Digital Signal Processing & Its Applications
Professor Vikram M. Gadre
Department of Electrical Engineering
Indian Institute of Technology, Bombay
Lecture – 22A
Review of Butterworth Filter design and Formulation of its System Function

So, warm welcome to the 22nd lecture on the subject of Digital Signal Processing and its Applications. We briefly recapitulate what we did in the previous lecture, and then go on the details of the current one. We have been discussing the whole theme of discrete time filter design for the past few lectures.

So, far we have come to the following point; we have agreed that we cannot meet the ideal specifications ever, for the piecewise constant, standard piecewise constant filters. We have also identified that we need to relax the ideal specifications by introducing tolerances and a transition band. We have also agreed subsequently that having made these relaxations, a realization is always possible. And the realization is possible either with infinite impulse response filters or with finite impulse response filters; both are possible.

We first decided to look at infinite impulse response filter design and that is because it takes a queue from standard analog designs. So, we can take advantage of the standard analog designs that are available in the literature, to design discrete time filters by an analog to discrete time transformation. And we identified what we call the bilinear transformation.

Subsequently, we said that having made a transformation from a discrete time filter to a corresponding analog filter that has to be designed. One needs to then convert from an arbitrary kind of analog filter; for example, a non low pass filter to a low pass filter. So, there is a transformation in the analog domain itself, which we will study later.

So, we will first look at low pass filter design and then we look at how we can generalize to other kinds of filters in the analog domain. And therefore, we were looking at the design of low pass analog filters, and there again we agreed there are four possibilities; depending on whether the pass band and the stop band are monotonic or non-monotonic.

And we said if it is non-monotonic, then the best choice is equilibrium. So, yesterday we looked at the case of monotonic stop band and monotonic pass band; and that corresponds to what is called the Butterworth filter design. And the Butterworth filter design, we have identified how to design the two parameters that characterized the Butterworth filter. The two parameters that characterized the Butterworth filter are as follows.

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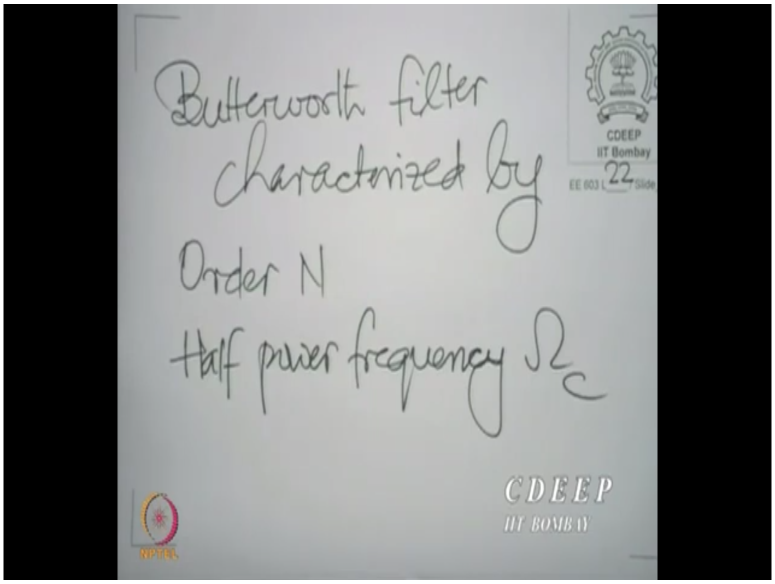

The parameter called the order N, and what is called the half power frequency or Ω_C . We have also written down the equations for obtaining the order and the half power frequency. Both of them have a range associated; so the order has a minimum that is required. And of course, you can take any order more than the minimum; but the most appropriate choice is to take the integer just above the quantity that we get. You see because the order corresponds to the use of resources; the more the order, the more the resources that you need to employ. And therefore, you would like to use as small in order as possible; but, sufficient to meet the specifications.

And therefore, you choose the ceiling or the integer just above a quantity that we calculated yesterday. Having chosen the order we can then choose the half path frequency again in a range; and that range comes again because of the operation of the ceiling. Now, having chosen the half power frequency, we then have the complete analog system function; so, the analog system, the analog magnitude function more appropriately.

