Electrical Equipment and Machines: Finite Element Analysis

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Lecture 05

Revisiting EM Concepts: Current Densities and Electric Fields in Materials

Good morning, and welcome to the fifth lecture of this course. Yesterday after the end of the lecture, some students asked me few doubts. I will try to explain them with some introductory remarks in the first two slides.

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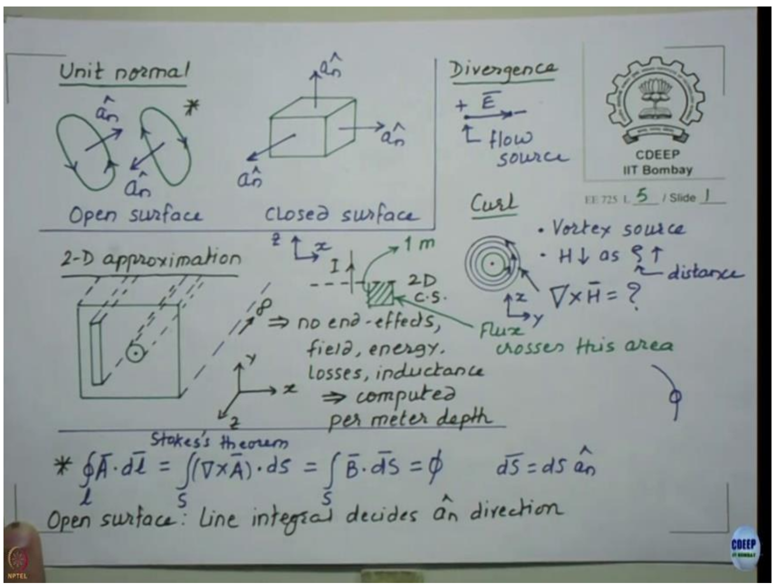
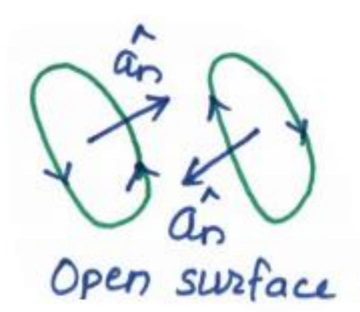
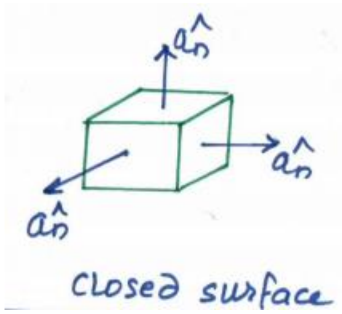


First of all, let us clearly understand, open surface, closed surface, and the unit normal concept. So, when you have an open surface, as I mentioned in the previous lecture, you have two possibilities, you can have a unit normal either in this direction (shown in the left-hand side part of the following figure) or in this direction (shown in the right-hand side part of the figure).



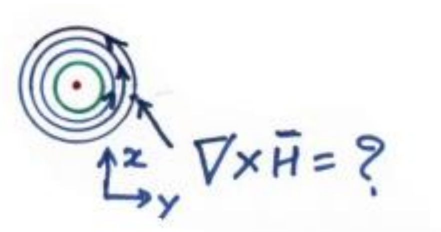
So, you have a choice. But if you are considering corresponding line or contour integral, for example, if you apply Stokes' theorem ($\oint \mathbf{A} \cdot d\mathbf{l} = \iint_S (\nabla \times \mathbf{A}) \cdot d\mathbf{S}$), then the corresponding contour integral decides the direction of this unit normal for the open surface. So, here you do not have a choice, if you are considering the corresponding line integral. So, that is the first point.

For a closed surface, it is always the outward normal. Consider the three surfaces I have shown here (shown in the following figure) and the corresponding normal is always the outward normal.



Now, let us consolidate our learning about divergence and curl. I was mentioning you in the last lecture that divergence of a vector is a flow source, why it is called as a flow source? Because at this positive charge, E vector is coming out and at the negative charge, this E vector is terminating and nothing is going out. So, at a positive charge, there is a positive flow source, and at a negative charge, there is a negative flow source that means flow stops or terminates at that point. So, divergence is little bit easy to understand.

