that C_{N+2} must be of degree N + 2; much more follows from this recursion.

We have seen that C_0 is odd and C_1 is even; let us assume that by again as an inductive basis that this is true for a particular N and N+1. You have to be assuming the basic step here is true for two successive values of the integer; and it is of course true for 0 and 1. So, suppose C_N and C_{N+1} obey this property that they are odd when N is odd; and even, when N is even; will prove that N + 2 also obeys the property.

So, you see let us, let N be odd, then N + 2 will also be odd. Now, this is then assumed to be an odd polynomial and this happens to be an even polynomial. But, an even polynomial multiplied by x makes it an odd polynomial again; and an odd polynomial minus an odd polynomial must give you an odd polynomial.

So, if C_N is odd then and C_{N+1} is even; then you go back to C_{N+2} being odd. And conversely, if N is even and then C_N is assumed to be an even polynomial; then C_{N+1} would be an odd polynomial. Multiplying an odd polynomial by x makes it even, and therefore an even polynomial minus an even polynomial will give you back an even polynomial.

And therefore, by mathematical induction C_N is an odd polynomial for odd N and an even polynomial for even N. All these are very interesting conclusions. Because we have seen so let us summarize these conclusions.

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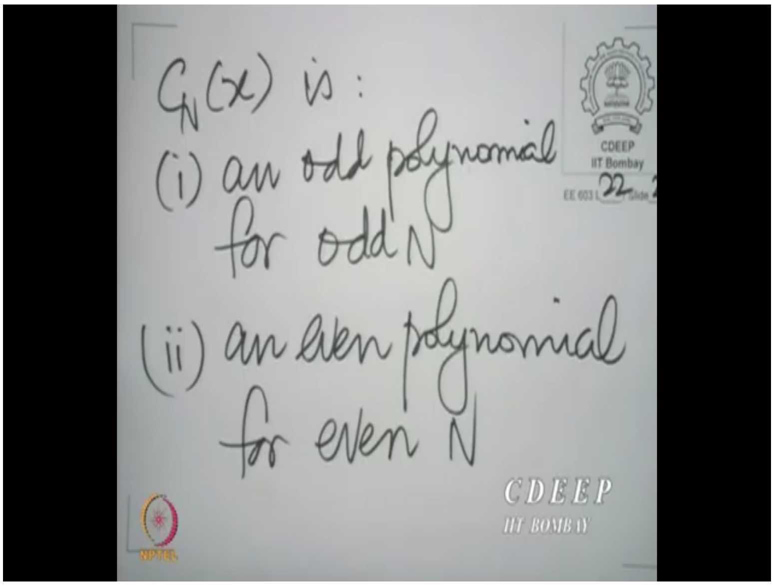

C_N (x) is an odd polynomial for odd N, an even polynomial for even N; and these are called the Chebyshev polynomials. In fact now, we can use two different expressions for the same polynomial; we will just repeat that once again. And the two different expressions are useful in different regions.

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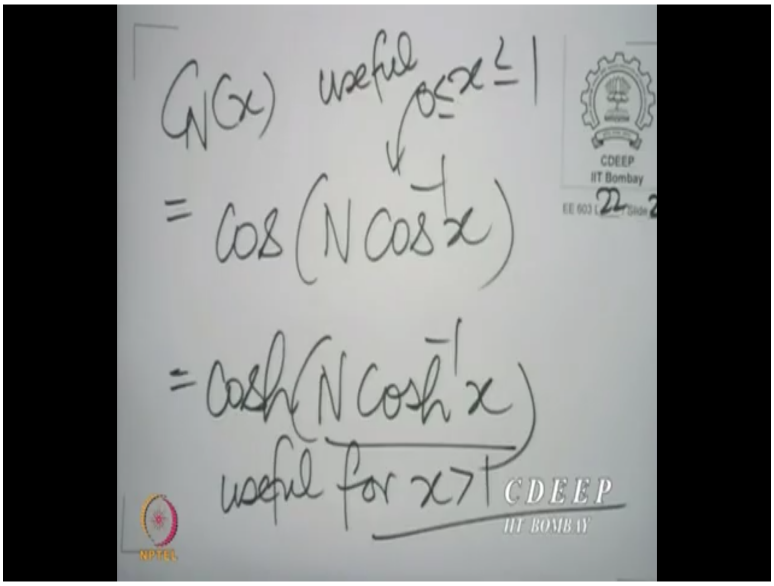

