Digital Signal Processing & Its Applications
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Lecture No. 09 b
Vectors and Inner Product

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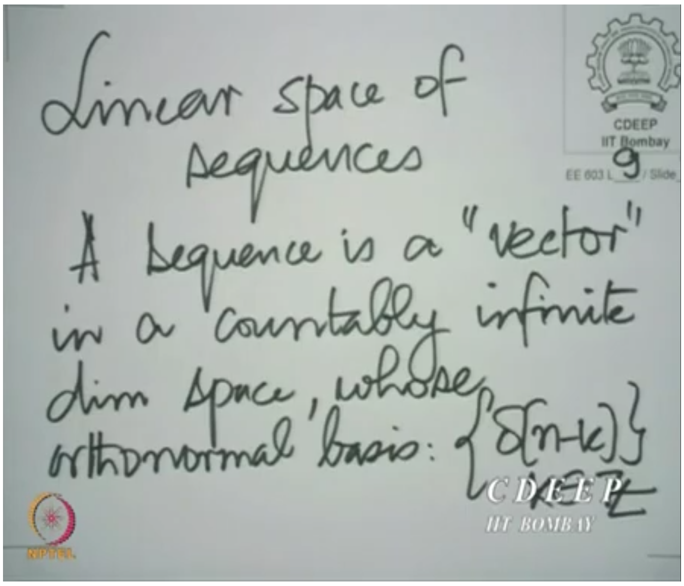

Any sequence is really like a vector in an infinite dimensional space, and the infinity of that dimension is the infinity of the integers. You can think of each sample of the sequence as a coordinate of that so-called vector.

So, a sequence is like a vector in countably infinite dimensional space, whose orthonormal basis is essentially the set of all unit impulse sequences. Put a unit impulse at each every integer location and each of these gives you a separate perpendicular vector. This is a generalization from what we have in this 5 dimensional space.

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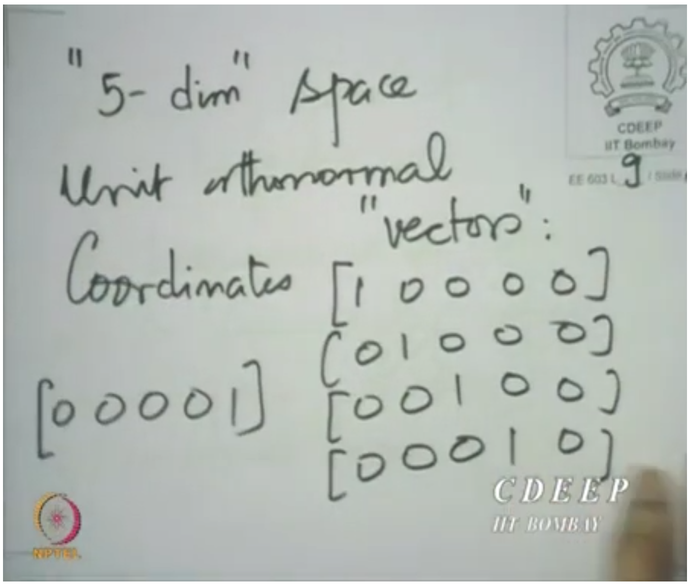

In 5 dimensional space we are putting the 1 respectively at the locations 0, 1, 2, 3 and 4. But you can generalize this to countably infinite dimensions.

You can put a one at each integer location and each of them gives you a new perpendicular vector, and that is how you have an orthonormal basis for the space of sequences. We already know how we can express every sequence in terms of this perpendicular basis, which we have already done when we were trying to derive the principle of convolution.

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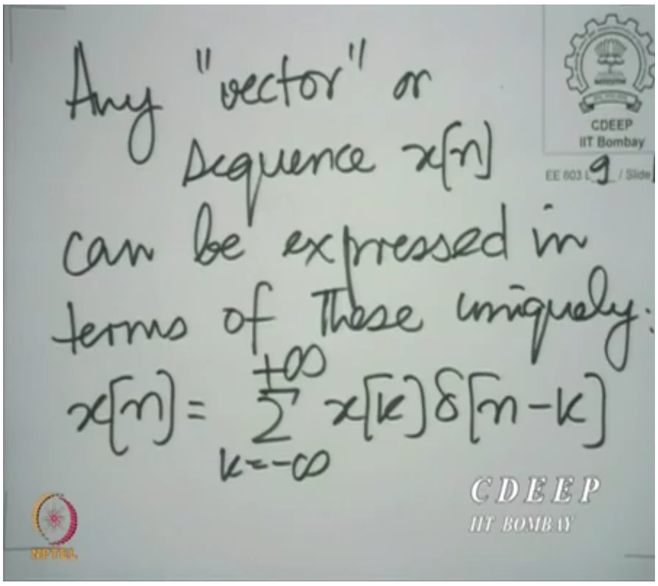

So, we saw that any vector, or any sequence $x[n]$ can be expressed in terms of the basis sequences as follows:

$$\text{x[n] is } x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Signal processing has an interesting similarity to poetry. The same line can have many meanings, the same equation can assume many meanings in signal processing in communication and one must learn to appreciate that. So, this very equation here had a different meaning when we talked about convolution.