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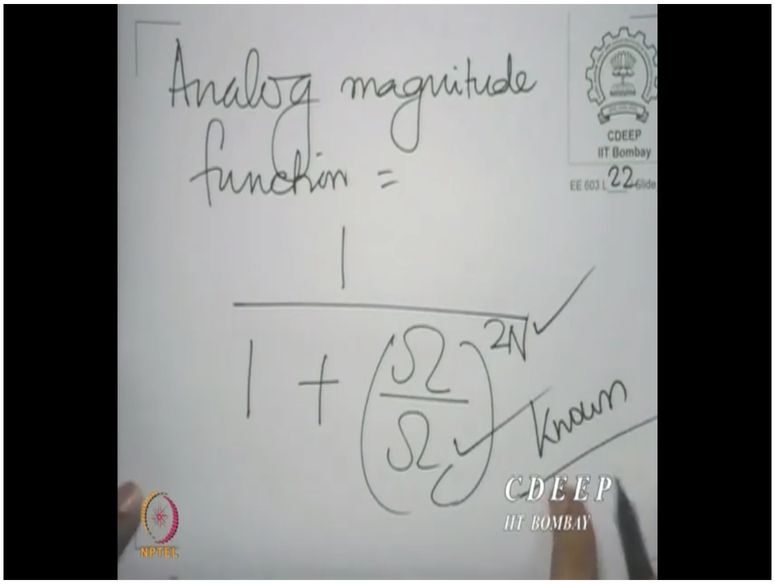

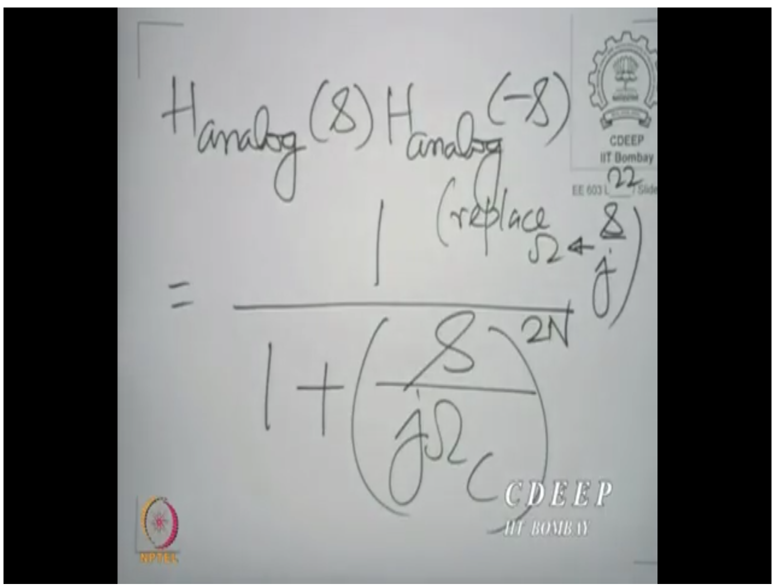

So, the analog magnitude function is $1/(1 + (\Omega/\Omega_C)^{2N})$. And this N is known and Ω_C is known; so the complete analog magnitude function is known. Now, we need to complete the design by identifying $H_{analog}(s)$; so once we know N and Ω_C. We know $H_{analog}(s)$ into $H_{analog}(-s)$, which is $1/(1 + (s/j\Omega_C)$. How do you get the analog system function? You get it by replacing Ω by S by j; you see s is equal to j Ω on the imaginary axis. When we wanted to find the frequency response, we replaced s by j Ω.

And therefore, to go back to the system function, we need to replace Ω by s by j. Now, of course this is acceptable if the function can be continued all over the complex plane as we have done here. In other words, you know the function as restricted to the imaginary axis, and you are assuming that it can be subjected to what is called analytic continuation. That means you may continue that function all over the complex plane with the same expression. If that is acceptable, then this replacement is valid. Now, here this replacement is valid, because this continuation is acceptable.

It is an analog, it is a analytic expression in the analog domain; and therefore it can be continued all over the complex plane. Anyway, once we have this, now it is very easy to identify the poles and zeros of this analog system function. All that we need to do, of course we can see very clearly there are no zeros at all. And the poles can be obtained by equating the denominator to 0.

