Digital Signal Processing and Its Applications
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Lecture No. 21b
Analog Filter Design using Butterworth Approximation

So, we will first look at the Butterworth possibility now. We will, we will look at the design of analog low pass filters using the Butterworth Approximation.

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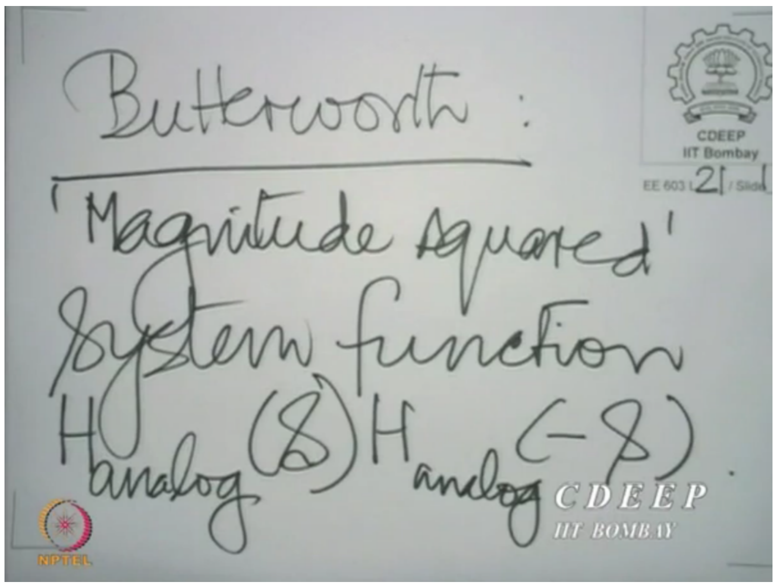

Now, you see in the Butterworth Approximation, you want the passband and the stopband both to be monotonic. So obviously if you want the passband to be monotonic and stopband to be monotonic the only sensible way of doing it is to let the magnitude decrease starting from zero frequency all the way down to the edge of the passband and then again starting from the edge of the passband all way down to infinity.

That seems to be the most sensible way to do it. Which means you need, you see now that is another interesting thing. Since we are dealing only with magnitudes, we shall not design the system function of the analog filter directly. What we will design instead is what is called the magnitude complex system function.

So, in general we introduce the idea of what is called a magnitude squared system function. And a magnitude squared system function is essentially the analog system function or the filter analog system which we will call $H_{analog}(s)H_{analog}(-s)$ and we will explain why that is the case. You see, you want to take the $\left|H_{analog}(j\Omega)\right|^2$. Is not it?

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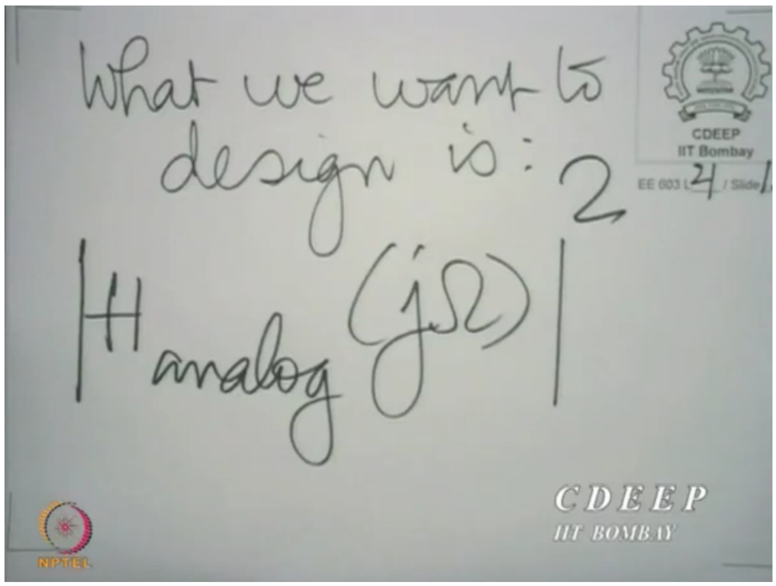

What you want is, what we want to design is $\left|H_{analog}(j\Omega)\right|^2$. Well, I mean you could say you want to design $\left|H_{analog}(j\Omega)\right|$ that is alright. But anyways it is much easier to do square as we see in a minute. They are equivalent because it is all non-negative.

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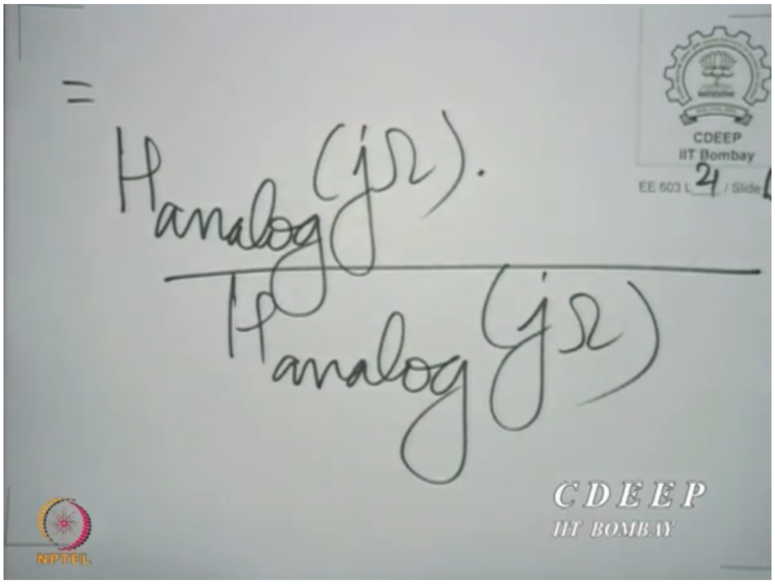

So, you see $\left|H_{analog}(j\Omega)\right|^2$ can also be re-written as $H_{analog}(j\Omega) * \overline{H_{analog}(j\Omega)}$. But remember we are going to have real coefficients in $H_{analog}(s)$. They are all going to be real coefficients in S.

So, when we take the complex conjugate, the real coefficients remain unchanged, but S gets complex conjugated. So, we could have written S conjugate or you see since we are talking about j Ω, the complex conjugate of j Ω is -j Ω. So, even if you write -s there, we would get the complex conjugate.

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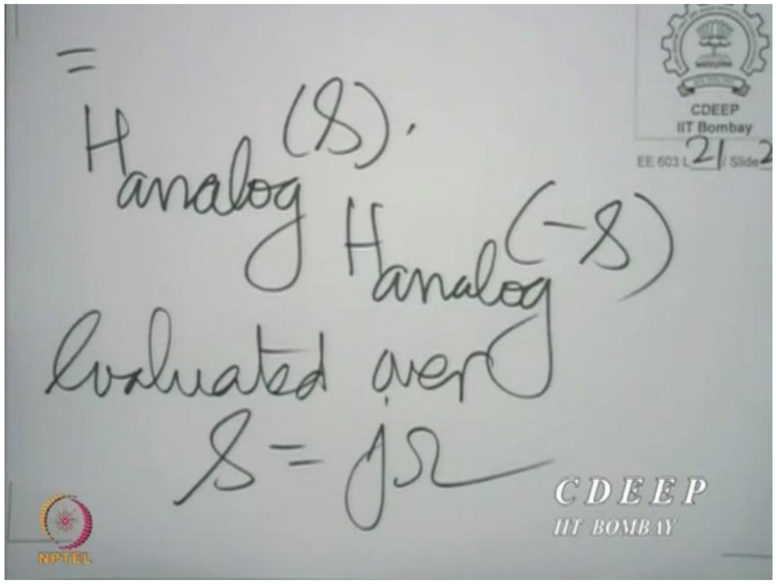

So, in fact, what we are saying is, we want $H_{analog}(s)$ into $H_{analog}(-s)$ evaluated over $s = j\Omega$. That is because taking the complex conjugate amount to replacing $1 - j\Omega$. Now you may wonder, why we did not write $\overline{H_{analog}(s)}$. You see, what we want to ultimately design is what is called an analytical function of S.