The same C_N (x) can be written as cos(N $\cos^{-1}$ x) = cosh(N $\cosh^{-1}$x). This is useful for x between 0 and 1; because then $\cos^{-1}$ x becomes real. And this is useful when x > 1, because cosh for a real argument is always greater than 1, greater than or equal to 1. Cosh x for real x is of course an even function; so cosh of x is equal to cosh of minus x. And therefore, we need only to look at the positive side of x.

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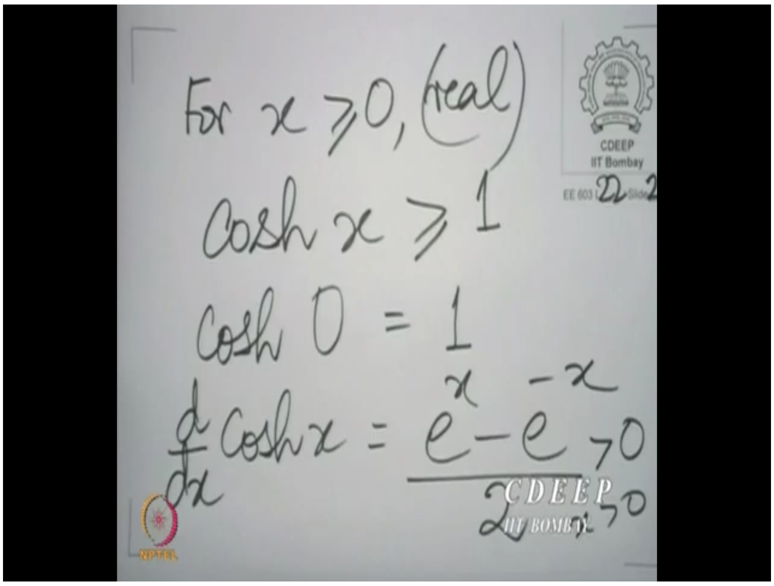

Cosh x is always greater than 1, x ≥ 0 ,real of course; cosh x ≥ 1. In fact, cosh 0 = 1 and one can easily see that $\frac{d}{dx}$ cosh x = $\frac{e^{x}-e^{-x}}{2}$. And this is always > 0, for x > 0, and therefore the derivative is always positive. And therefore cosh x strictly increases beyond 1. So, for x > 1, we can use the expression involving hyperbolic cosine; and for x < 1, we use the trigonometric expression. And that gives us an insight into the behavior of the Chebyshev filter. Now, let us make use of this.

See, what we want is to use that 0 to 1 region to create a passband, and the beyond 1 region to create the transition band and the stopband. So, let us do the following as we did for the Butterworth case. What we want is up to between 0 and 1, this Chebyshev polynomial is going to oscillate or alternate.

So, it is going to be equiripple; and it is not going to go beyond a certain range, depending on what you might see. You can always multiply this Chebyshev polynomial by a constant by a tolerance to keep it within a certain range. And beyond x equal to 1 of course, you can allow it to increase monotonically; so that you go into the transition band and the stopband.

The only catch is the Chebyshev polynomial itself is increasing, but you want the resultant expression to decrease; so you can put it in the denominator. But, if you put in the denominator, you do not want to create a situation; at 0, you want to take the value 1. So, we use the same strategy as we did for the Butterworth filter.