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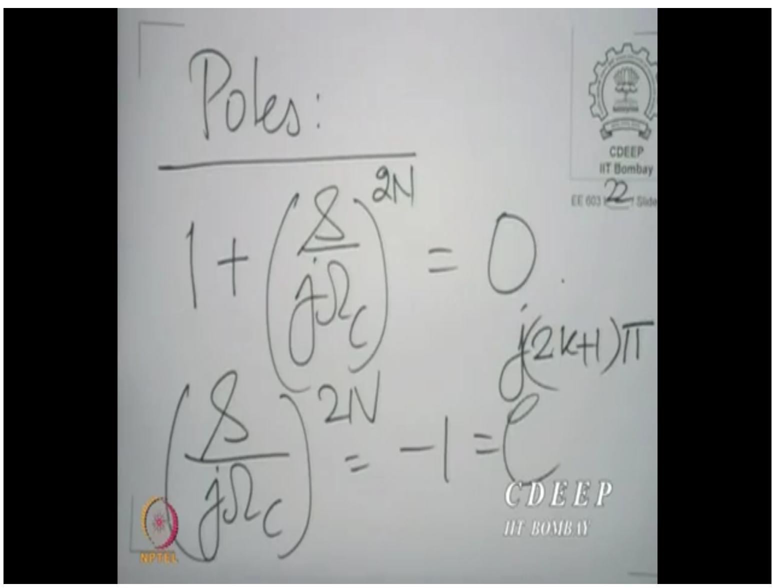

And that is very easy to solve. $(s/j\Omega_C)^{2N} = -1$; and you see you need to take the 2Nth root of both sides. So, you need to write minus 1 in a way in which you can identify all the 2Nth roots. And therefore, you should write $-1 = e^{j(2k+1)\pi}$, an odd multiple of π,. Any odd multiple of π in the argument creates minus 1.

Now, you need to write this because you need to identify that when we take the 2Nth root on both sides; you are going to have 2N possible solutions. And to identify these distinct solutions, you need to identify the multiplicity in the phase of minus 1; so, now let us solve this equation.

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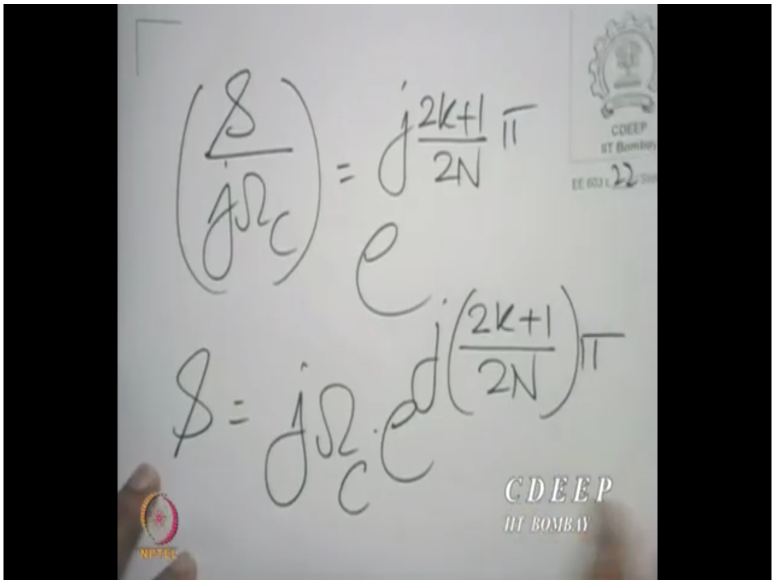

So, therefore, $s/j\Omega_C = e^{j(2k+1)/2N^*\pi}$ simple. And therefore,$s = j\Omega_C e^{j(2k+1)/2N^*\pi}$; now you can see the angle, it is (2k + 1) /2N * π. So, of course you could write this in the form in its polar form. Clearly, this contributes this and this contribute the magnitude of 1; and this really contributes the magnitude, and therefore the polar form.

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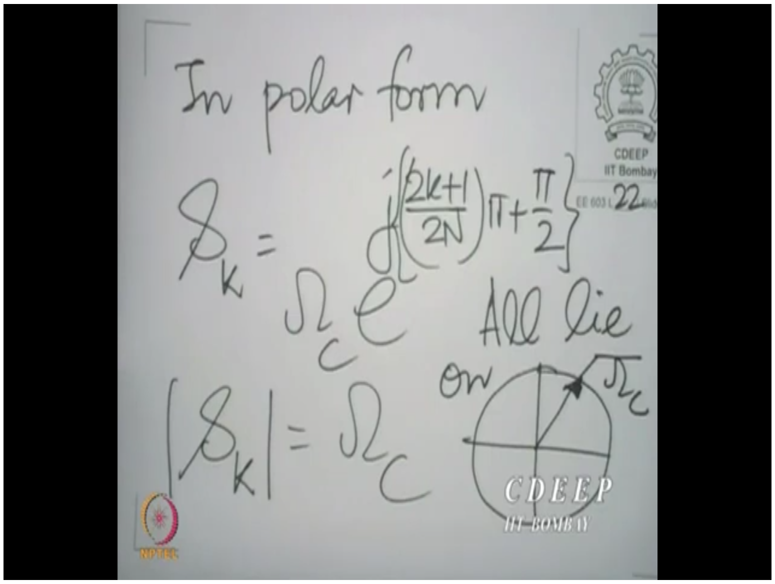