Now, if we use a complex conjugate on s, then what we get is a non-analytical function. The moment you complex conjugation does not allow for derivatives in the first place forget about in the higher order differentiation. So, we cannot use s complex conjugate but we can use -s. -s will retain the analyticity and therefore, we choose -s over s conjugate. Anyway, so we have chosen to design $H_{analog}(s)$into $H_{analog}(-s)$.

And of course, what we have to do is design this compound system and then we have to identify which is its zeros and poles correspond to $H_{analog}(s)$ and which of its zeros and poles correspond to $H_{analog}(-s)$. But that is not a difficult job at all. We know that we want the filter to be stable and the analog domain is plane if stability is the case. In fact, not just stability, we want to be positive real. And positive realness means all the zeros and all the poles must be on the left half side of the S plane.

So, once we have the collection of zeros and poles corresponding to this product, H analog S into $H_{analog}(-s)$, it is very easy to identity what is $H_{analog}(s)$. Take all the zeros and the poles in the left half of the plane and put them into H analog. Automatically all the -s go into $H_{analog}(-s)$. That is easy.

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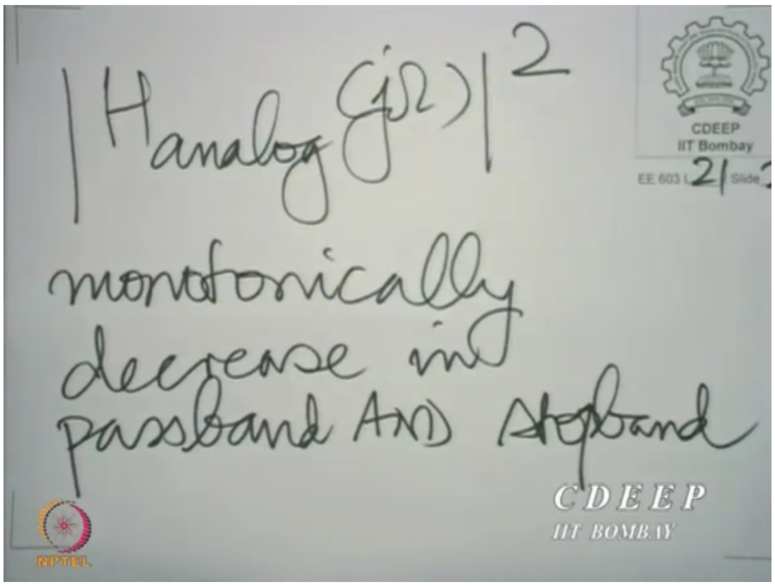

So, you see, if you want a monotonic response for – you see you want $\left|H_{analog}(j\Omega)\right|$ to monotonically decrease in passband and stopband. So it must have its maximum at $\Omega = 0$ and it must steadily decrease as you move from zero to infinity. That is that is the kind of response that we want.

Now, of course, the response at zero should not go up without bar. It must be finite. In fact, it is most convenient to make it 1 and then we want the response to die. So, what can be envisaged as a function of Ω? Now remember, we are taking $H_{analog}(j\Omega)$ into $H_{analog}(-j\Omega)$. So, therefore, the product is - a magnitude squared function is necessarily going to be a real function of Ω.

So, I need to look for a real function of Ω and the simplest real function of Ω, which monotonically decreases as you go from zero to infinity essentially a function where the denominator monotonically increases and you want it to be rational so denominator must of course be rational function in fact, why rational?

The simplest thing to do is to choose the denominator to be a polynomial. And what simpler polynomial can you have; the polynomial which has a constant term and other term which increases monotonically with Ω.

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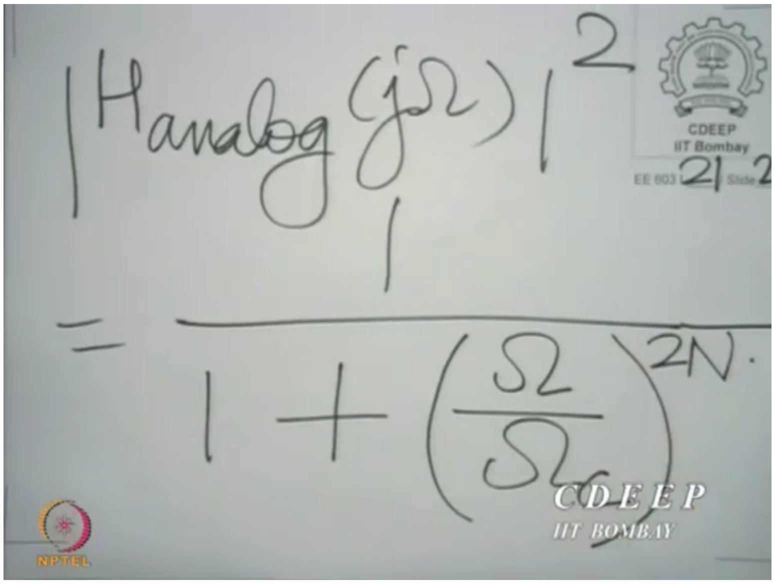

In other words, what we are envisaging is a magnitude squared function that looks like this. 1 by 1 plus, you want it to be non-negative. So, you have Ω by same parameter, positive parameter, the whole to the power 2N. Now, let us understand this. You see, why have we chosen this? As we anticipated, at $\Omega = 0$ this is 1 and therefore, the response pars at 1.

As Ω goes from zero to infinity, this monotonous, the denominator monotonically decreases from zero to infinity and therefore, this function monotonically decreases from 1 towards zero. And further this Ω_C, why have we introduced Ω_C and why have we introduced N? You see N controls the rate at which the response drops.

And $\Omega_C$controls the point at which the response reaches a certain value. For example, $\Omega = \Omega_C$, this is 1 and therefore, this magnitude squared reaches what is called the half power point. The magnitude squared becomes half and therefore, that is called the half power point if you were to put in a sinusoid in the unlock domain into the filter and that sinusoid will come out with half power at $\Omega = \Omega_C$.

So, point of half power changes as you change Ω_C. And how fast the response drops after Ω_C is controlled by N. The larger the N, the faster it drops. Now, you see, just these 2 parameters, Ω_c and N. We shall be able to meet any magnitude specifications that we choose. Let us see how.