There we were trying to express all sequences in terms of the unit impulse. Now here you are trying to show that you have an orthonormal basis (and in fact we will show in a moment that it is orthonormal) and that the orthonormal basis can allow you to expand any vector in terms of it.

So one equation can have several meanings or several different interpretations and one must learn to appreciate that as we go along. Anyway so coming back to this, it is very clear that we can express $x[n]$ in terms of this orthonormal basis. But what we all want to see is why are we calling it orthonormal.

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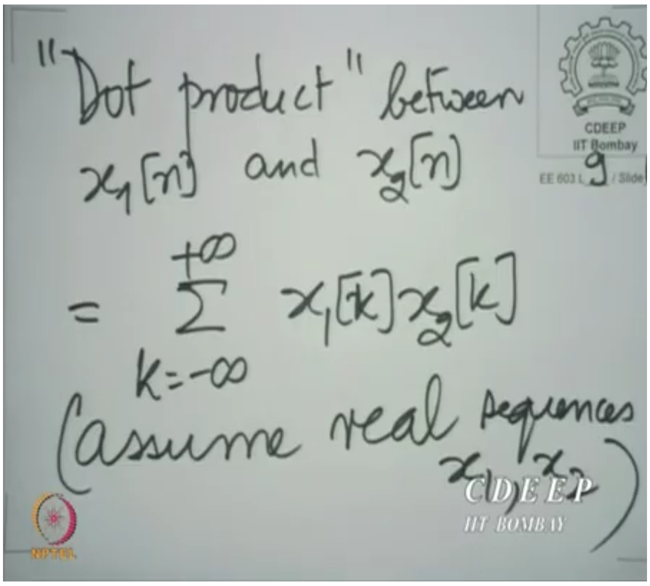

So, we need to talk about a dot product. As we generalize from finite dimensional spaces, the dot product between two sequences would essentially be a sum over all the coordinates. You see each case a coordinate now, each location is a coordinate. $x_1[n]$, $x_1[k]$, $x_2[k]$ and for the moment let us assume real sequences x_1, x_2 at the moment.

We are assuming real sequences and we are assuming the coordinates are real. So we need essentially a generalization from what we did in finite dimensional spaces. We took the product of corresponding coordinates and we added up over all the coordinates. But now we need to ask whether this definition is acceptable when the sequences are complex. So for that we need to ask what more do we want from a dot product.

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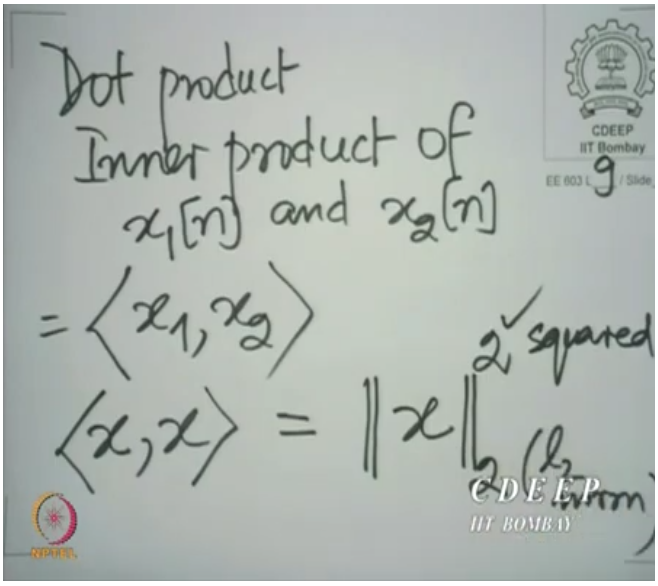

Before we do that since we are generalizing the idea for dot product we would like to use a slightly more general notation for dot product. The dot product is also more generally called the inner product and it is denoted by $\langle x_1, x_2 \rangle$ using triangular brackets. Now you see, if you take the dot product of x with x, what do you get? In conventional vector algebra we get the magnitude square of the vector.