Now consider curl of a vector, the well-known example to explain curl is field distribution around a current-carrying conductor shown in the following figure by a red dot and consider that the current is coming out.



The convention is always such that dot represents current coming out and cross represents current going into the paper. Here, flux contours are shown. Now, this is a vortex source (red dot in the above figure). Now, what is a vortex source? Basically, in many books it is explained by using the example of dropping a stone in water, then waves will be produced and those waves will basically go outward from that point where the stone was dropped and the intensity of waves will go on reducing as you go away from the point where the stone was dropped (vortex source).

So, exactly the same thing happens with the current carrying conductor, as you go away from this current source, the magnitude of field intensity reduces gradually because later on we will see $H = \frac{I}{2\pi R} or \frac{I}{2\pi\rho}$ in cylindrical system, where ρ is the distance from the source. So, another point to be noted here is, $\nabla \times \mathbf{H} = \mathbf{J}$ which is the point form of Ampere's law (one of the Maxwell's equations).

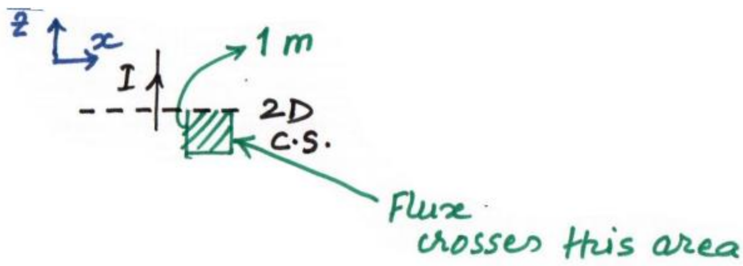
Now, if I ask you a question that there is an arbitrary point here somewhere (outside the conductor which is indicated in the above figure) and what is the value of $\nabla \times \mathbf{H}$**?** The way I have asked question, it has only two possible answers: either $\nabla \times \mathbf{H} = \mathbf{J}$ or $\nabla \times \mathbf{H} = 0$. So, we should remember that $\nabla \times \mathbf{H} = \mathbf{J}$ is the point form of Ampere's law.

When we ask a question, what is the $\nabla \times \mathbf{H}$ at a point? If J exist at that point, then $\nabla \times \mathbf{H}$ = J and not equal to 0. So, in this diagram $\nabla \times \mathbf{H} \neq \mathbf{0}$ only inside the conductor and it is 0 outside. Although you may feel that there is encircling of this field contours around the current source, and why should $\nabla \times \mathbf{H}$ be 0 at that point outside the conductor. Remember the basic definition of curl that we saw yesterday, curl exist only if there is a curling in the vicinity of that point because curl is defined at a point in space.

So, a question we have to ask here is whether there is curling around that point or not. For any point outside this current source, there is no local curling effect (no current source). What we see here is that these contours encircling the current source is an integral effect. In the denominator of definition of curl (given below), you have ΔS and we said $\Delta S \to 0$.

$$\nabla \times \bar{A} = \left[\lim_{\Delta S \to 0} \frac{\oint_L \bar{A} \cdot \bar{dl}}{\Delta S}\right]_{max} \hat{a}_n$$

So, if you take the third circle (in the above figure) and draw it here. We have to ask a question. Is there a curling effect around that point? There is no curling, that is why curl is 0. Whereas curl of H will be non-zero inside the conductor because current and J exist. So, this is the point to be noted. Mathematically also you can easily prove this, if you calculate curl at any outside point, curl will come identically equal to 0. So, that was the second point. Now, another thing we have discussed is 2D approximation. Let us consider the part a figure shown here.



