Electrical Equipment and Machines: Finite Element Analysis
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Lecture No: 09
Revisiting EM Concepts: Time Varying Fields

Good morning and welcome to the 9th lecture of this course. In the previous two lectures we saw magnetostatic fields and in that we studied magnetic vector potential, magnetic scalar potential, magnetic forces, and boundary condition in case of static magnetic fields. We will continue that discussion for 1 or 2 slides in this lecture and then we will go into time varying fields.

Now it is appropriate time to just summarize the analogy between electric and magnetic fields or circuits.

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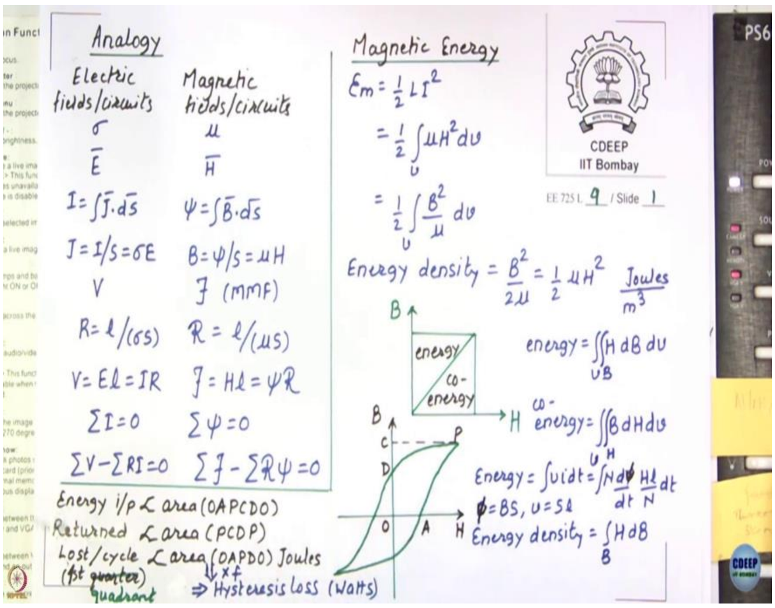
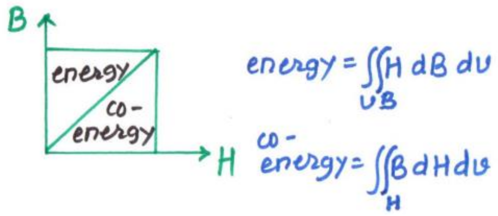


The table in the above slide shows the corresponding parameters of two fields or domains which are analogues to each other, so, conductivity and permeability, electric field intensity and magnetic field intensity, current which is $\iint_S \mathbf{J} \cdot d\mathbf{S}$ and flux which is $\iint_S \mathbf{B} \cdot d\mathbf{S}$, current density J is $I/S = \sigma E$ which is point form of Ohm's law, whereas B is $\frac{\psi}{S} = \mu H$, etc., are analogous as shown in the table.

The KVL for electric fields and the corresponding circuit is $\sum V - \sum RI = 0$. In case of magnetic fields, it is $\sum \mathcal{F} - \sum \mathcal{R}\psi = 0$. So, this analogy of KVL is important for understanding various electromagnetic devices. Now, we will go over to magnetic energy.

In this course we will be dealing with magnetic energy for at least 2 lectures. We will also be talking of co-energy later when we are calculating forces. So, magnetic energy $\mathcal{E}_m$, the symbol of magnetic energy I have purposely drawn like this to differentiate with respect to E which denotes electric fields. Magnetic energy is defined as $\frac{1}{2}LI^2$ and that is also equal to $\iiint_v \frac{1}{2}\mu H^2 dv = \iiint_v \frac{1}{2\mu} B^2 dv$,where v is the volume.

The above expressions of energy in terms of field quantities have volume integrals because $\frac{1}{2\mu}B^2$ or $\frac{1}{2}\mu H^2$ represent energy density with unit as J/m^3. Now, here for a linear material wherein B-H characteristic is linear as shown in the following figure, we have what are known as energy and co-energy given by the triangles shown in the figure.



Now, let us understand little bit about the energy and co-energy terms. Later on, we will talk more when we do calculation of forces in one of the lectures when we start looking at application of finite element method to calculate forces. So the expression of energy given in the above figure has two integrals, one is volume integral and other is of integration over B, because, there is an independent variable dB. Here, the area of the triangle that corresponds to energy is calculated by integrating with respect to B as the variable so, the area under the linear BH characteristics is the area of the upper triangle in the above figure.

The integration over the volume is required because, for a material there are so many points of interest and at each point, you will have different values of B and H, so, you have to integrate the $\int HdB$ over the whole volume, so there are 2 integrals involved. The next question is why HdB is called as energy and BdH is called as co-energy?