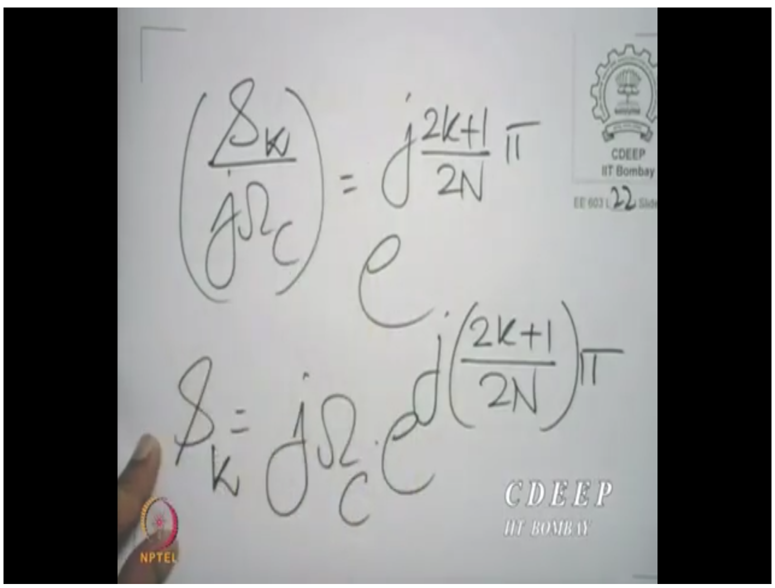

So, in fact we should we should identify there are roots indexed by k. So, we should write s_k here; there are the roots are indexed by the integer k. $s_k = j\Omega_C e^{j(2k+1)\pi/2N}$, plus $\pi/2$. The $+\ \pi/2$comes from this term j here; this adds the phase of $\pi/2$; so, this is the magnitude and the phase.

So, it is very clear that all the s_k have a magnitude of Ω_C; and therefore all these roots lie on a circle with radius Ω_C. A circle centered at the origin with radius Ω_C in the complex plane. In fact, just to get a feel of how the angles change; of course, they all have the same magnitude.

But, just to get a feel of how the angles change; let us enumerate the various angles for the case of N odd and N even. So, let us take the case of N equal to 3 and N equal to 4; that will enable us to see how these angles are distributed.

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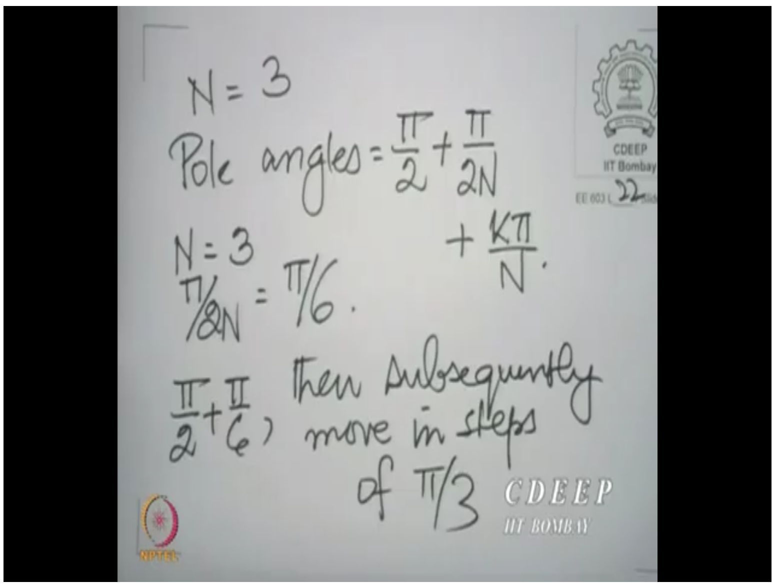

So, for the case n equal to 3, let us write down the angles, pole angles. The pole angles are of course given by π by 2 plus π by 2N, plus k π by N. And if N is equal to 3, then π by 2N is π by 6. So, you can of course you see all that you need to do is to identify that there are 2N distinct poles here.

So, if you run the integer k over 2N consecutive values; you will be covering all the poles. So, you could without any loss of generality start from k equal to 0; and in this case it could run from k equal to 0 to k equal to five. So, in general you could have started with k equal to 1 it does not matter.