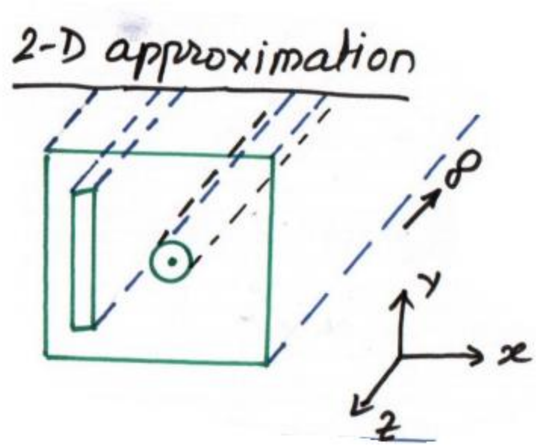
(a)

(b)

When we do 2D approximation we always take cross-section as xy plane and it is perpendicular to the current which is in z direction for most of the equipment and machine analysis. So, we take always cross- section normal to the current direction.

So, now we need to visualize this correctly, consider that current (shown in the above figure) is in z direction. In the figure, I have marked z, x directions, and y direction is into the paper. As I said, current is in z direction and we have shown the paper as xz plane (in the part a of the figure). So, y in this case is into the paper. So, now when current is in z direction the flux crosses this hashed area in the above figure. Flux is always identified with the corresponding crossing surface. Flux always flows through or crosses a surface. Now, in part b of the figure area is not visible because one of the dimensions of the area is into the paper. So, that is shown in part a of the figure where cross-sectional area is shown. The current is in z direction and one dimension is along x. The other dimension is into the paper (z in part b of the figure). So, the flux always will cross this area which is shown by this hashed area.

So, now coming to 2D approximation, suppose, you have a current source in the vicinity of a metallic bar as shown in the following figure and if you are interested to find how much flux is linked with this bar and how much loss is occurring in it in case of time varying fields.



So, remember in these initial introductory concepts that I am explaining, I am not going in a textbook kind of course content, like only considering electrostatics, then magnetostatics like that.

I am taking help of some concepts of time varying fields also while explaining these initial concepts. So, here, when you do 2D approximation for this problem, what essentially you will be doing in the FEM analysis is you will actually consider this z dimension as infinite as indicated in the above figure, that means whatever you see on this xy plane on the paper same thing repeats in the z direction up to infinity.

That means effectively there are no end effects. So, whatever field distribution you see in this x- y plane same thing will continue through and through at any z into the paper. So, that is the meaning of 2D approximation. When we do that, there are no end effects, and field, energy, losses, inductance and any other performance parameter are calculated per meter depth.

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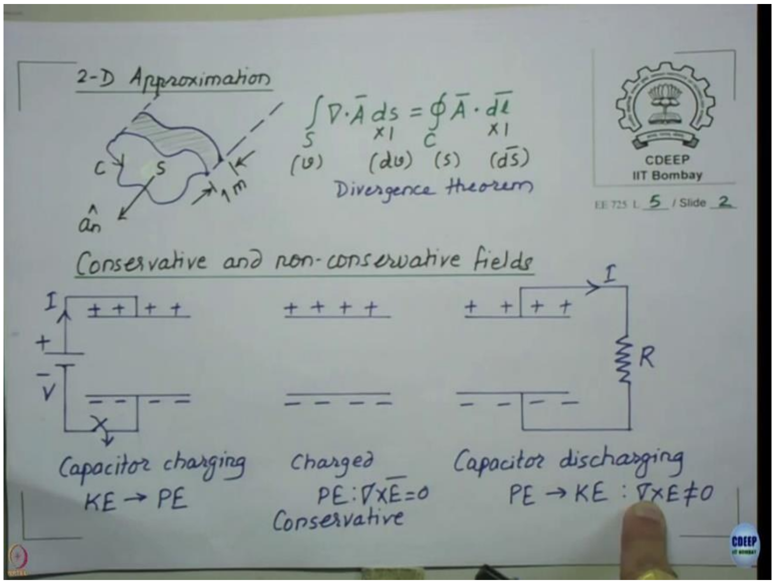
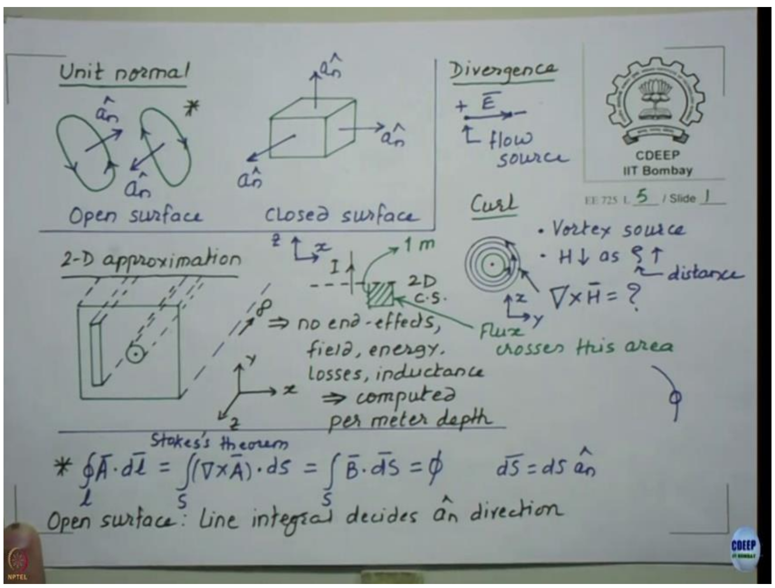
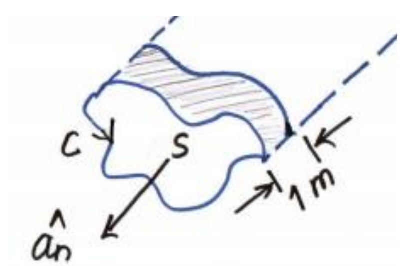