Here we would like to use the norm notation. Instead of saying magnitude for a vector in general, or for the generalized class of vectors, we call it the norm. In fact we call it the 2 norm. Do not worry too much about this lower 2, but the upper 2 is essentially a square. This is what is called the L2 norm. This is a technicality, let us not worry too much about it but let us write it down for completeness. And you could also have other numbers in place of 2.

However, in our discussion in this case, we shall not keep writing the subscript 2 after this, we shall understand that it is the 2 norm.

Now you see one thing, this norm is the generalization of the idea of magnitude. So instead of using the word magnitude for this generalized class of vectors we call it the norm, the same idea.

But you see what do we want of a norm? We definitely want a norm or a norm squared to be non-negative. In fact we want the norm to be not only non-negative, if the vector is 0 only then we would allow it to be 0. If there is a non-zero vector we definitely do not expect that it should have a 0 magnitude.

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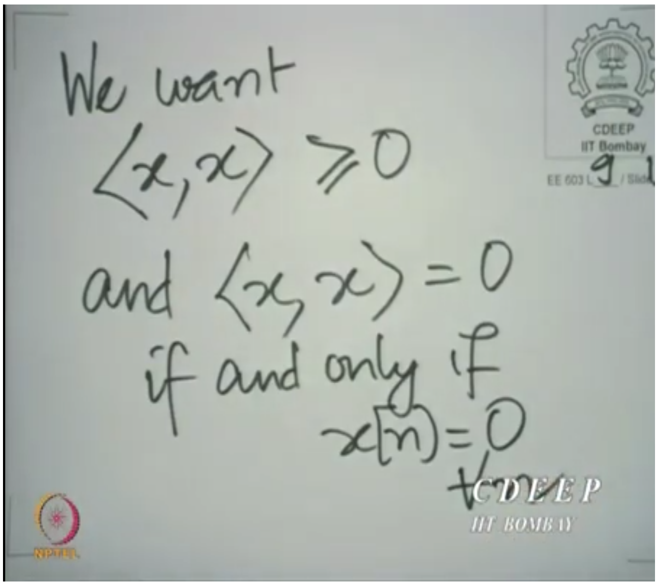

So, we want a dot product of x with x to be greater than equal to 0 and we want the dot product of x with x to be equal to 0 if and only if the vector x is itself 0 which means x[n] is equal to 0 for all n. Is that correct? But you see if you take the definition of the dot product that is the inner product that we just used a few minutes ago.

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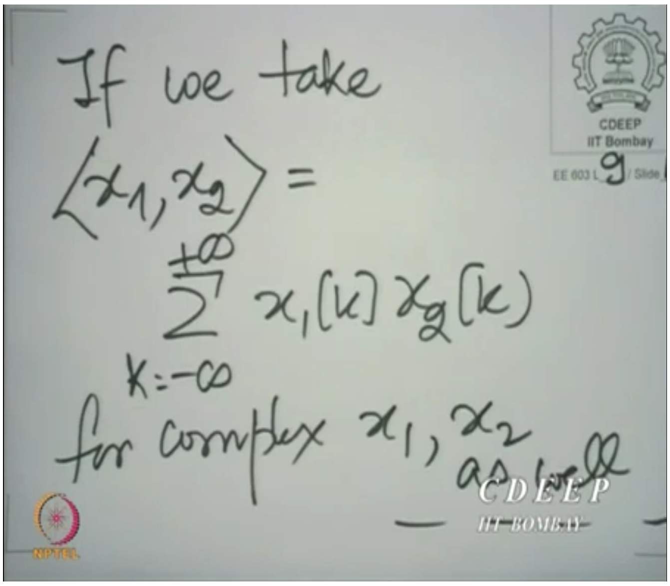

If we take the dot product or the inner product of x_1, x_2 to be $< x_1, x_2 > \; = \sum_{k=-\infty}^{\infty} x_1[k]x_2[k]$ for complex $x_1,\ x_2$ as well. If we do this then what are we going to land up with?

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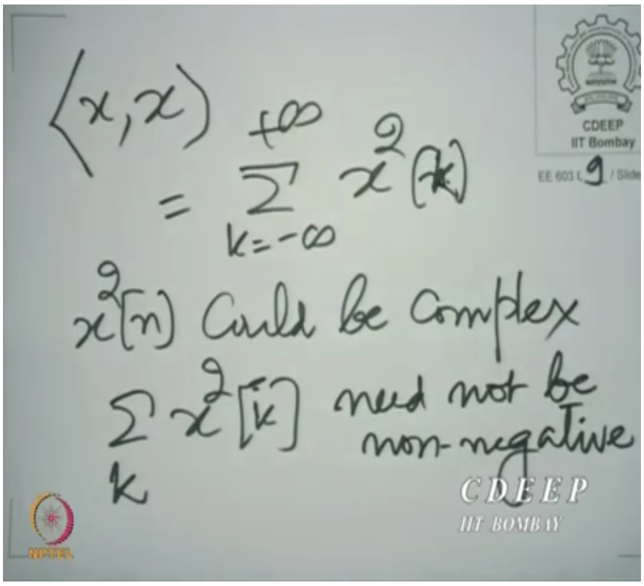