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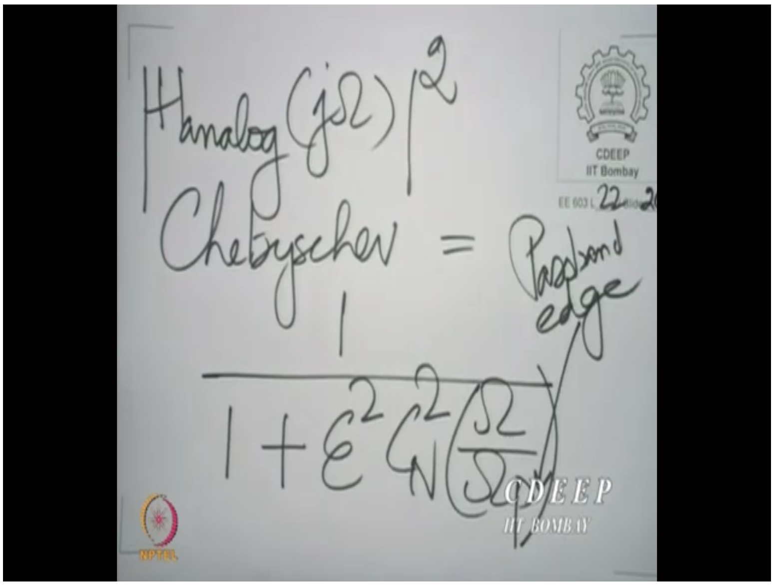

We create the expression $|H_{analog}(j\Omega)|^2$ for the Chebyshev case is $= \frac{1}{1+\epsilon^2 {C_N}^2(\frac{\Omega}{\Omega_p})}$. Now, why are we doing this? For Ω between Ωp by the way is the passband edge. Why are we doing this? For Ω between 0 and Ωp, this argument is between 0 and 1. And therefore you are employing the equiripple part of the Chebyshev response. For $\Omega > \Omega p$, this quantity becomes greater than 1; so you are employing the monotonic part of the Chebyshev polynomial.

And of course, for $\Omega > \Omega p$, this quantity would monotonically increase; and therefore the denominator monotonically increases leading to a monotonic decrease of this expression. And therefore, we have achieved what we wanted. An equiripple behavior in the passband and a monotonically decreasing behavior starting from the edge of the passband; all through the transition band and then down into the stopband. Of course the quantity $\mathcal{E}^2$ allows you the passband tolerance; so we say $\mathcal{E}^2$ because $\mathcal{E}$ is assumed to be real. And therefore we take a square of a real number to ensure it is positive. That is what we really mean by saying $\mathcal{E}^2$.

So, therefore let us now write down the two requirements; this is this is the magnitude squared expression for a Chebyshev filter. Let us write down the two requirements on; of course here what is to be designed. If you want to design a Chebyshev filter, so designing a Chebyshev filter.

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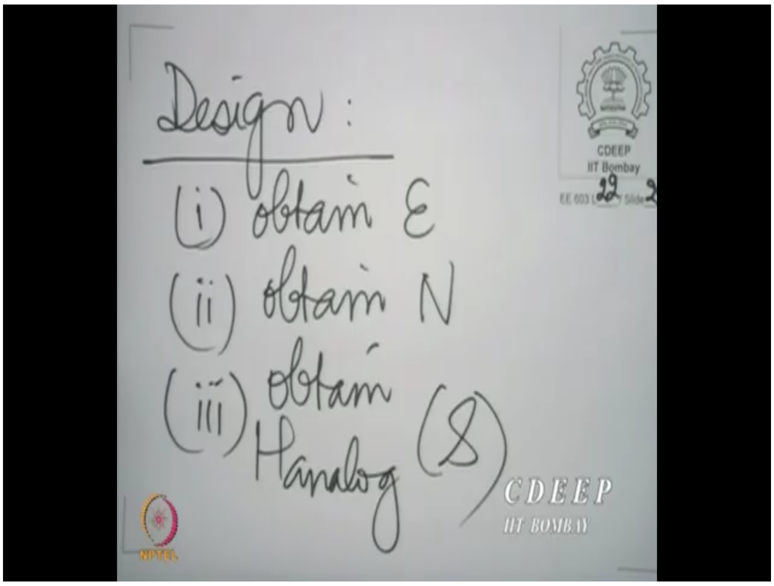

Design means the following: obtain Ɛ, obtain N, and then obtain therefore obtain $H_{analog}(S)$ And how would you obtain $H_{analog}(S)$? You would have to identify the poles and then segregate those poles that correspond to $H_{analog}(S)$, by removing the poles that correspond to $H_{analog}(-S)$. So, let us obtain Ɛ first, Ɛ is very easy.