Going further into this 2D approximation. Now, consider an arbitrary surface which is indicated by this S and the corresponding contour is marked by C in the following figure. So, again to re-emphasize the point of 2D approximation, I have just drawn an arbitrary surface (shown below) and this is 1 meter depth (hashed region in the figure) in z direction.

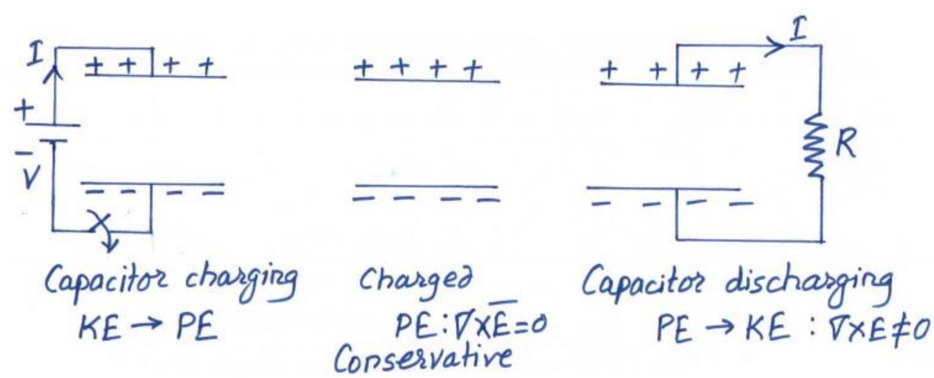


So, surface S is in the xy plane and $\hat{\mathbf{a}}_n$is in z direction. Now, we have $\iiint_v \nabla \cdot \mathbf{A}\, dv = \oiint_S \mathbf{A} \cdot d\mathbf{S}$ as the original form of divergence theorem. But here when we are doing 2D approximation, we are writing that dv as $dS \times 1$.

So, this $dS \times 1$ is the complete volume. Similarly, the right hand side of the above equation, which was originally $\oiint_S \mathbf{A} \cdot d\mathbf{S}$ but in 2D that $d\mathbf{S}$ is written as $dl \times 1$. So, when you have 2D approximation, the surface integral reduces to line integral. Note the difference between this divergence theorem applied with 2D approximation versus this Stokes' theorem in which an open surface integral is converted to a closed contour integral. Here also by using Stokes' theorem you are going from surface integration to line integration.