Co-energy as I said is $\int BdH$ and then again this integral is integrated over a volume. So, if you see, energy is $\int vidt$ because, vi is power and multiplication of power and time is energy. Now if we express v as $N\frac{d\psi}{dt}$ which is Faraday's law and in fact we are going to see in the next slide.

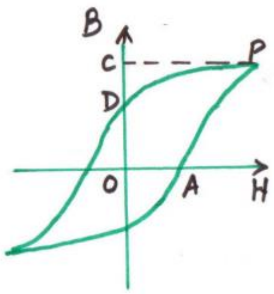
$N\frac{d\psi}{dt}$ is voltage and $I = \frac{Hl}{N}$ where Hl is MMF and that divided by number of turns will give you I, so in the expression of energy, in terms of circuit quantities if we substitute v and i, we obtain, $\int vidt = \int N\frac{d\psi}{dt}\frac{Hl}{N}dt$ now, if you substitute $\psi = BS$, the expression of energy is modified as given below.

$$Energy = \int vidt = \int N\frac{d\psi}{dt}\frac{Hl}{N}dt = \left(\int HdB\right)Sl = \left(\int HdB\right)volume$$

So, energy density is $\int HdB$, because, voltage in case of all these magnetic induction problems is always $\frac{d\psi}{dt}$ or $S\frac{dB}{dt}$and energy is related to $\frac{dB}{dt}$, so, that dB in the energy density term comes from that $d\psi$ term.

So, the energy has a physical significance, whereas co-energy is given by the area of the lower triangle in the above figure or the area under the linear BH characteristics with H as the reference axis as co-energy is defined as $\int BdH$. For a linear material, energy and co-energy are equal but, for nonlinear material they are not equal so, we will see more about this when we see the virtual work approach for force calculation.

Now, we will understand little bit more on hysteresis of ferromagnetic materials which we covered in the previous lecture. Consider a typical hysteresis loop as shown in the following figure.



The area of this loop gives what is known as hysteresis loss.

So, let us understand why this is loss, when we say loss, it is some input minus some output. Let us understand the input to the magnetic circuit, the energy given back to the source and the difference of the two energies which is dissipated as loss. Now, consider the curve in the first quadrant of the above figure. In the first quadrant when the excitation is increased and the curve follows path AP, expression for energy is $\int_v \left(\int_B H dB \right) dv$.

The energy that is given to the magnetic circuit will be the area under this whole curve in the first quadrant that means the area under the curve which is OAPCDO shown in the above figure.

The energy is calculated with B as the reference axis. So, that is the input to the magnetic circuit when the excitation goes from A to P, when you reverse the excitation, the curve does not follow the same path and that is why we have this memory effect or hysteresis effect and then it follows the path P to D, the energy is represented by this area PCDP and this energy returned back to the source, why this energy is returned back? Because, now the sign of dB is negative.

Because, we are going from P to D, D to C, and C to P, dB is negative, so, the sign of the energy or sign of the power is reversed. That means when the sign of power is reversed, the power is flowing back to the source. So, that means when the curve is traversed from P to D the energy or the area corresponding to this PCDP is returned back to the source. Now, in the first quadrant what is the energy lost?

Area under the curve formed by the path OAPDO is the energy lost and this is for the first quadrant, you can then do the similar way for the all 4 quadrants and all these 4 areas will give you the total hysteresis loss, this loss per cycle for the first quadrant that I just explained to you is proportional to OAPDO and that is in Joules.

Now, when you add all these 4 areas corresponding to the four quadrants, you will get area of the loop that is energy loss per cycle, that if you multiply by frequency (f), cycles per second it represents the energy lost in f cycles. Suppose, if it is 50 Hz, 50 cycles per second, when you multiply the area of the loop with 50 you will get the hysteresis loss in Watts, because, this energy times f cycles per second is energy per second which represents power. So, that is why hysteresis loss is proportional to f whereas, later we will see eddy current loss is proportional to f^2 square.