But, you need to run over consecutive 2N values; so let us begin with k equal to 0. You have πby 2 plus πby 6; then you have and then subsequently move in steps of πby 3; that is what this says.

Each time you increment k by 1, you are adding an angle of πby 3. So, let us sketch these poles on the complex plane; this is the complex plane.

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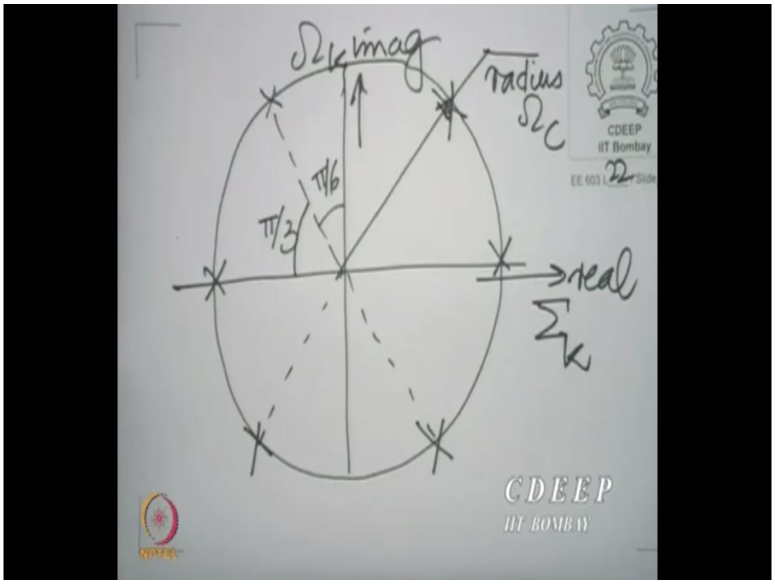

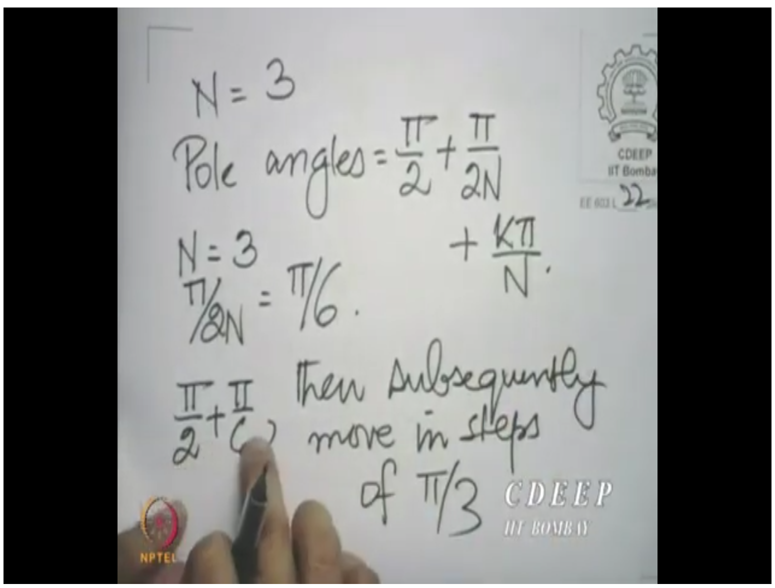

This is the radius Ω_C, this is the first pole located; this is the real axis and this is the imaginary axis. So, we could call this sigma k and capital Ω_K to denote the real part of the pole and the imaginary part of the pole. So, the first angle as you see is πby 2 plus πby 6; so π by 2 and then another angle of 30 degrees here. So, here we are, this is the first pole; this angle is πby 6. Subsequently, we move in steps of πby 3; so when you move in the first step of πby 3, you would reach here. Move the second step of π by 3 and you would take the complex conjugate of this.

Move in one more step of π by 3 and you reach here; and then of course you have one more there. So, as you can see there are six poles and they are located at angles 0-π, π by 3 to 2 π by 3. And you of course calculate the remaining ones. Now, it is very interesting and quite satisfying to see that the poles are symmetric with respect to the real axis; and also the imaginary axis. The symmetry with respect to the real axis is to be expected. That is because you want the poles to occur in complex conjugate pairs; you want the coefficients of $H_{analog}(s)$ to be real.

And therefore, the poles must occur in complex conjugate pairs; and they do occur indeed in complex conjugate pairs. Here of course we do not need a complex conjugate; wherever there are real poles, they occur singly. And wherever there are complex poles, they occur in conjugate pairs; that is to be expected.