Same thing is done using the divergence theorem with 2D approximation. In this case also, you can go from an open surface integral to a closed contour integral. Now, again yesterday one of the participants asked question about these conservative and non-conservative fields. And I thought I will just explain little bit more elaborately about these two fields.

Three cases are shown in the following figure, wherein first the capacitor is charged by a battery by closing the switch. So here the current flows and capacitor gets charged.



When the current was flowing, the source is transferring the energy (potential energy) to this capacitor through the movement of electrons. So, always remember as I mentioned to you in the previous class, the energy gets transferred or work is done when electrons move and the current flows.

So, during the time when the capacitor is charged the source is transferring the energy through movement of these electrons and kinetic energy is stored as potential energy, when the capacitor is fully charged, current becomes 0, and there is no longer transfer of energy or there is no longer work done.

So, that is why the second case of the above figure is a representative of conservative field where the field is electrostatic field, and there is only potential energy which is stored in this capacitor and the corresponding $\nabla \times \mathbf{E} = 0$. So, it is a conservative field. This charged capacitor will not be able to do any useful work, unless it is connected to some circuit and current flows.

So, in the third circuit of the above figure, this capacitor is connected across a resistor, and now the capacitor discharges. This potential energy is converted into the kinetic energy of the electrons. Here, the electrons move, the current flows and then here work is done in terms of heat.

Remember, the manifestation of work in practical life can be done in at least three different ways, the first one is heat, second is motion for example in rotating machines and those examples, and third is light. Now, here in this case, heat is not a useful work because you are just wasting the energy but if it is some heating application, it will be a useful energy.

So, when this capacitor is being discharged for that time period the current will be flowing and again $\nabla \times \mathbf{E} \neq 0$ because some work is being done. And therefore, it is a non conservative field. And this is in sync with Maxwell's equation. That is whenever current flow is there, in fact here this current will be a function of time, because, initially the current will be high and at the end the current will become 0.

Since, I is changing with time, B also will change with time. So, $\frac{\partial B}{\partial t}$ will be non-zero during the transient period and therefore, $\nabla \times \mathbf{E} \neq 0$. So, I think we have discussed quite in detail about conservative and non-conservative fields and I hope this will be useful to you to remember.

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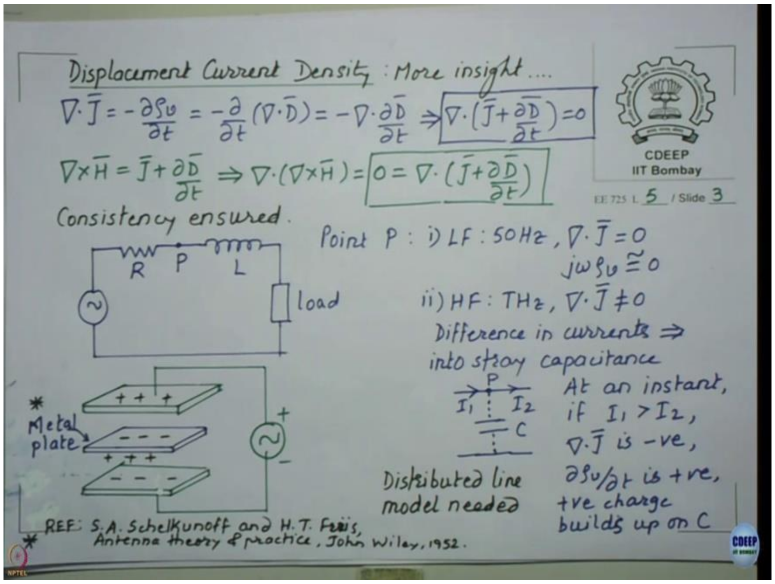


Now, in the last lecture we saw displacement current density. Displacement current density is applicable for time varying fields because it is $\frac{\partial D}{\partial t}$ but we are pre-empting that discussion because we are in general discussing current densities. We just made a comment in the last lecture that Maxwell introduced this $\frac{\partial D}{\partial t}$ term to make the whole system of equations consistent with continuity equation.