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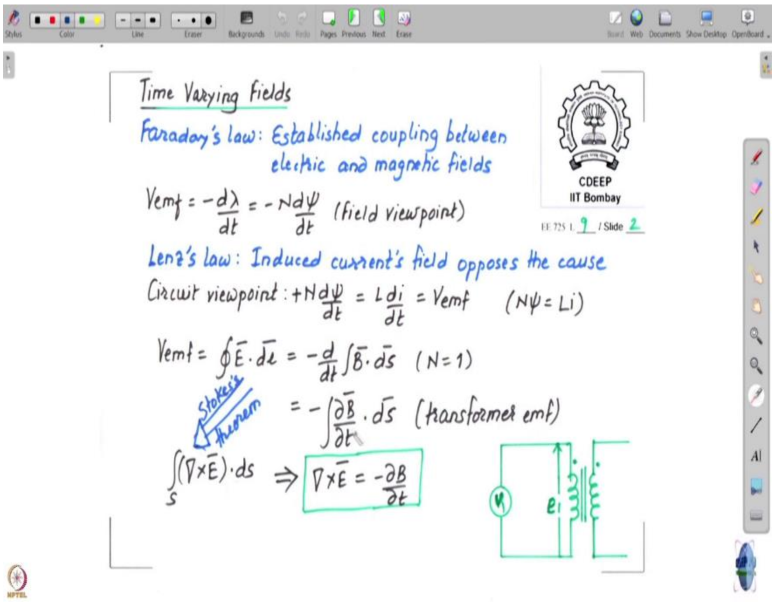


So, now we will see time varying fields. The most important law in electrical and electronics engineering which makes this whole world move is Faraday's law and it established coupling between electric and magnetic fields. The law says that for any closed circuit if the flux linkages change with time then voltage will be induced and that voltage is called as EMF, electro-motive force and that is given by this famous equation which is given below.

$$V_{emf} = -\frac{d\lambda}{dt} = -N\frac{d\psi}{dt} \quad (\text{field viewpoint})$$

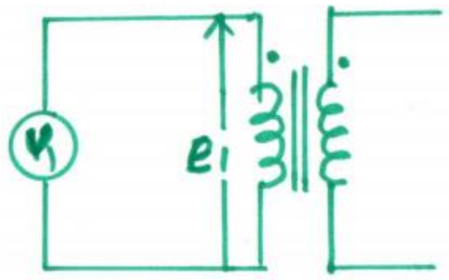
where $\lambda = N\psi$ is the flux linkage.

In the above equation, $V_{emf} = -\frac{d\lambda}{dt}$ is field viewpoint and why it is field viewpoint? Because of this Lenz's law which says field created by the induced current opposes the applied field (cause). So, in the closed circuit the corresponding induced current will flow in such a way that the flux produced by it will oppose the cause which is this changing flux.

The opposition is represented by this minus sign. Now, in circuit viewpoint, we do not consider this minus sign because, we consider induced emf as just $N\frac{d\psi}{dt}$ and then $N\psi$ is nothing but Li

because, $L = \frac{N\psi}{i}$ which is flux linkage upon current. So, then you have $N\frac{d\psi}{dt} = L\frac{di}{dt}$ and that we take it as induced voltage.

This induced voltage is opposing the source voltage. Let us consider a transformer under no load condition as shown in the following figure



The induced voltage (e_1) opposes V_1 applied to the primary circuit and the primary current depends on the difference between these two voltages, so, the induced voltage is opposing the source voltage so, that is why that opposition is inherent in the circuit representation and that is why we do not put minus sign here.

Going further, this induced voltage is represented as $\oint E \cdot dl$ and then by applying Stokes' theorem, this equation reduces to $V_{emf} = \iint_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -N\frac{d\psi}{dt}$ and then here, this ψ can be represented as $\iint_S \mathbf{B} \cdot d\mathbf{S}$ and we are taking N number of turns as equal to 1 and then the right hand side of the equation simplifies to $-\frac{d}{dt}(\mathbf{B} \cdot d\mathbf{S})$.

Now, here the derivative with respect to time is made to act only on B, because, we are talking about stationary circuit, dS is not changing with time, so, in a way this is a transformer emf, because, in case of a transformer the flux that goes through the core is changing, and everything is stationary there, so, that is a transformer emf and then when the integrands of the above equation are equated, we get the famous Faraday's law (in differential form) as given below.

$$(\nabla \times \mathbf{E}) = -\frac{d\mathbf{B}}{dt}$$

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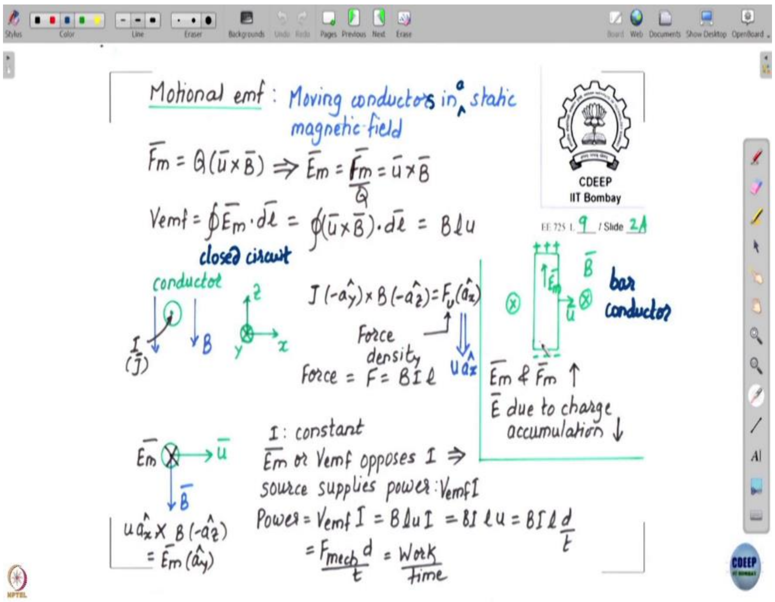