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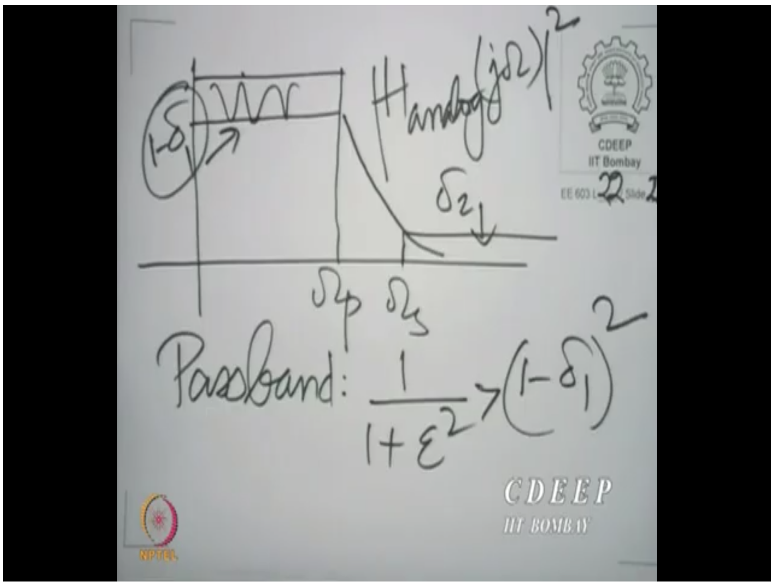

You see, it is very clear that there would be an oscillatory behavior in the passband. I am plotting $|H_{analog}(j\Omega)|^2$, and then from the edge of the passband; and downwards there would be a monotonic behavior. And the oscillatory behavior essentially for how much is the oscillation, how.

Well, when $C_N = 0$, of course you reach this point; and when $C_N = 1$; that is the maximum value it can take. When the cosine takes the value of 1; you reach this point. And therefore the passband constraint, you want this to be the tolerance here, this should not go below $1 - \delta_1$.

And this should not go above δ_2 that is what you want.

So, passband requires that $\frac{1}{1+\varepsilon^2} > (1 - \delta_1)^2$. Remember you are talking about the squared magnitude function; and therefore we have a constraint on ϵ. If you look at it this does not involve N at all that is interesting; the passband has very little to do with N, seems so.

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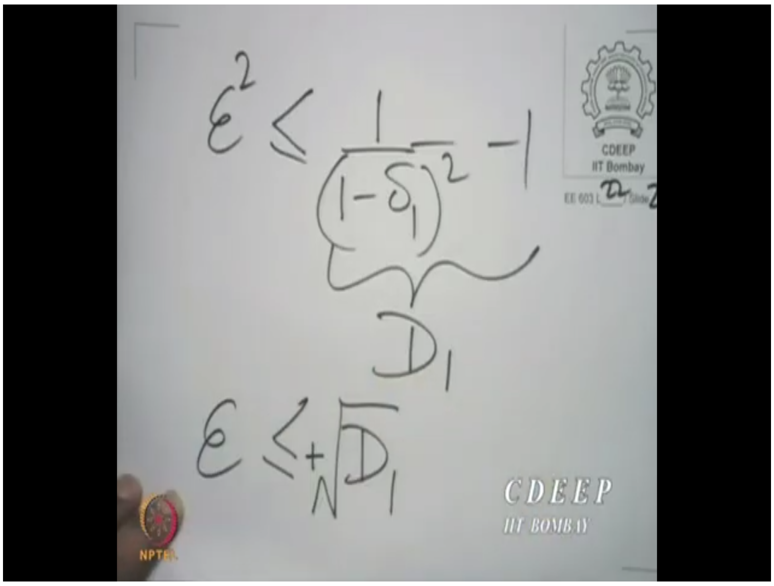