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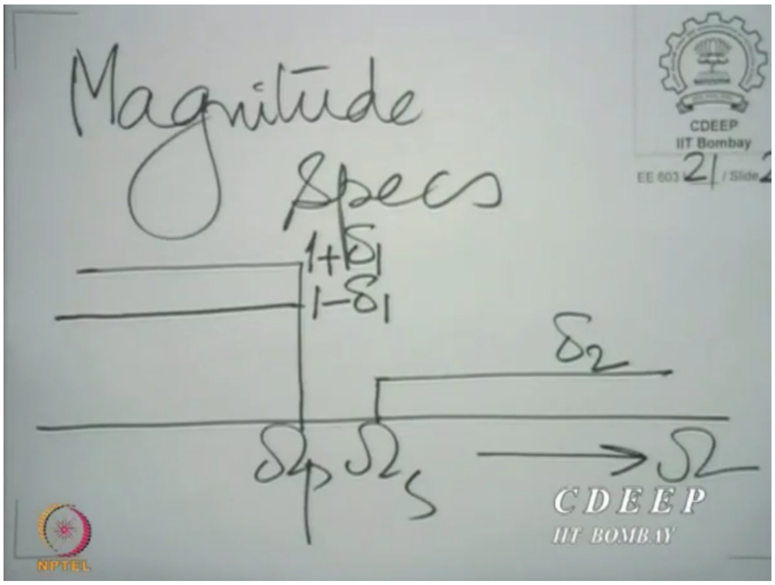

So, let us assume that you have arrived at. The magnitude specification is as follows: This I am talking about the analog domain. So, our passband S, stopband S. In the passband, we have agreed that you want the response to be between $1 + \delta_1$ and $1 - \delta_1$. And in the stopband, you want it to be beyond δ_2.

And the passband p is at Ω_H and the stopband s is at Ω_S. And this is the analog frequency Ω. Now, we shall see in a minute that we can meet this by using the Butterworth filter. Now, in fact, we can do a little better. In the Butterworth filter, we do not even need this.

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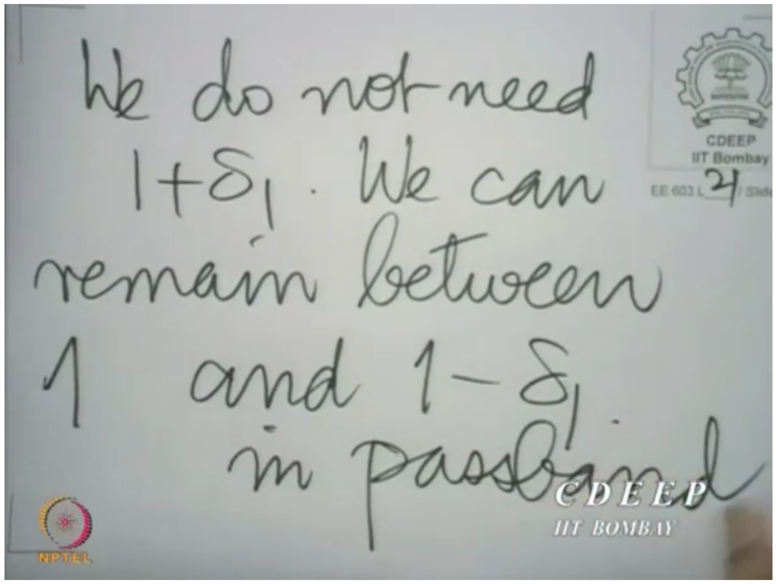

We do not even need $1 + \delta_1$. We can remain between 1 and $1 - \delta_1$ in the passband. And in fact, it is very easy. We will write down the equations that we need to satisfy.

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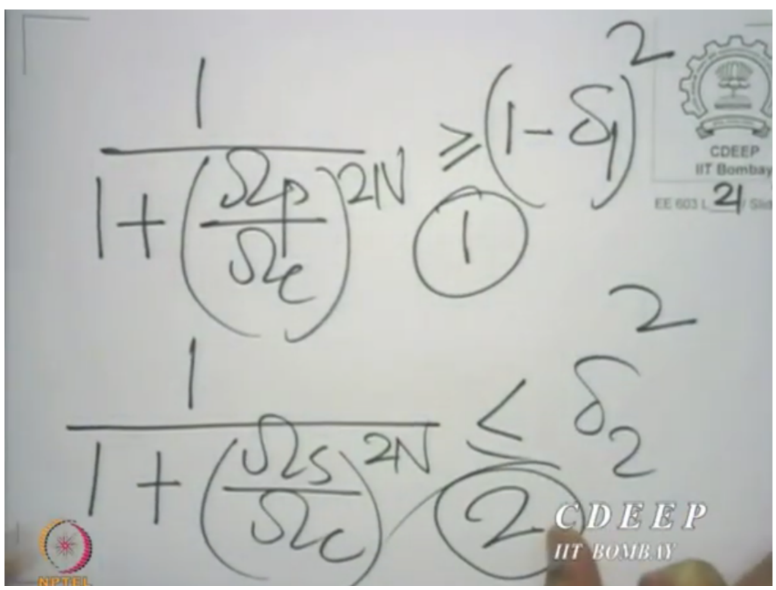

You see, what we want is at the passband edge, $1/(1 + (\Omega_P/\Omega_C)^{2N}) \geq (1 - \delta_1)^2$. Remember we are talking about squared magnitude. And the second equation that we have is at the edge of the stopband at Ω_S, it must be $\leq \delta_2^2$. And now, it is very easy to solve these 2 equations.

In fact, these are all non-negative quantities. So, if we multiply both sides for non-negative numbers then the – you see, remember, we must not forget, we are dealing with inequalities here. So when we multiply both sides of an inequality, we have to be careful whether we are multiplying by a non-negative number or a negative number. Here, luckily all of them happen to be non-negative. Anyway, you know let us take the first inequality and let us note that – let us give them number. Let us call this inequality 1 and let us call this inequality 2.

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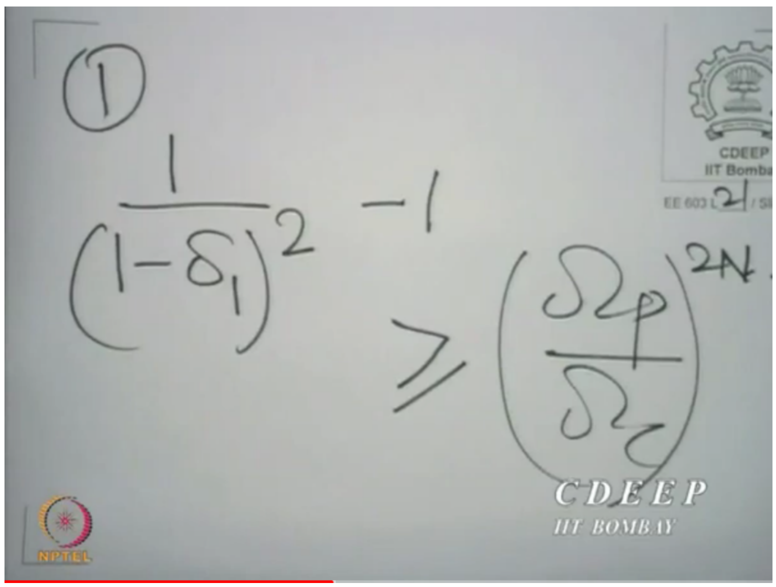

For inequality 1, we have $1/(1-\delta_1)^2 - 1 \geq (\Omega_p/\Omega_C)^{2N}$. And in the second inequality, we take the reciprocal. And when we take the reciprocal, of course, the inequalities reverse. They are all non-negative quantities. If I take the reciprocal of both sides, the inequalities reverse. And of course, luckily n1 of the sides is zero, so reciprocal is acceptable.