Now, in addition there is a symmetry with respect to the imaginary axis. Now, why do we have a symmetry? In fact, the symmetry more than the imaginary axis is with respect to the point 0 or the origin. Now, why is that symmetry there? Because you have already created $H_{analog}(s)$ into $H_{analog}(-\ s)$.

So, for every pole at S there is a pole at minus S; and therefore there is a symmetry with respect to the origin. And that symmetry reflects as a symmetry with respect to real axis and a symmetry with respect to the imaginary axis. Now, of course as we expect there are no poles on the imaginary axis; if there were a pole on the imaginary axis, stability would be affected. The filter could never be stable, just as the discrete system. If there is a pole on the unit circle, the filter cannot possibly be stable; and therefore, here too there is no pole on the imaginary axis.

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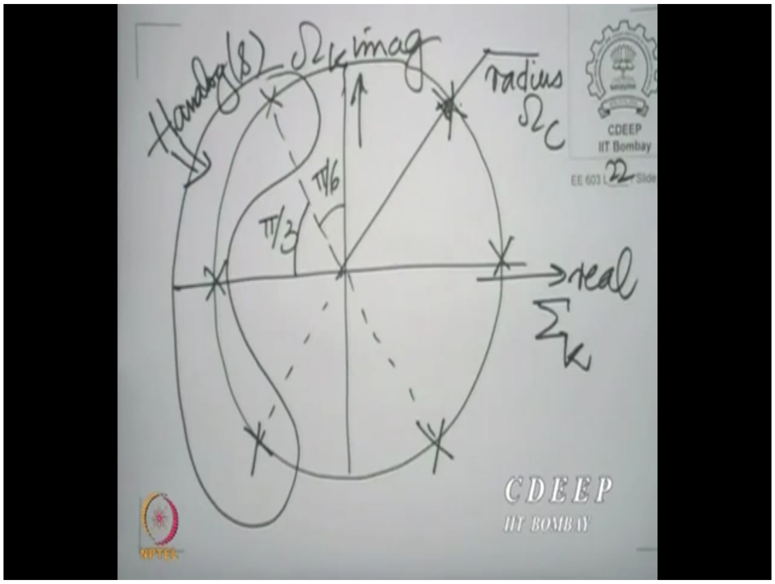

And all that we need to do now to identify $H_{analog}(s)$ is to take away the poles in the left half plane. So, these poles, these three poles give us the poles corresponding to $H_{analog}(s)$; and therefore of course naturally these three poles are the poles corresponding to $H_{analog}(-s)$. Now, you see let us write down therefore the system function corresponding to $H_{analog}(s)$.

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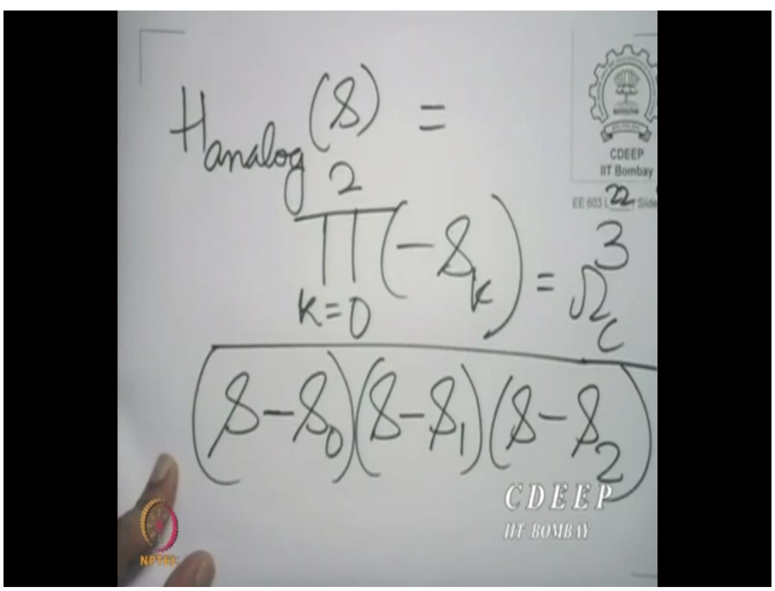