Now after seeing transformer emf, we will see motional emf, which gets induced when conductors move in a static magnetic field, so, the corresponding mechanical force (F_m) due to this induced emf and corresponding induced electric field intensity (E_m) are given by the following equation

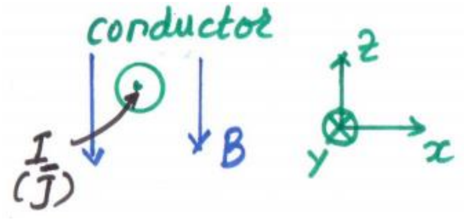
$$\bar{F}_m = Q(\bar{u}\times\bar{B}) \Rightarrow \bar{E}_m = \frac{\bar{F}_m}{Q} = \bar{u}\times\bar{B}$$

The induced electric field intensity due to the motional emf or due to the motion is given by F_m/Q that is given in the above equation.

Now, induced voltage will be again given by integral $\oint \mathbf{E}_m \cdot d\mathbf{l}$ and then in place of $\mathbf{E}_m$ you substitute $\mathbf{u} \times \mathbf{B}$ and then, if $\mathbf{u} \times \mathbf{B}$ is in direction of $d\mathbf{l}$ then this reduces to simple product Blu as given in the following equation. Here integral dl will be l and this l will be the length of the total closed circuit.

$$V_{emf} = \oint \bar{E}_m \cdot d\bar{l} = \oint(\bar{u}\times\bar{B})\cdot d\bar{l} = Blu$$

Now, let us go further and understand what happens to a conductor placed in static magnetic field which is shown in the following figure.



In this case, B is vertically down and conductor is placed there. Now, moment it is kept there you will have this B interact with this current through the following equation and the force density is given by the following equation.

$$J(-\hat{a}_y) \times B(-\hat{a}_z) = F_v(\hat{a}_x)$$

Force density

$$\Downarrow u\hat{a}_x$$

$$\text{Force} = F = BIl$$

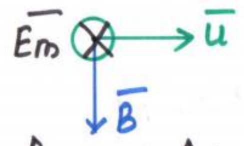
So, J is in $-\hat{\mathbf{a}}_y$ direction because, in the above figure, y is into the paper. J and current is coming out so, J cross B, B is in $-\hat{\mathbf{a}}_z$ direction, because, it is vertically down and that will give you force density in x direction as given in the above equation. Now, this force density will move the conductor in x direction with velocity $\mathbf{u}$. Earlier we saw how we can visualize the movement of current-carrying conductor in the vicinity of magnetic field.

Because moment you have this current in the vicinity of a field, it will have its own field around it and on one of its sides the field will get enhanced and on the other side field will get reduced, so, it will have some kind of stretching action of the conductor, here in x direction that is why this conductor would move.

And remember this force is acting on the free charges in the conductor because Lorentz force is defined on Q. So, the force is acting on free charges which constitute this current I. Going further, we already have derived the expression of force, i.e, BIl in one of the earlier lectures. In the above figure, the charges are moving in $-\hat{\mathbf{a}}_y$ direction and force is in x direction.

So, this force in x direction is not going to change the kinetic energy of the free charges in the conductor and that is why it is only going to change the direction but, it is not going to show any impact on kinetic energy. So, if the energy associated with that conductor is not going to change then, there won't be any work done by the magnetic force on the charges. Then, the question arises that who is doing the work to move the conductor?

Now, let us go further, when this conductor starts moving with velocity **u**, as per the formula $\mathbf{u} \times \mathbf{B}$, motional electric field intensity will be induced in this conductor as shown in the following figure.



$$u\hat{a}_x \times B(-\hat{a}_z) = \bar{E}_m(\hat{a}_y)$$

So, now this electric field intensity (E_m) will act in a direction to oppose the current as governed by Lenz's law. In the above case, current is coming out so, E_m direction will be going in.