So, let us write down $\varepsilon^2 \leq \frac{1}{(1-\delta_1)^2} - 1$ And recall that this is the D_1 that we saw in the Butterworth filter; $\varepsilon \leq +\sqrt{D_1}$. Now, you see it looks like you have a range for ε ; I mean in principle you could even choose $\varepsilon = 0$. So, nothing comes for free.

As I said it looks as if N has nothing to do with the passband; but it does indirectly. If you choose ε smaller, you will see that you will actually be pushing for a larger N. So, now let us of course we need to obey this constraint; $\varepsilon \leq +\sqrt{D_1}$. But, it is in our interest to choose the largest possible ε , which is essentially $\varepsilon = \sqrt{D_1}$.we will see that in a minute.

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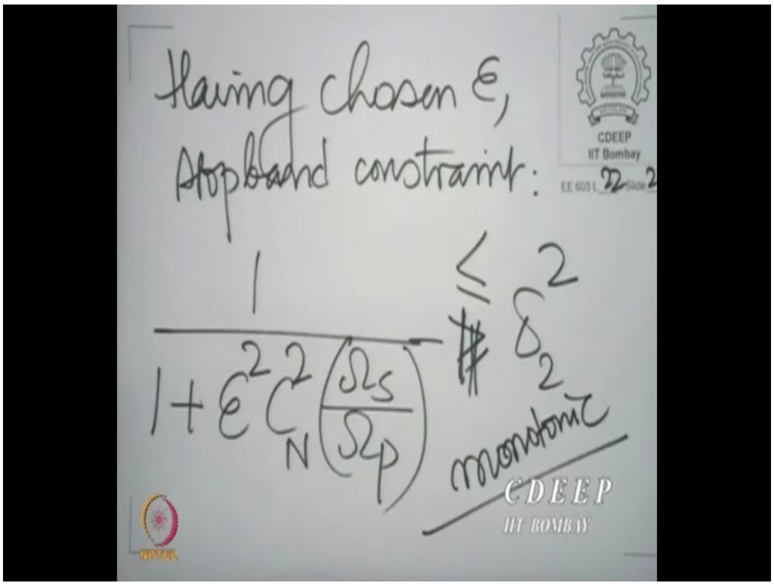

So, having chosen the ε stopband constraint says $\frac{1}{1+\varepsilon^2 C_N^2(\frac{\Omega_s}{\Omega_p})}$ ≥ I am sorry is ≤ δ_2^2; it must be within the stopband. So, at the edge of the stop, you see because because of monotonicity. So, if the requirement is obeyed at the edge of the stopband; it is obeyed all over the stopband. If you have ensured that at the edge of the stopband, you have reached the value ≤ δ_2^2; you are bound to remain within δ_2^2 all over the stopband.

So, you need to check only at the edge of the stopband. Now, we can solve this very easily; but here remember we have $(\frac{\Omega_s}{\Omega_p}) > 1$. And therefore we must make use of the cos hyperbolic or cosh expression and there we have.

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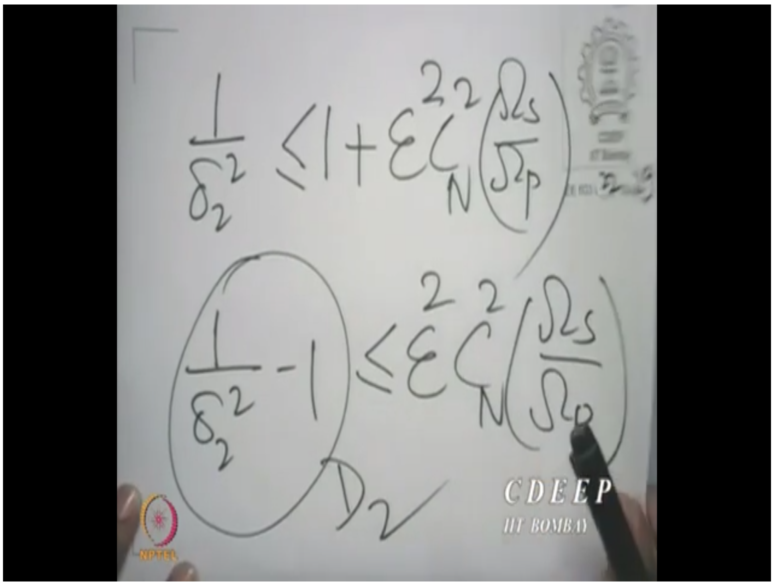