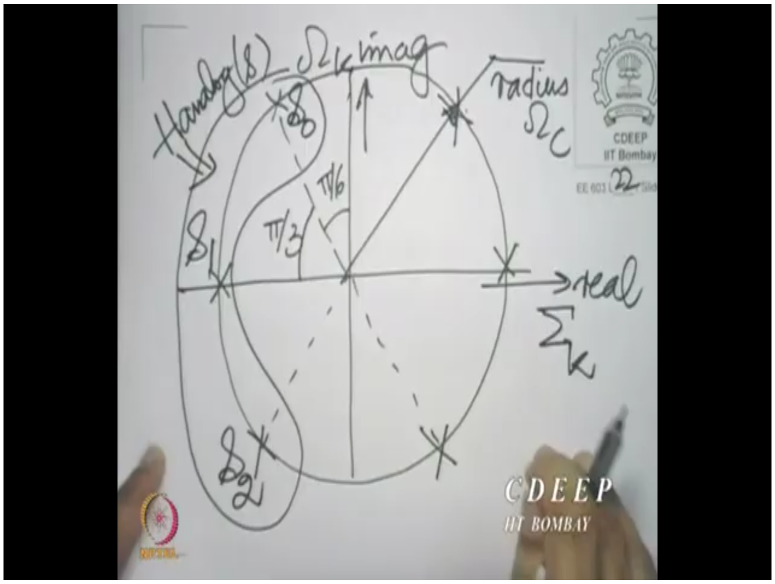

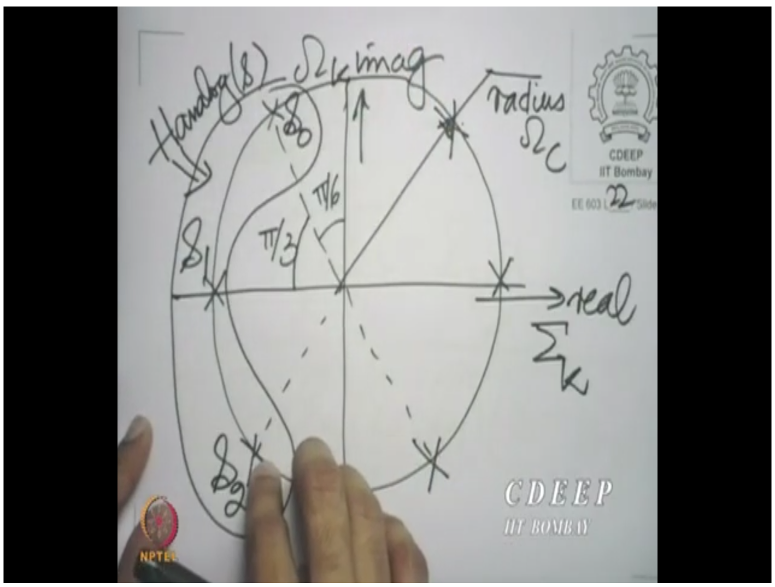

In fact, let us call these poles since we index them with k, this is s_0, this iss_1, this iss_2. So, $H_{analog}(s)$ is equal to something divided by $(s - s_0)(s - s_1)(s - s_2)$. Now, what must be come in the numerator? Of course, there are no zeros; so you need a constant; you need to put a constant.

Now, what constant need you to put? You see the constant should be such that when you put $s = j\Omega$; and when $\Omega = 0$. In other words, when you put $s = 0$, this should evaluate to 1. Now, when you put $s = 0$ what you have here is the product; we write product like this, product k going from 0 to 2 of minus s_k.

Because, we want this to evaluate to 1 at $s = 0$; and it is very easy to identify this product. You see if you look at it, $-s_1$ is of course Ω_C; $-s_0$ into $-s_2$ is $|s_0|^2$. Because they are complex conjugates; and therefore that is Ω_C the whole square. And therefore, this product is essentially equal to Ω_C^3, very simple. In fact, I leave it to you as an exercise to work out the corresponding set of poles for capital N equal to 4. I shall straightway draw them; I leave it to you as an exercise to work out these poles.

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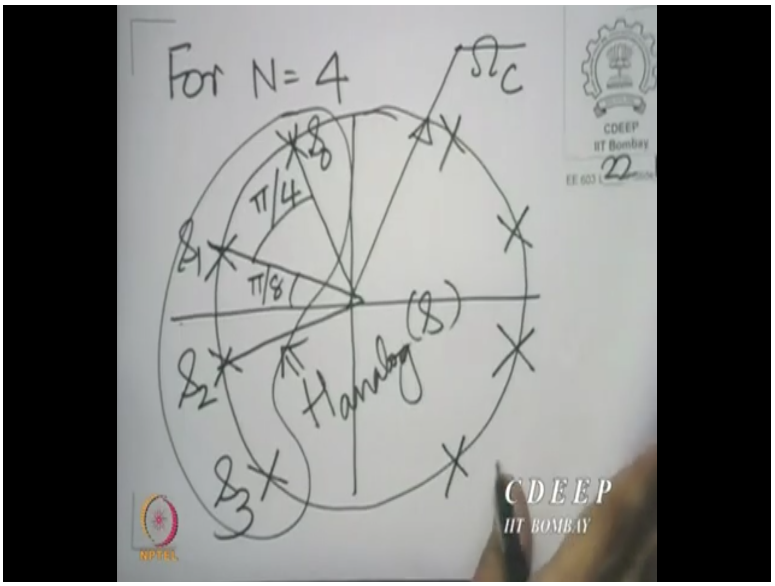