Now, the moment E_m appears and it opposes the current and if we assume that I is constant, because when you are explaining some phenomena by some intuitive explanation we have to assume something as constant. So, here if we assume I as constant, that means the source which is supplying the power to this conductor is going to maintain current constant and then the corresponding voltage of the induced electric field intensity will oppose the source current flow and it will try to reduce the current.

So, this source then has to come into action and it will try to maintain the current as constant, in other words, it has to supply the corresponding power $V_{emf}I$, earlier when this V_{emf} was not there, the source was only supplying the I^2R loss kind of power, but, now this $V_{emf}I$ will be the additional power due to the induced emf that has to be supplied by the source. So, $V_{emf}I$ is the extra power that will be supplied by the source which is connected to this conductor and that will maintain constant current in the conductor.

The V_{emf} is derived as Blu. Substituting this $V_{emf} = Blu$ in the expression of power as given below.

$$\text{Power} = V_{emf}I = BluI = BIlu = BIl\frac{d}{t}$$
$$= \frac{F_{mech}\,d}{t} = \frac{\text{Work}}{\text{time}}$$

In the above derivation then rearranging this $BluI$ as $BIlu$ and BIl is nothing but mechanical force which we already saw in the previous lecture. Also, we have seen it before $u = \frac{d}{t}$ where d is the distance in the direction of motion of the conductor divided by time.

So, now you can see, this is the F_{mech} (= BIl) times distance upon time is work upon time, as given in the above derivation. This work is done by the corresponding power source which is supplying the conductor and that power is nothing but, $V_{emf}I$ and that is numerically equal to mechanical work done done in moving this conductor forward in the x direction divided by time.

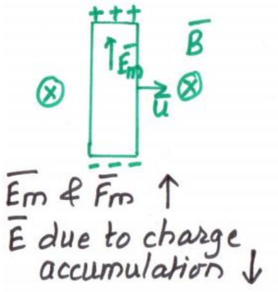
So, this explains that in this case the work is done by the source, which is maintaining the current constant, and the direction of the induced electric field intensity is given by $\mathbf{u} \times \mathbf{B}$.

Now, suppose if we want to say that this current is not constant then also it can be proved that it is the source that is doing work. How? The induced voltage and the corresponding opposition to the current will reduce the kinetic energy associated with the free charges in the conductor.

And that reduction in kinetic energy can be associated or can be equated to the mechanical work done and the corresponding kinetic energy gained by the conductor while moving it in the x direction. So, that way also it can be explained that magnetic fields do no work, if we do not assume that current is constant.

But remember this is all intuitive explanation and some assumptions are involved and for more detailed and comprehensive discussions on this topic you can refer Griffiths book on introduction to electrodynamics, 4th edition and you will find a comprehensive discussion on this whole topic with many case studies being explained.

A final point in this which I want to say is, till now what we sort of understood intuitively is that magnetic forces do not act on or do not do work on free currents which are due to the movement of free charges in the conductor as discussed in the previous case. But, if magnetic monopoles exist, if somebody finds magnetic monopoles in future, then the magnetic force can act on those individual magnetic charges (monopoles) as in case of electric field acting on charges electric charges. So, magnetic forces can act on monopoles (if they exist) and do work on them. Till now we saw a case of closed conductor that means conductor was part of a closed-circuit. Now, we will see what happens when we have a conductor which is like a open circuit, that means the conductor is a bar conductor and it is not connected to any closed circuit and it does not form a closed circuit.



Now let us assume that the conductor is placed in a region with B field going into the paper as shown in the above figure , here the B field is into the paper and let us assume that this conductor is moving in x direction and with velocity $\mathbf{u}$, in this case also, again $\mathbf{u} \times \mathbf{B}$ will result into the corresponding induced electric field intensity and the corresponding force on free charges in the conductor is in vertically upward direction.

For this conductor, there are many free charges and the E field will act on positive charges and they will move vertically up and negative charges will move vertically down as shown in the above figure. So, you will have charge accumulation at the two ends and this process will continue till the potential difference due to this charge accumulation which is vertically down will balance the corresponding force which is vertically up due to this charge separation on account of the motion of the conductor

So, this will continue until there is a balance between the 2 forces. So, here I wanted to bring forth this point that motional emf is generally in direction opposite to the electric field intensity due to the accumulation of charges.