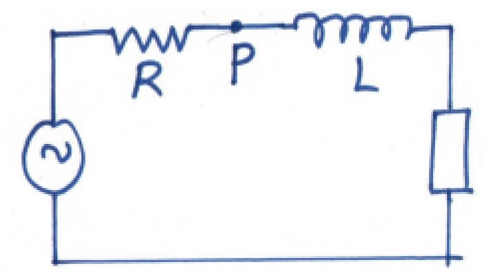
So, in just two steps it is proved here that yes indeed the consistency is obtained by adding $\frac{\partial D}{\partial t}$ term. Starting from $\nabla\cdot\mathbf{J} = -\frac{\partial\rho_v}{\partial t}$ which is continuity equation, in place of ρ_v in the equation you substitute $\nabla\cdot\mathbf{D}$, this leads to $\nabla\cdot\mathbf{J} = -\frac{\partial}{\partial t}\nabla\cdot\mathbf{D}$ and then you interchange the operators ∇ and $\frac{\partial}{\partial t}$ in the RHS. And why you can do that? Because they are sort of independent and whether you do divergence operation first or $\frac{\partial}{\partial t}$ operation first, the result is not going to change. Now the equation is simplified as

$$\nabla\cdot\mathbf{J} = -\nabla\cdot\frac{\partial\mathbf{D}}{\partial t}$$

After further rearranging you get $\nabla\cdot\left(\mathbf{J}+\frac{\partial\mathbf{D}}{\partial t}\right) = 0$ Now, for the point form of Ampere's law, which was $\nabla\times\mathbf{H} = \mathbf{J}$, if you add $\frac{\partial\mathbf{D}}{\partial t}$ to the RHS, and you take divergence and we know divergence of curl is always equal to 0.

So, that is why the divergence of this $\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ term will come equal to 0, which matches with the equation $\nabla \cdot \left(\mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}\right) = 0$ derived from the continuity equation. So, that makes the last Maxwell's equation (Ampere's law) consistent with the continuity equation which is the law of conservation of charge and it cannot be violated. Now, let us understand what this law of conservation of charge is because we are all familiar with Kirchhoff's current law.

Let us understand this displacement current density from the point of view of KCL. So, suppose you take a simple circuit (shown below) in which the source is connected to the load by some kind of transmission line.



Now, at low frequencies you can approximate it as simple R and L elements in series. And then at low frequency say at 50 Hz, $\nabla \cdot \mathbf{J} = 0$, because whatever current is coming in to the point P is going out.

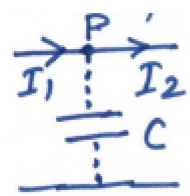
So, by definition of divergence, at this point divergence of current density will be 0 and that gets confirmed. Because $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$ and this has to be 0 at low frequencies because of this fact that current is same before and after the point P. So, indeed this is true because in frequency domain $\frac{\partial}{\partial t}$ is $j\omega$. So, this term (RHS of the continuity equation) basically becomes $j\omega\rho_v$ and this indeed is negligible or equal to 0 because the frequency $\omega(= 2\pi f)$ is very small.

Because remember ρ_v volume charge density or charges generally in practice are in nano or pico coulombs. In practice, you will never find 1 coulomb charge or 5 coulomb charge. So, this ρ_v is a very small number and if this ρ_v is multiplied with $2\pi 50$, it is going to become a negligible number.

So, it justifies that $\nabla \cdot \mathbf{J} = 0$ at low frequency. But now if frequency of this source is changed to 10^{13} or 10^{14} Hz then $j\omega\rho_v$ is not going to be a negligible number. That means $\nabla \cdot \mathbf{J} \neq 0$ at high frequencies.

Now, the question to be asked is $\nabla \cdot \mathbf{J}$ at this point is not 0 that means, whatever current is coming into point P is not equal to the current going out of that point. Always remember, you have to analyse divergence at a point because it involves partial derivatives.

That means the net current at point P has to go somewhere. The only place it can go is through this stray capacitance C (shown below).