We are going to land up in the troublesome situation that the inner product of x with x is simply $< x, x > \; = \sum_{k=-\infty}^{\infty} x^2[k]$. Well if x[n] is complex, $x^2[k]$ could be complex and $\sum_{k=-\infty}^{\infty} x^2[k]$ need not be 0, need not be non-negative. In fact it can be complex in general and we definitely do not want that.

We wanted it to be a magnitude square so that it can only be non-negative and it should be 0 only if x itself is 0. Is that correct? Now you see how do we then correct the situation? It is very clear that we want each of these terms in the summation to contribute a non-negative quantity and that can be done only if you have a modulus in each of the term and that modulus can appear only if you bring in a complex conjugate.

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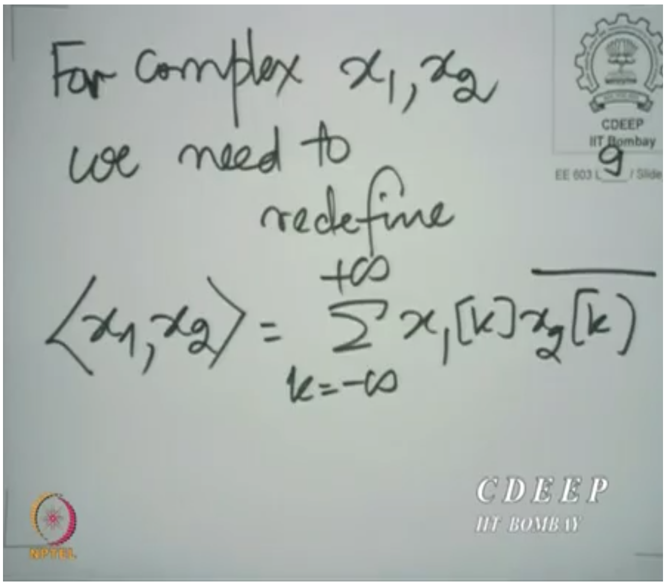

And therefore, we redefine the dot product in general for complex x_1, x_2. We need to redefine the dot product $< x_1, x_2 > \; = \sum_{k=-\infty}^{\infty} x_1[k]\overline{x_2[k]}$. No doubt this makes no difference for real x_1, x_2 at all. The definition for real x_1, x_2 is as it is. Is that clear? Yes? You see now it is very clear.

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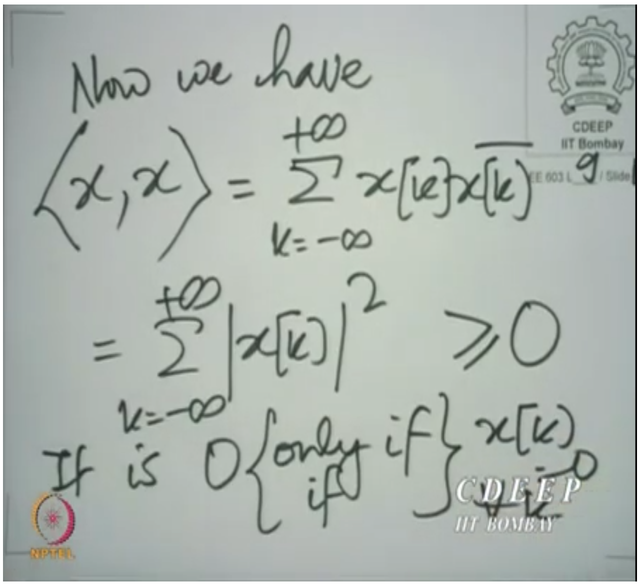

Now of course, once we have done this we have taken care of the problem. So now we have the inner product $\langle x, x\rangle \; = \sum_{k=-\infty}^{\infty} |x[k]|^2$ which is of course greater than equal to 0. It is 0 only if, in fact if and only if $x[k] = 0 \; \forall \; k$ unless each of those terms is 0, this cannot come to zero because each term there is non-negative.

And therefore, we have achieved what we want out of the dot product. We have been able to redefine the dot product in such a way that it takes care of this use of the dot product for calculating the magnitude square of a vector.