$\frac{1}{\delta_2^2} \le 1+ \varepsilon^2 C_N^2(\frac{\Omega_s}{\Omega_p})$ Or, in other words $\frac{1}{\delta_2^2} - 1 \le \varepsilon^2 C_N^2(\frac{\Omega_s}{\Omega_p})$; and there again we land up with the familiar D_2 here, D_2 from the Butterworth filter. So, remember it is very interesting that the same quantities occur both in the Butterworth filter and the Chebyshev filter. Now, in place of C_N, we must make use of the hyperbolic cosine expression.

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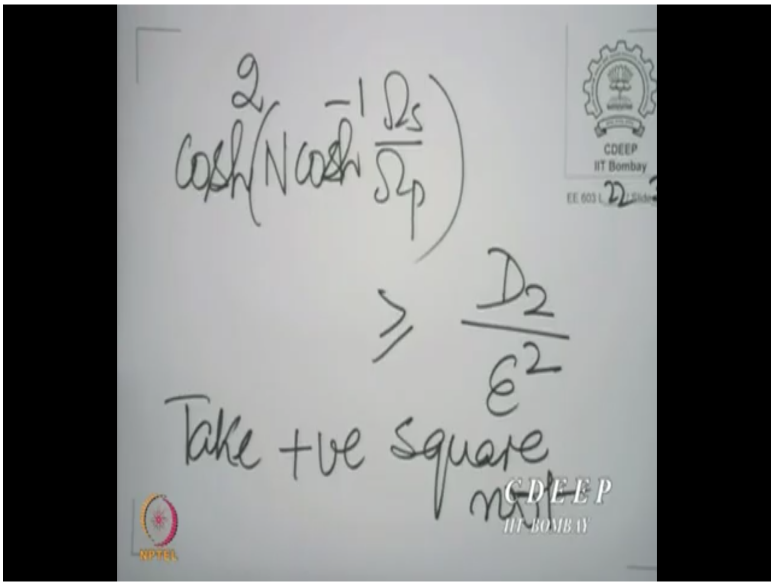

And therefore, we have $\cosh^2 (N \cosh^{-1} \frac{\Omega_s}{\Omega_p}) \ge \frac{D_2}{\varepsilon^2}$. And here we can take the positive square root on both sides. If we take the positive square root, square root is a monotonically increasing operation. And you have if you have positive quantities on both sides. So, if you take the square root on both sides, the inequality is preserved; but, you must remember to take the positive square root.

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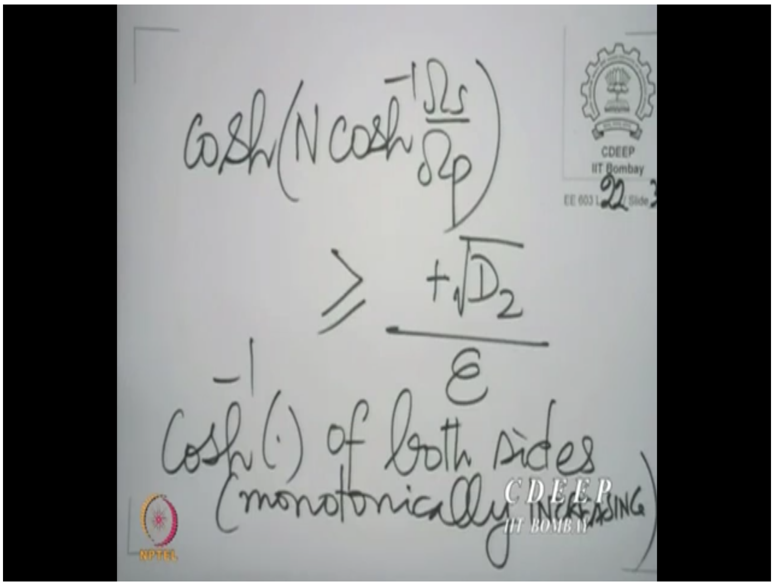