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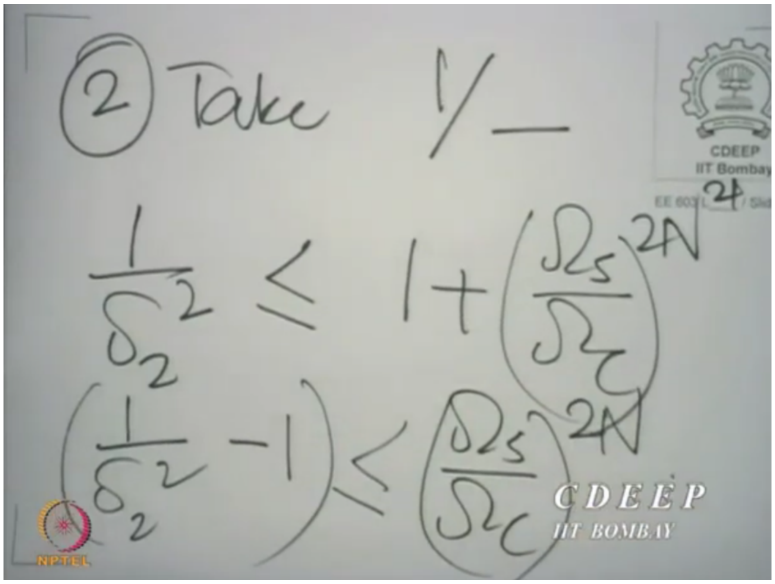

So, we take the reciprocal of the second. And you get $1/\delta_2^{\ 2} \leq 1 + (\Omega_S/\Omega_C)^{2N}$. Therefore, we have $1/\delta_2^{\ 2} - 1 \leq (\Omega_S/\Omega_C)^{2N}$. So, now we have 2 inequalities here. You see what we need to do is to first eliminate 1, we have to solve for 2 quantities $\Omega_C$and $\Omega - \Omega_C$and N.

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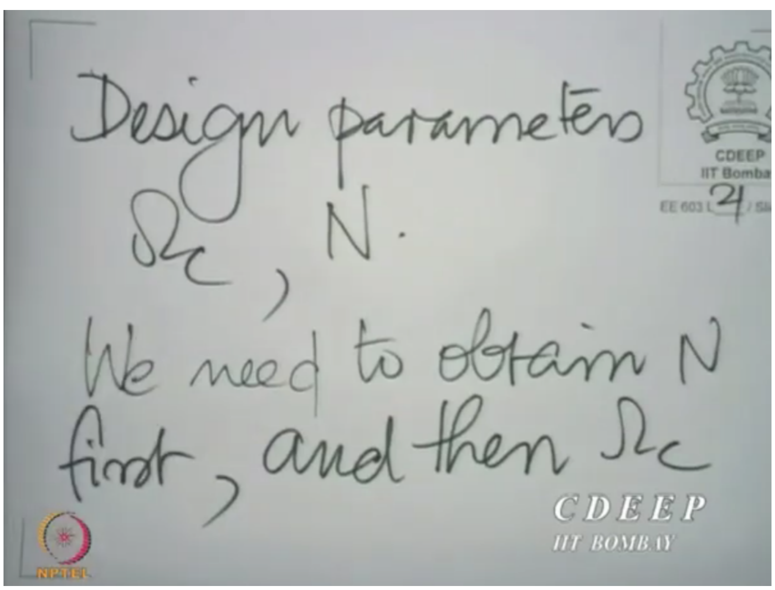

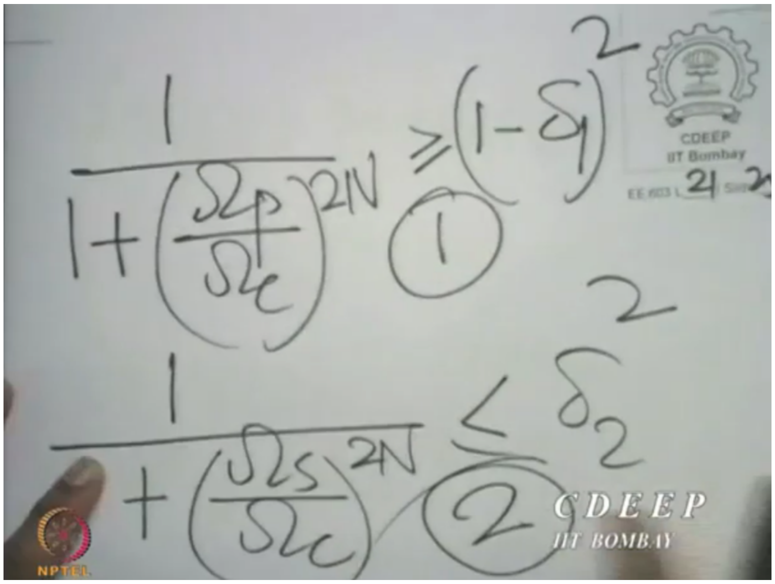

You see, so what are the design parameters here, Ω_c and N. You see, we can, let us give these quantities names. So, now let us observe these 2 inequalities that we have here. Let us write them down again. So, we will proceed as follow. We will obtain N first and then Ω_c. So to obtain N first, we need to do away with Ω_c first. From the 2, let me flash the 2 equations before you again. So, we have these 2 equations here, which we have converted back into these 2 equations here.

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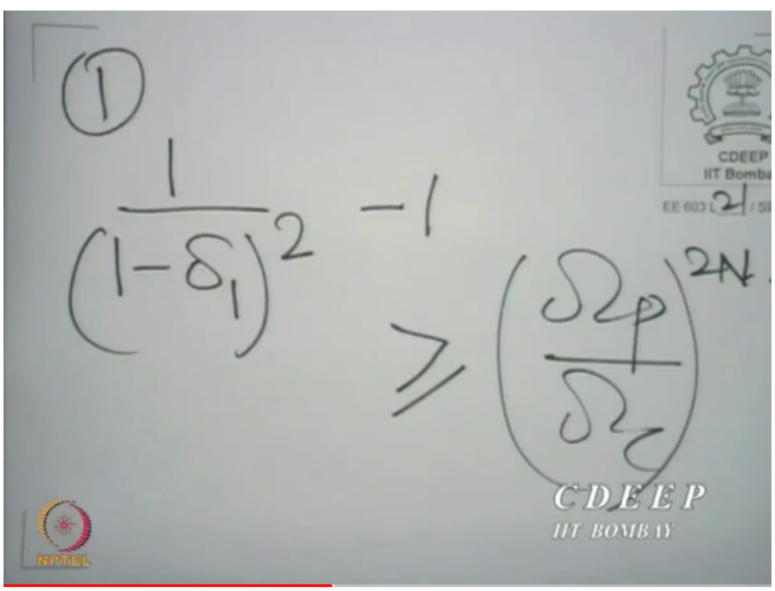

Now, please note, let us make a few observations about these quantities here. 1 minus delta 1 squared is a quantity less than 1 so therefore, this quantity is going to be greater than 1 for sure. And minus 1 is therefore going to be positive.