In this case you would have two poles in each quadrant, and they would have a separation of pi by 4, which is 45 degrees. So, this I will just mark some of them, this is pi by 8, this angle is pi by 8. And this angle could be pi by 4; in fact all the poles will be separated by an angle of pi by 4; and of course you can draw the others. So, this together would amount to H analog S and here too you can number the poles. So, you can call this S0, then S1 and so on; and therefore we can now write down H analog of S. That essentially takes these 4 pole; because they are in the left half plane.

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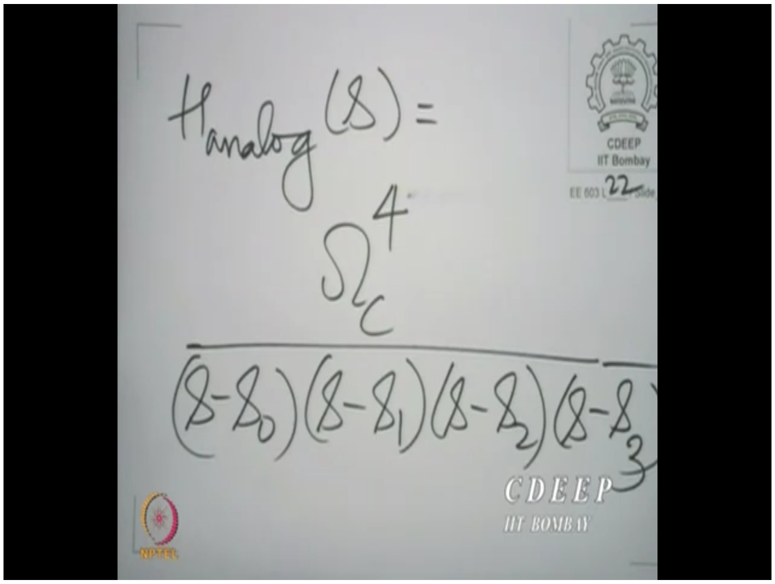

So, $H_{analog}(s) = \Omega_c^4/(s - s_0)(s - s_1)(s - s_2)(s - s_3)$. I leave it to you as an exercise to complete the details of this. Now, having written down $H_{analog}(s)$, our job is almost done; the next step of course is to replace S by $1 - z^{-1}/1 + z^{-1}$.

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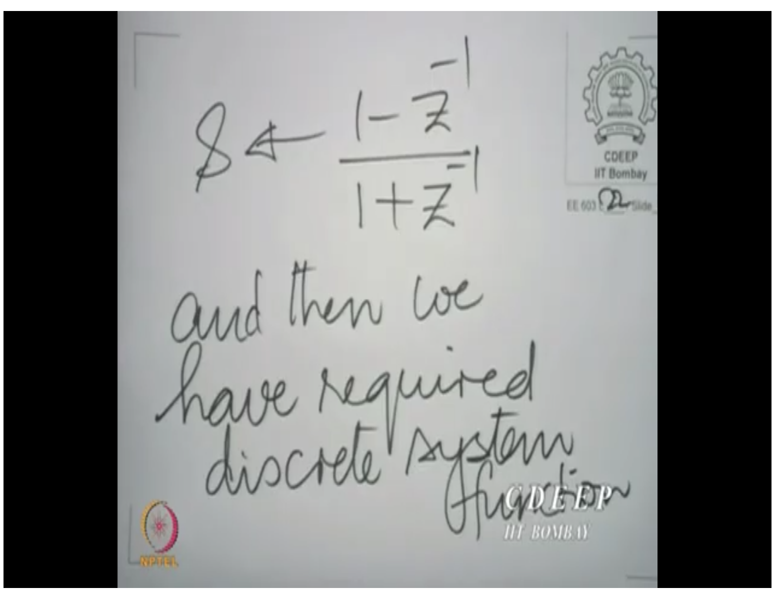

And then we have the discrete system function; and that completes the Butterworth design. So, much so for the design of Butterworth low pass filters.