That gives us cosh (N $\cosh^{-1} \frac{\Omega_s}{\Omega_p}) \geq \frac{+\sqrt{D_2}}{\varepsilon}$ And of course, $\cosh^{-1}$ (.) of both sides; it is valid to take $\cosh^{-1}$ of both sides. Because $\cosh^{-1}$ and cosh are both monotonically increasing functions. When is it valid to take a function of both sides of an inequality? It is valid if the function is monotonically increasing. If a > b, then fa is > fb; if f is a monotonically increasing function of its argument.

If it is a monotonically decreasing function of the argument; then the inequality is reversed. If the function is neither monotonically increasing nor monotonically decreasing; then it is invalid to take the function of both sides of an inequality that is the beauty. You can operate a function on both sides of an equality without any concern; but when there is an inequality, you have to be worried about whether the function is monotonic or not.

If it is not monotonic, then the inequality can either be destroyed entirely or reversed or preserved. So, here of course $\cosh^{-1}$ cosh itself is a monotonically increasing function. So, its inverse also is going to be a monotonically increasing function. So, since it is a monotonically increasing function, we can take $\cosh^{-1}$ on both sides, and that gives us.

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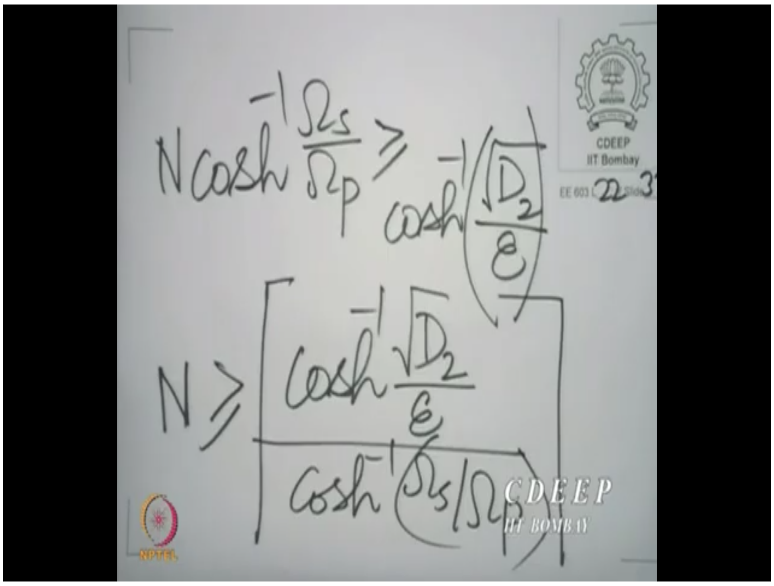

$N \cosh^{-1} \frac{\Omega_s}{\Omega_p} \geq \cosh^{-1}(\frac{\sqrt{D_2}}{\varepsilon})$. $N \geq [\frac{\cosh^{-1} \frac{\sqrt{D_2}}{\varepsilon}}{\cosh^{-1}(\frac{\Omega_s}{\Omega_p})}]$. As expected there is a minimum value of the order; and that minimum value is given by this expression. And of course you must put here a ceiling; because N cannot be non-integral. So, you need to put there the integer just above that quantity. If this quantity works out to be 7.3, the ceiling is 8 and so on. Now, of course naturally you would choose N equal to the ceiling; you do not want to invest more resources than required.