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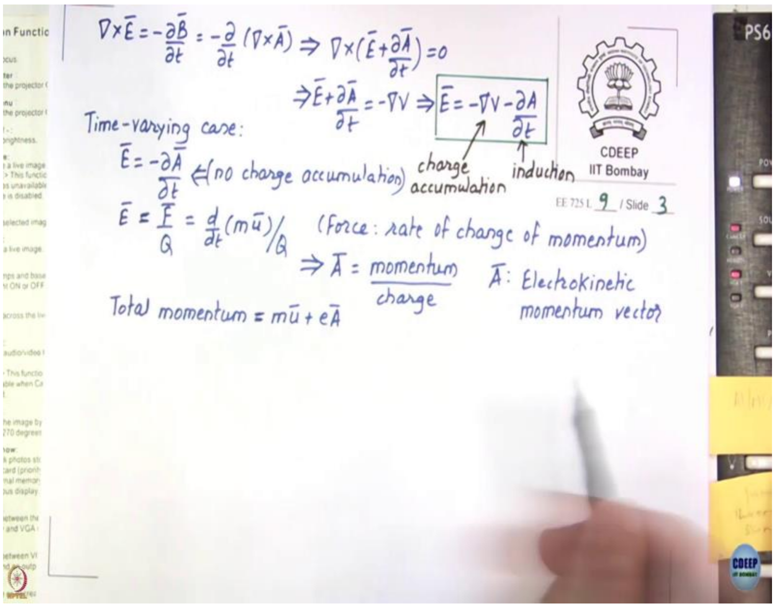

Now, we will see a little bit more on magnetic vector potential, because, this vector potential is used in our course when we are solving magnetic field problem by using FEM. Those formulations will be based on magnetic vector potential (A). So, it is worth spending a little bit more time on understanding magnetic vector potential.

Consider $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$ which is Faraday's law in point form. Now, replace B with $\nabla\times\mathbf{A}$ and then rearrange the expression, it gives the following equation:

$$\nabla\times\bar{E} = -\frac{\partial\bar{B}}{\partial t} = -\frac{\partial}{\partial t}(\nabla\times\bar{A}) \Rightarrow \nabla\times\left(\bar{E}+\frac{\partial\bar{A}}{\partial t}\right) = 0$$

We know the curl of a gradient is always 0. So, now this $\mathbf{E}+\frac{\partial\mathbf{A}}{\partial t}$ can be expressed as a gradient of some scalar potential as given below.

$$\Rightarrow \bar{E}+\frac{\partial\bar{A}}{\partial t} = -\nabla V \Rightarrow \boxed{\bar{E} = -\underset{\text{charge accumulation}}{\nabla V} - \underset{\text{induction}}{\frac{\partial A}{\partial t}}}$$

The above choice of potential is not arbitrary, it could have been any other scalar potential, but, we have purposely taken V here because, when we reduce the above equation to static case $\frac{\partial A}{\partial t} = 0$ and then $\mathbf{E} = -\nabla V$ and that is consistent with electrostatics.

So, **E** has to be gradient of V in case of electrostatics that is static fields and that is why we are choosing $\mathbf{E} + \frac{\partial \mathbf{A}}{\partial t} = -\nabla V$ and then you can rearrange this as given below:

$$\boxed{\bar{E} = -\nabla V - \frac{\partial A}{\partial t}}$$

charge accumulation   induction

The above equation is a sort of the total definition of E. The question asked by a student is why we are considering the definition of E as given in the above equation.

Because, this definition is consistent with the E field definition in electrostatics that we have already covered as explained earlier.

In the above equation, the $-\nabla V$ term is representing the charge accumulation and the $\frac{\partial A}{\partial t}$ term is representing the induction.

In most of the cases we analyze either of the 2 cases, that means when we solve problems either we are dealing with electrostatics then we are only considering $\mathbf{E} = -\nabla V$ or when we are doing magnetic field analysis, we do not consider charge accumulation. We purely consider the current flowing, induced voltages, forces, losses and all that so there, we are not considering the charge accumulation.

So, we will use either $-\nabla V$ term or $-\frac{\partial A}{\partial t}$ term whenever we deal with any practical problem in FE analysis. Now we will see a little bit more on the physical interpretation of magnetic vector potential, if you take no charge accumulation case then $\nabla V = 0$, so, then E reduces to $-\frac{\partial A}{\partial t}$. Actually E is also equal to F/Q, because F = QE. This F can act on static charge or moving charge, so, that is why this equation can be used for time varying case when Q is moving with varying speed. We also know that in mechanics force can be represented as the rate of change of momentum. So, the force can be expressed as given below, in the following equation where **u** is the velocity.

$$\bar{E} = \frac{\bar{F}}{Q} = \frac{d}{dt}(m\bar{u})/Q$$

Now, you compare the above equation with the following expression of **E** in terms of **A**.

$$\bar{E} = -\frac{\partial \bar{A}}{\partial t}$$

With this comparison you can infer that **A** is nothing but momentum per charge, that is the reason, Maxwell originally called magnetic vector potential (A) as electrokinetic momentum vector, which is the correct definition of **A**, unfortunately, what has happened is magnetic vector potential term has got established over the last few decades and but, it is not potential, and it is electrokinetic momentum vector and whenever we are considering particle dynamics, whenever electrons are moving with velocity **u** and the electron is having charge e then the total momentum of that electron will be given by the following equation.

$$\text{Total momentum} = m\bar{u} + e\bar{A}$$

In the above equation, m**u** is normal momentum because of its motion and in this e**A**, **A** is momentum per charge times charge e, will give you the momentum. Of course, the momentum m**u** we are not going to use in this course, but, those who are interested in particle dynamics this equation would be required at some point of time.