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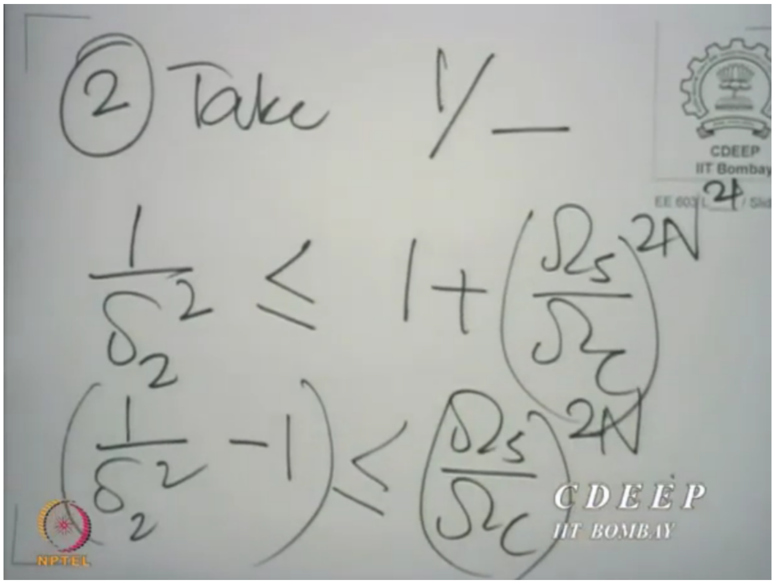

On the other hand, if you look at this quantity here, $\delta_2^2 \leq 1$. So, $1/\delta_2^2 \geq 1$. And therefore, this -1 is going to be positive. So, both of these are positive. Now, we can take the logarithm on both sides. In fact, you could. The logarithm is a non-negative operation. Is that right?

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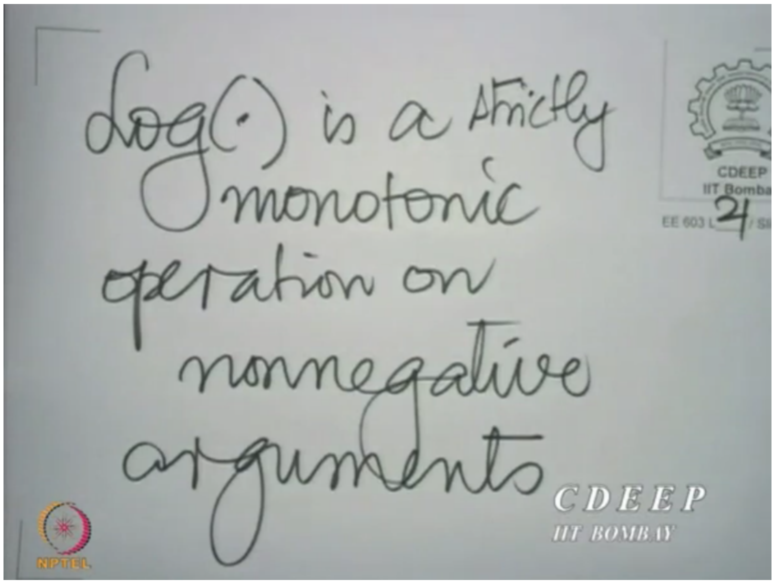

So, logarithm is a monotonic operation which works on non-negative arguments. In fact, it is what is called the strictly monotonic. So, there is no place where it stabilizes or flattens. So, if I take the logarithm on both sides of an inequality of the negative quantities, that inequality is preserved. So, let me take the logarithm on both sides.

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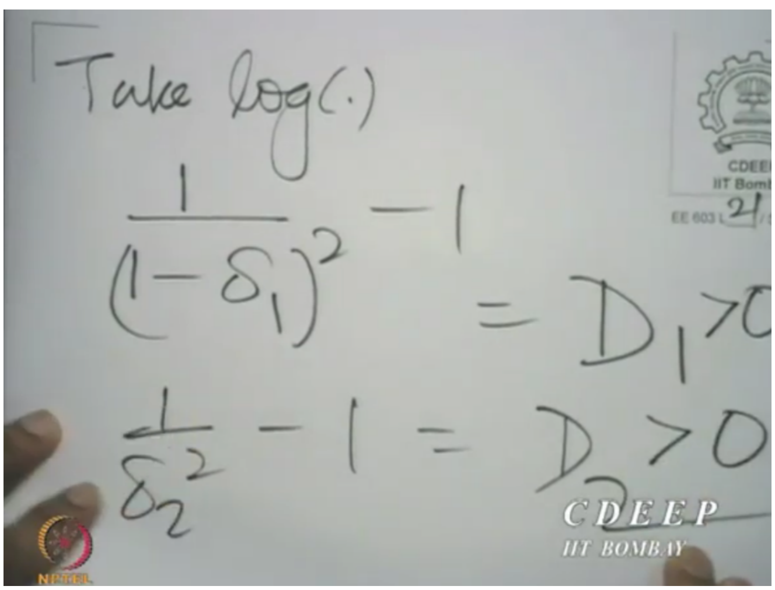

And let us give the name $1/(1-\delta_1)^2 - 1 = D_1$ and $1/\delta_2^{\,2} - 1 = D_2$. So we have, and of course, both of them are greater than zero as we have observed.

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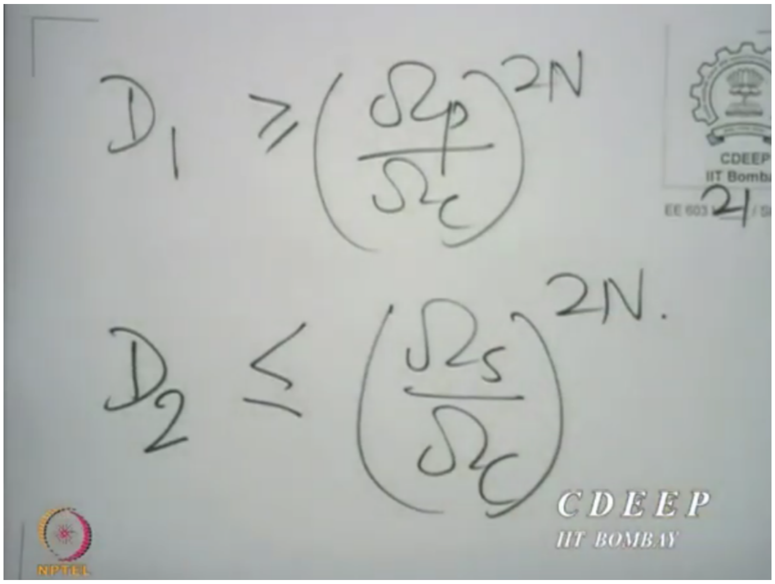

So, the first equation that we have is $D_1 \leq (\Omega_P/\Omega_C)^{2N}$. And the second equation that we have is $D_2 \leq (\Omega_S/\Omega_C)^{2N}$.

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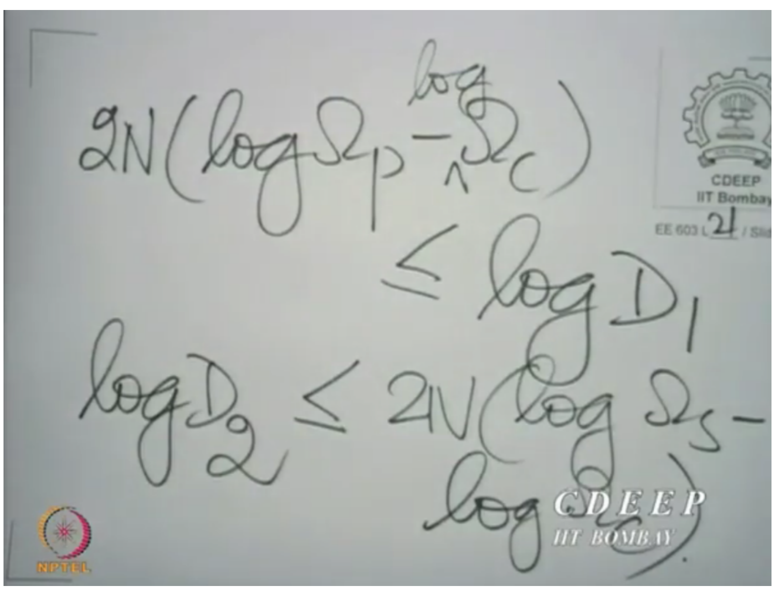

Now, if we take the logarithm as we do. We have $2N(log(\Omega_P) - log(\Omega_C) \leq log(D_1)$. And of course $log(D_2) \leq 2N(log(\Omega_S) - log(\Omega_C)$. See, what we want to do is to subtract these 2 inequalities, to get rid of $log(\Omega_C)$ in a way. And the most convenient thing to do is to reverse this inequality by taking the negative on both sides.