But, this is what requires to be obeyed by the order. And now it is also very clear why ε indirectly plays a role in the order. You see, $\cosh^{-1}$ is a monotonically increasing function of its argument; the smaller you choose ε , the larger this argument; and the larger the $\cosh^{-1}$, and therefore the larger the requirement on the order. So, it is in our interest to choose as large an ε as we can; and that is $\sqrt{D_1}$. So, let us make a remark.

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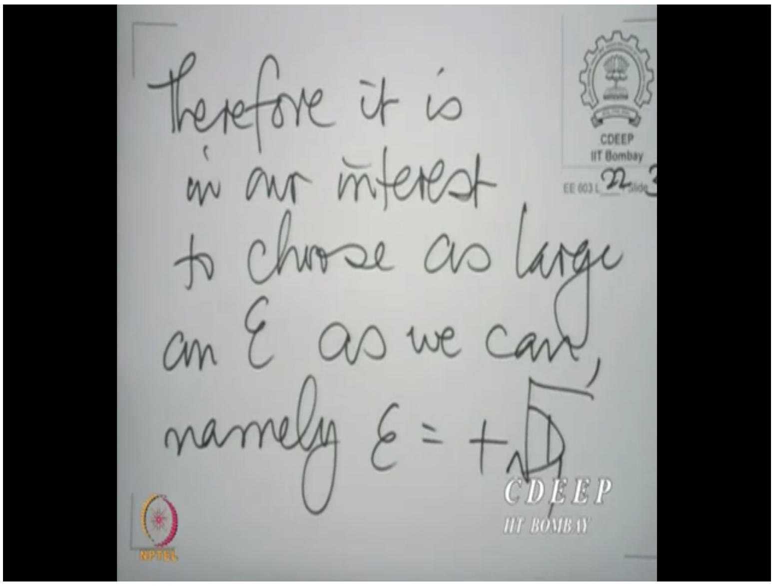

Therefore, it is in our interest to choose as large an ε as we can, namely ε = $+\sqrt{D_1}$. And having made that choice, then of course we have a simple expression for the order.

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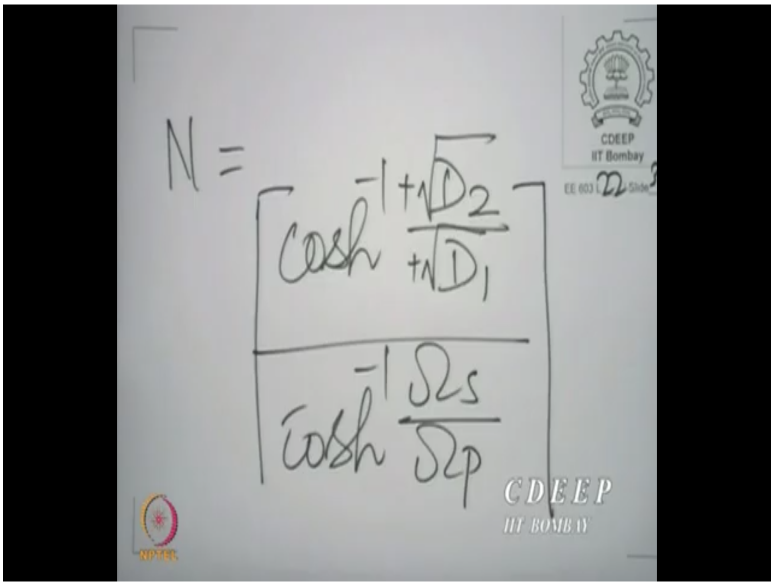

And that order is N = $[\frac{cosh^{-1}\frac{\sqrt{D_2}}{\sqrt{D_1}}}{cosh-1\,\frac{\Omega_s}{\Omega_p}}]$. So, much so then for the design of ε and N; we shall proceed in the next lecture to remark on the behavior of this choice as we change δ_1 and δ_2 and what happens once we have made this choice of N, as far as the choice of system function is concerned. So, we will do that in the next lecture.