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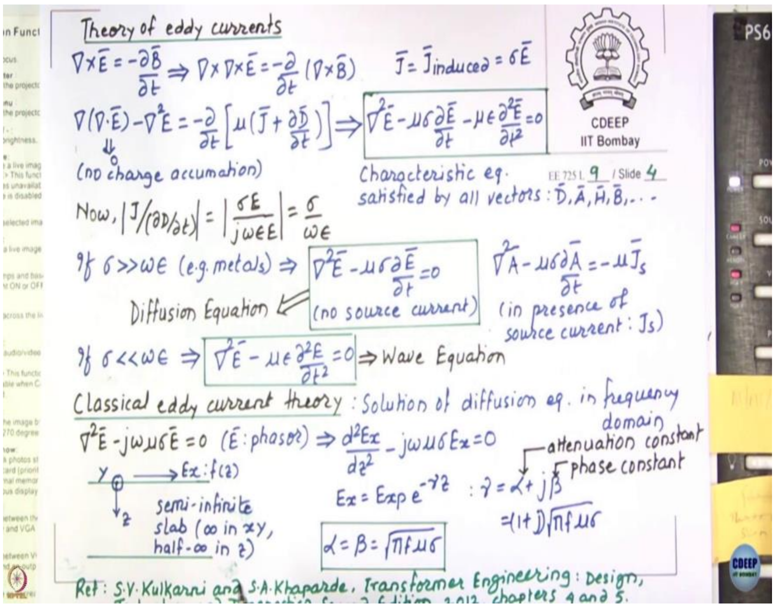

Now, we go to the theory of eddy currents, which is a very important topic while dealing with electrical machines, because we are dealing with induced voltages and currents and the corresponding losses that are occurring in machines. They are important to be estimated so that temperature rise within the electromagnetic device under investigation does not exceed certain limits.

Now, here we start the discussion again with one of the Maxwell’s equations that is Faraday's law $\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$ and then we take curl of both sides of the equation and then by applying vector identity of $\nabla \times \nabla$, the point form of Faraday's law is simplified as given below.

$$\nabla\times\bar{E} = -\frac{\partial \bar{B}}{\partial t} \Rightarrow \nabla\times\nabla\times\bar{E} = -\frac{\partial}{\partial t}(\nabla\times\bar{B})$$

$$\nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}\left[\mu\left(\bar{J} + \frac{\partial\bar{D}}{\partial t}\right)\right]$$

Here, we are talking of eddy currents. We will not consider charge accumulation, so, that is why this $\nabla.\mathbf{E}$ which is nothing but $\frac{\rho_v}{\epsilon} = 0$

So, the above expression reduces to the following equation.

$$\nabla(\nabla\cdot\bar{E}) - \nabla^2\bar{E} = -\frac{\partial}{\partial t}\left[\mu\left(\bar{J} + \frac{\partial \bar{D}}{\partial t}\right)\right] \Rightarrow \boxed{\nabla^2\bar{E} - \mu\sigma\frac{\partial \bar{E}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} = 0}$$

In the above equation on the left hand side, we have considered **J** as only induced current density. I will tell you the what are the other possible current densities and how the equation gets modified little later in this slide itself. So, **J** is induced current density, which is given by σ**E** . Here, **E** represents the electric field intensity which in turn represents induced voltage.

Now the final expression on the right hand side is a characteristic equation. We started with Faraday's law and arrived at this equation. If you start with some other equation and sort of eliminate all other variables except some maybe **B** variable, we will also get the same equation in B, similarly we can get the equation in **H** or **D**, so, the above equation is satisfied by all the vectors **D, A, H, B** and so on that we encounter in electromagnetics. So, this characteristic equation is also satisfied by **A**.