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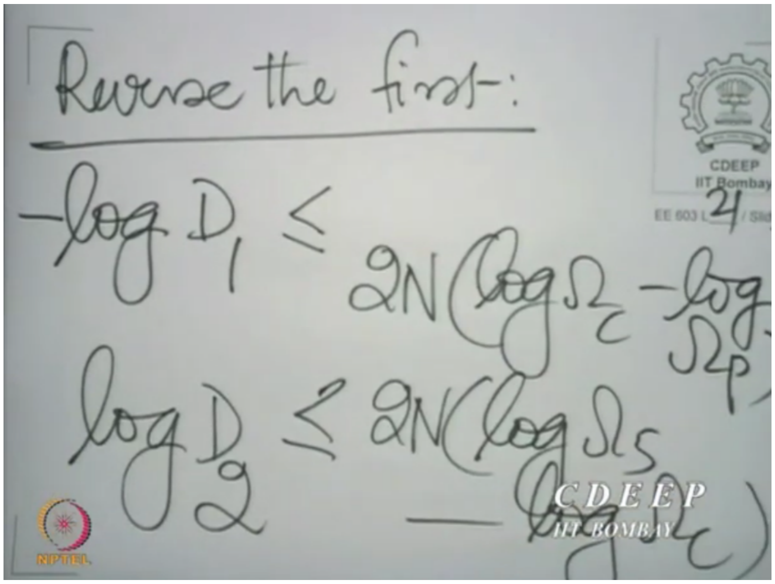

So, let us reverse the inequalities. So we have $-\ log(D_1) \leq 2N(log(\Omega_C) - log(\Omega_P)$ and $log(D_2) \leq 2N(log(\Omega_S) - log(\Omega_C)$. Simple. And now you can just add.

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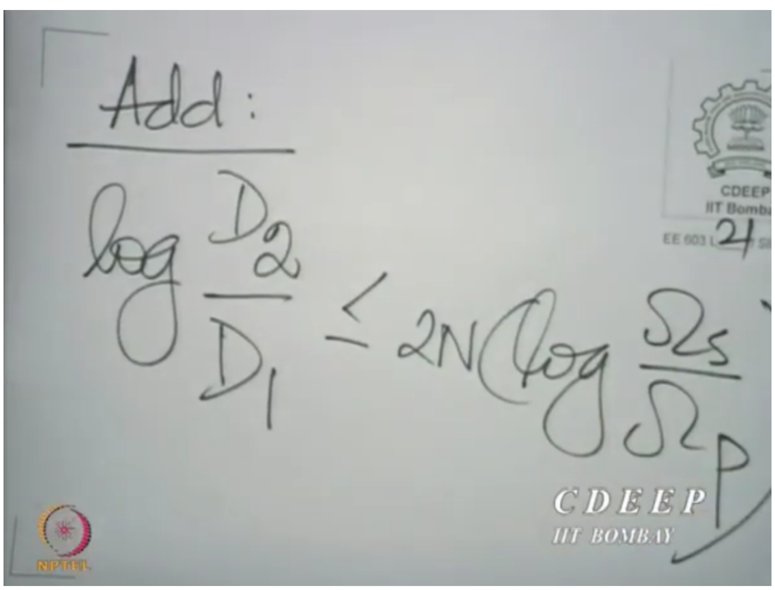

Now we can just add these inequalities. So, is that right? And now does away with. You see $log(D_2) - log(D_1) = log(D_2/D_1)$. And this is less than equal to $2Nlog(\Omega_S/\Omega_P)$. And why we have chosen this form is because Ω_S is greater than Ω_P. So, this logarithm is positive. Ω_S is, of course, the stopband is more than the, the stopband edge is more than the passband edge.

And therefore, Ω_S has a quantity greater than 1 so its logarithm would be positive. And now the question is what about D_2 and D_1? Remember $D_2 = 1/\delta_2^2 - 1$. Now, δ_2^2 is much less, or less definitely than $1/\delta_1$, $1/\delta_1^2$. So, D_2 is definitely greater than D_1.

$D_2 = 1/\delta_2^2 - 1$. $\delta_1$is lower than minus 1 $\delta_1$squared. So, $1/\delta_2^2 \geq 1/(1-\delta_1)^2$. And therefore, D_2 is greater than D_1 as well. And therefore, we have positive quantities on both sides of this inequality here. So, we are in good shape. And now, we have a very simple relationship.

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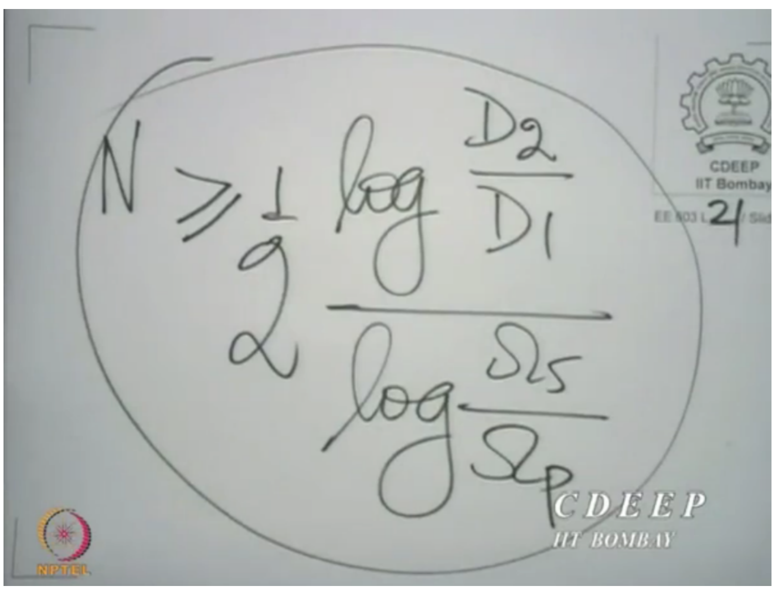

$N \geq \frac{1}{2}\frac{log(D_2/D_1)}{log(\Omega_S/\Omega_P)}$ and this is what governs the choice of what is called the order. N is the rate at which the response decays and N has a direct relationship to how much of resource you are going to require to realize the filter. That is not too difficult to see.

You see, once you have N, it tells you what is the power of s. You see, after all Ω or $j\Omega$ is s. So a higher power of s means a higher power of z. And a higher power of z means more delays, more additions, more multiplications. So, N has a direct bearing on the amount of resource that you require to realize the filter.

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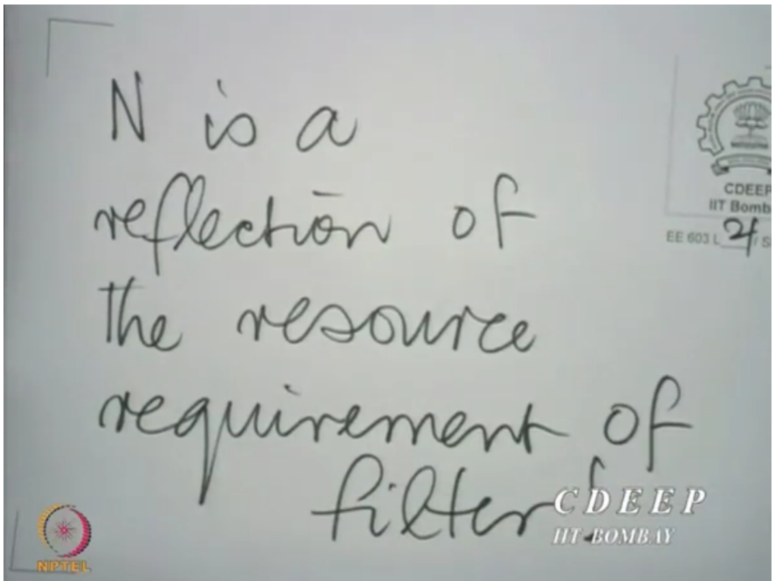

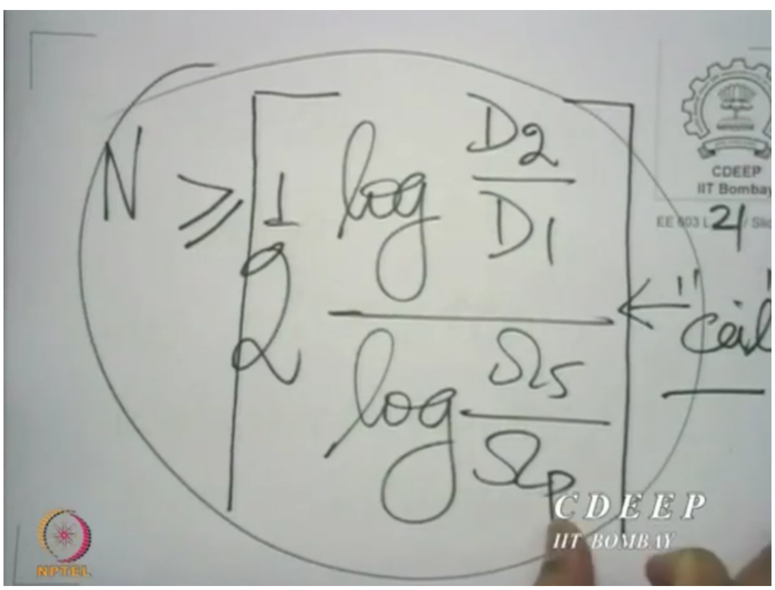

So N, let us make a remark here. N is a reflection of the resource requirement of the filter. The more the N, the more resources you will need to invest in realizing the filter. And in fact, if you look just for a minute at the expression for N, then you will see that it is very, very intuitively clear why this is a reflection of resource.

Let us see how N, now remember N is greater than or equal to some quantity. Now, this quantity need not be an integer. So, what must you do? You must put here what you call a ceiling operation. A ceiling operation means the integer just above that. For example, this quantity works out to be 6.4, then the ceiling for that is 7.

If it works out to be 7.9, then the ceiling is 8. If it works out to be 5.1, the ceiling is 6. So, if it is a little bit above 5, the ceiling goes up by 1 step. The ceiling is the integer just above. So, N must be greater than or equal to the ceiling of that quantity. Now, let us look at the quantity itself for a minute and make some inferences.

You see, it is very clear that N, the requirement of N is going to be more if the numerator is more or if the denominator is less. Let us first look at the denominator. We need the denominator to be less. The denominator would be less if Ω_S and Ω_P are across to 1 another. If $\Omega = \Omega_S$ is far away from Ω_P the denominator is more.

And therefore, the requirement of order goes down. So, asking for a very sharp transition band means asking for a higher order. Now let us look at the numerator. In the numerator, D_1 – you see when will the numerator be more? Either when D_2 is more or when D_1 is less. If D_2 is more, you are saying that $1/\delta_2^2$ is more.

And that means δ_2^2 is less that means you are asking for smaller tolerance in the stopband. If you are asking for smaller tolerance in the stopband, you are demanding more and therefore, you have to invest more resources. On the other hand, if $D_1 \leq 1/(1 - \delta_1)^2$. That means δ_1^2 is more. That means we are reducing the tolerance in the passband.

And that means you are asking for more and therefore, you will have to invest more. Engineering design whether it is in discrete time system design or in any other branch of engineering, is always a game of ask for more, invest more. When you ask for more, you have to invest more and that is true for all the three kinds of asking here.

If you ask for a sharper transition band, you have to invest more. If you ask for a smaller stopband tolerance, you will have to invest more. If you ask for a smaller passband tolerance, you will have to invest more. Ask for more, invest more, simple. Now, of course, once you have completed the choice of N and now you understand why you had a 2N there.

You had a 2N because you took $\left|H_{analog}(s)\right|$ into $\left|H_{analog}(-s)\right|$. So, whatever poles and zeros you have in the original H analog are doubled when you go into $\left|H_{analog}(s)\right|$ into $\left|H_{analog}(-s)\right|$. That is why you put 2N there. And anyway, you are considering the magnitude squared function, so you need to double and that is N is called the order.

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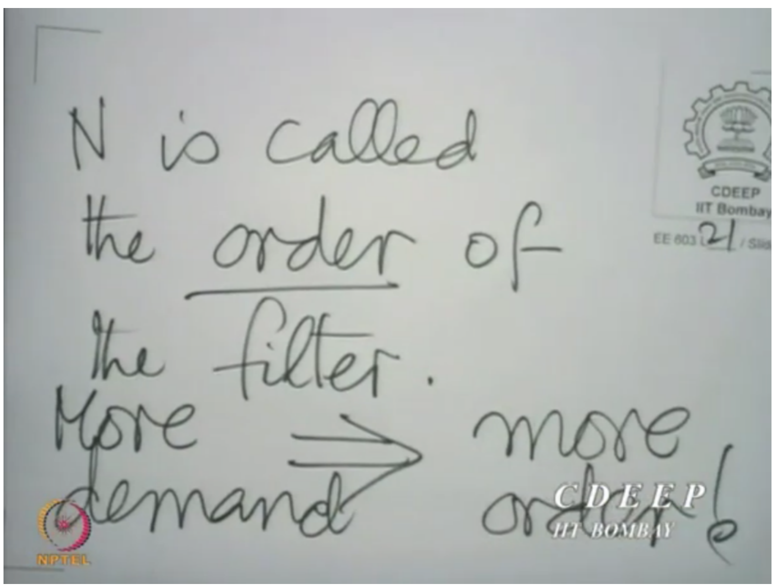

We will give it a name. N is called the order of the filter. More demand, more order. Now, of course, finding out Ω_C is very easy. Finding out Ω_C is very easy. You see, we could go back to the equation that we have here. Once you found Ω, once you found N, Ω_C is very easy.