The characteristic equation in **A** that will be mostly used when we do two dimensional magnetic field calculations in this course is given below.

$$\nabla^2\bar{A} - \mu\sigma\frac{\partial \bar{A}}{\partial t} = -\mu\bar{J}_s$$

(in presence of source current : J_s)

So, before we understand the above equation, let us understand the following equation and its importance like when to use which term when we are dealing with either low frequency electromagnetics or high frequency electromagnetics.

$$\nabla^2\bar{E} - \mu\sigma\frac{\partial \bar{E}}{\partial t} - \mu\epsilon\frac{\partial^2\bar{E}}{\partial t^2} = 0$$

So, if you take the ratio of conduction to displacement current densities which is given by the following equation.

$$\left|J/(\partial D/\partial t)\right| = \left|\frac{\sigma E}{j\omega\epsilon E}\right| = \frac{\sigma}{\omega\epsilon}$$

In the above equation, σ is the conductivity, $\omega(= 2\pi f)$ is the frequency and ϵ is the permittivity.

Now there are two cases, if σ is much greater than $\omega\epsilon$, which is typically valid for metals being analyzed at the low frequency range. When I say low frequency, I am not talking like 1 Hz and I am talking about power frequency that means 50 Hz or 60 Hz, even 1 kHz is also low frequency as compared to GHz, THz which are in high frequency spectrum.

If for metals, we are interested in finding induced eddy currents and all that, typically the conductivity of metals is high, frequency is not that high and ϵ is quite low. Because the value of ϵ_0 is 8.8×10^{-12}, ϵ_r for a typical dielectric material used in electric equipment is less than 10 and that makes the ϵ value as a small number. So, the value of σ is much greater than the value of $\omega\epsilon$. So the last term of the characteristic equation with double derivative in time almost reduces to 0, because, that is representing the displacement current density. Because, that term is coming from the $\frac{\partial D}{\partial t}$ term of Ampere's law. So, we neglect the $\mu\epsilon\frac{\partial^2 E}{\partial t^2}$ term when we are analyzing eddy currents and the equation reduces to the following equation what is known as diffusion equation.

$$\nabla^2 \bar{E} - \mu\sigma\frac{\partial \bar{E}}{\partial t} = 0$$

And as I said earlier, if we had started with some other Maxwell's equation or did some substitutions then, we would have got diffusion equation in terms of **A** which is given in the previous page. In that equation, $\mu\sigma\frac{\partial A}{\partial t}$ and $\mathbf{J}_s$ represent eddy current loss and source current density respectively.

Because, in the previous slide itself we saw $E = -\frac{\partial A}{\partial t}$ is representing the induced effects. So, if the diffusion equation is expressed in terms of A, then $\sigma\frac{\partial A}{\partial t}$ represents the induced currents. In the diffusion equation in terms of E, there is no source current, now, you may wonder if there is no source how can eddy current be there?

For example, if there is some metal plate and then if there is some time varying excitation on that metal that will be produced by some source and that result in certain distribution of time varying field on the conductor surface and if you are analyzing the eddy currents in the metal plate then you do not require to model source current and what you need to model is only the boundary conditions. We have seen earlier to analyze fields by properly setting up the boundary conditions. So, similar thing is used here, in fact, classical eddy currents theory that we are going to see next will make use of diffusion equation in terms of E without source current and with only boundary conditions.

But there could be some cases where you may have source current as well as induced current, that means, suppose there is a current flowing in a conductor and in its vicinity there is some metal plate and eddy currents are induced in the plate. So, then you need to analyze both induced currents as well as the source current, in that case you may need to model the whole system. Then, both induced and source currents can be represented using the diffusion equation in terms of **A**.

Now, I think we covered Poison's equation in magnetostatics which is $\nabla^2 \mathbf{A} = -\mu \mathbf{J}$.

$$\nabla^2 \bar{A} - \mu\sigma \frac{\partial \bar{A}}{\partial t} = -\mu \bar{J}_s$$

In the above equation, if there is no time varying field term, then it will be $\nabla^2 \mathbf{A} = -\mu \mathbf{J}$ which is vector Poison's equation which we have seen earlier.

So the above equation is the more general form wherein both source current and eddy current are there. So, if you transfer the induced current term in the above equation on to the right hand side the sign of that will become plus. So, the sign of eddy current is plus and the sign of source and induced currents will be opposite which should be the case. Because, induced current generally opposes the source current, so, that is also consistent here.

The next possibility is σ is much less than $\omega\epsilon$, which is used to analyse dielectrics, in that case the second term is neglected and then what we get is wave equation which is given below.

$$\text{If } \sigma << \omega\epsilon \Rightarrow \boxed{\nabla^2 \bar{E} - \mu\epsilon \frac{\partial^2 \bar{E}}{\partial t^2} = 0}$$

And this wave equation is the starting point for analysing waveguides, antennas, transmission lines at high frequencies. The high frequency electromagnetics basically starts from this wave equation. We will stop here and continue this in the next lecture. Thank you.

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