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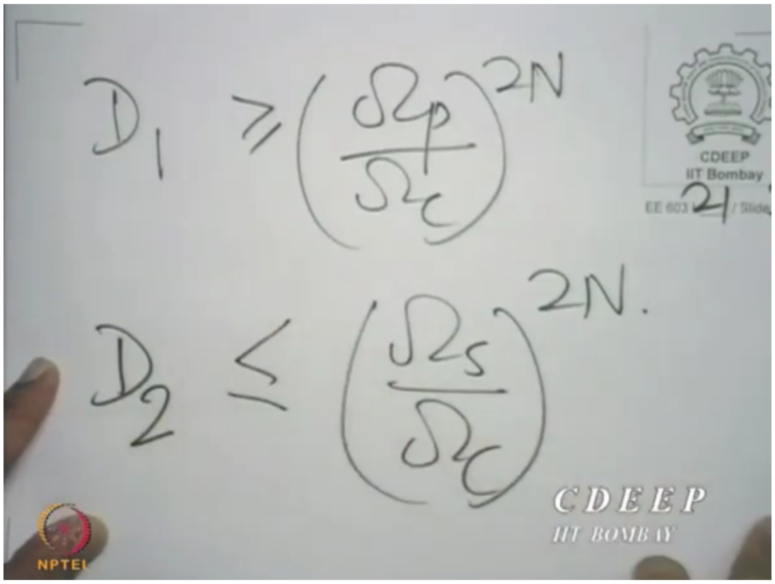

You had this equation here earlier. D_1D1 is greater or equal to $(\Omega_p/\Omega_C)^{2N}$ and this is true. Now all that we need to do is to take Ω_C to the other side in both of them. We have from the first equation, from this equation

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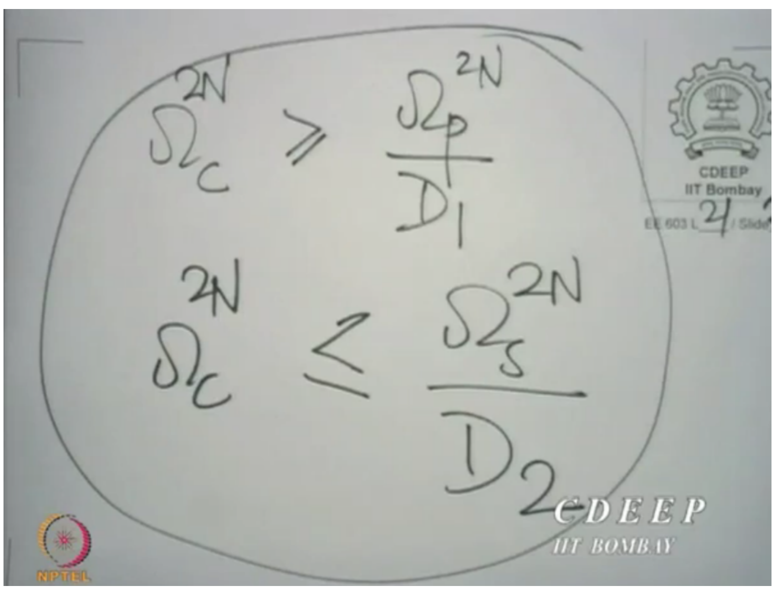

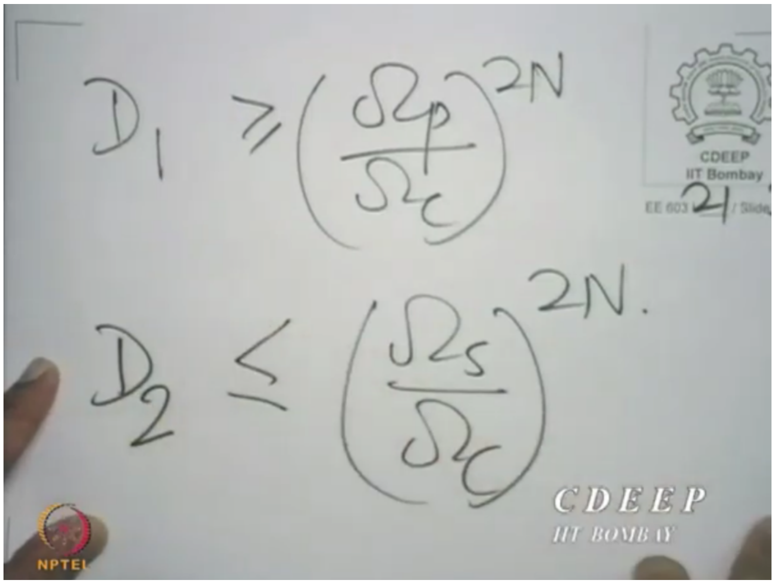

$\Omega_C^{2N} \geq \Omega_P^{2N}/D_1$. And for the second equation, taking $\Omega_C^{2N} \leq \Omega_S^{2N}/D_1$.

So, now you have a beauty range in which Ω_C can lie. Of course, Ω_C is a positive quantity. So, if you know, Ω_C to the power 2N, you can find Ω_C by taking the 1 by 2 nth root or you can use logarithm. That is not a problem.

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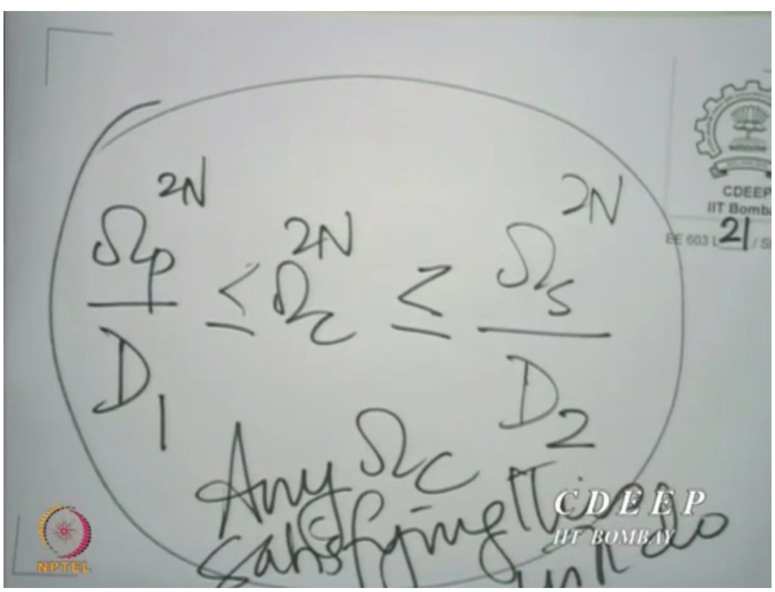

But what we can see from here is that Ω_C to the power 2N must lie between Ω_S to the power 2N by D2 and Ω_P to the power 2N by D1. And any other Ω_C satisfying this will do. Any Ω_C which satisfies this inequality will do. You may wonder why you got a range of Ω_C which is acceptable. That is because N took a ceiling operation.

So, N took a ceiling operation, you introduce some amount of tolerance there and therefore, you got a range of Ω_C which is possible. If that, when you had the expression for N, if the right hand side happen to be an integer then you would get no tolerance, you would get a fixed value of Ω_C.

But because you have got a ceiling operation, you have introduced some tolerance. Therefore, we have a range of Ω_C to choose from. Any Ω_C which satisfies this inequality will do. Is that right? So yes, is there any question?

Professor: Well, the question is does Ω_C lie between Ω_P and Ω_S? Well, that is not necessary. It depends on the tolerances that you have given. Ω_C is the half power point. If your passband tolerance is more than half point power, of course Ω_C lies inside the passband. So Ω_C is only the half power point. That is all that can be said. Anyway, with this then we come to the end of this lecture. In the next lecture, we will proceed further with to complete the design of the Butterworth filter. Thank you.