So, for every transmission line, we know that it has stray capacitance, why stray because this upper line is at some potential and this lower line is at some other potential. So, there is an invisible stray capacitance and the net current will go through that. So, at any instant if suppose, we assume I_1 is greater than I_2 that means divergence at this point is negative, higher current is coming in and lower current is coming out. That means $\nabla \cdot \mathbf{J}$ is negative. So, if $\nabla \cdot \mathbf{J}$ is negative, then $\frac{\partial \rho_v}{\partial t}$ will become positive, which means at this point the charges are flowing into the capacitor and ρ_v is increasing with time. And positive charge builds up across this stray capacitance. But remember when we are analysing this circuit, we are actually analysing only a point in space. Effectively, what you can understand is this charge is moving out from this point and it is charging this capacitor.

Another issue here is, at higher frequencies this model is definitely not good, because there are concepts like transit time effect and electrical line length, wavelength (λ), etc., which I would have covered in detail if we were discussing high frequency electromagnetics, but since this course is on low frequency electromagnetics and electrical machines and equipment, I will make some passing comments that you have to use a distributed line model here.

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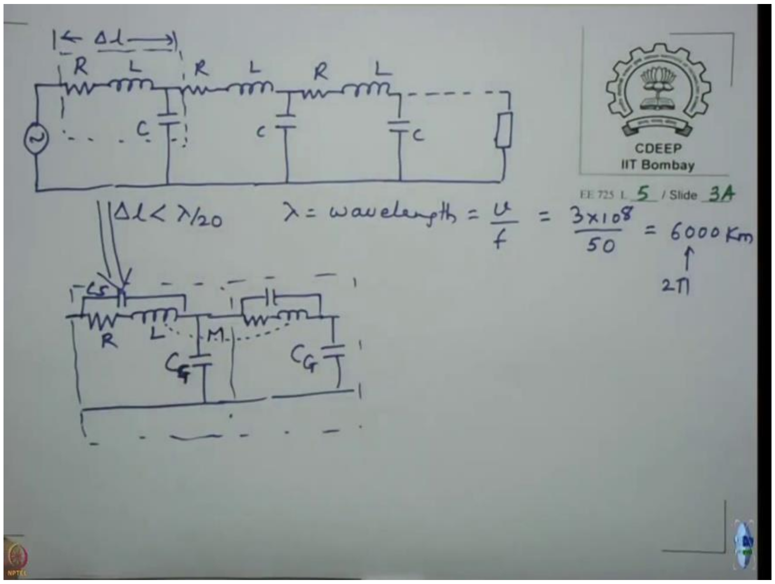

Now, this slide here shows distributed model of a transmission line wherein these R, L, and C are per unit length parameters of the line, and now here this whole transmission line is divided into number of sections. And the length of each section (Δl) has to be appropriately decided. Now, this Δl should be less than $\frac{\lambda}{20}$.

Now, what is λ? λ is the wavelength which is equal to $\frac{v}{f}$, where v is velocity of wave in the medium. Because $v = f\lambda$ and since we are talking in free space, energy and power travels in free space around the line. Later on, when we discuss Poynting vector, I will explain to you how the power flows along a transmission line.

So, till that time let us defer that discussion. So, since this is free space, here we have 3×10^8 m/s as velocity of light divided it by frequency, say 50 Hz, then λ is actually in km, $\frac{v}{f}$ of course in m but if we convert this into km, $\lambda = 6000$ km.

So, what is the meaning of λ then? In 6000 km, there is a phase change of 2π radians, but when transmission line is analysed from the point of view of transients, then we have to take the highest transient frequency of interest and then correspondingly this wavelength will reduce and then this Δl for each entity has to be less than $\frac{\lambda}{20}$ and λ being the wavelength corresponding to the highest frequency of interest in transient conditions.

Now, when this transmission line model is used for high frequency communication circuits. There of course, the frequency of operation is high (in MHz and GHz) and then accordingly the same rule applies. Then the corresponding λ will be much smaller and then this Δl should be less than $\frac{\lambda}{20}$ for accurate results, so, that this circuit analysis is very close to field analysis.

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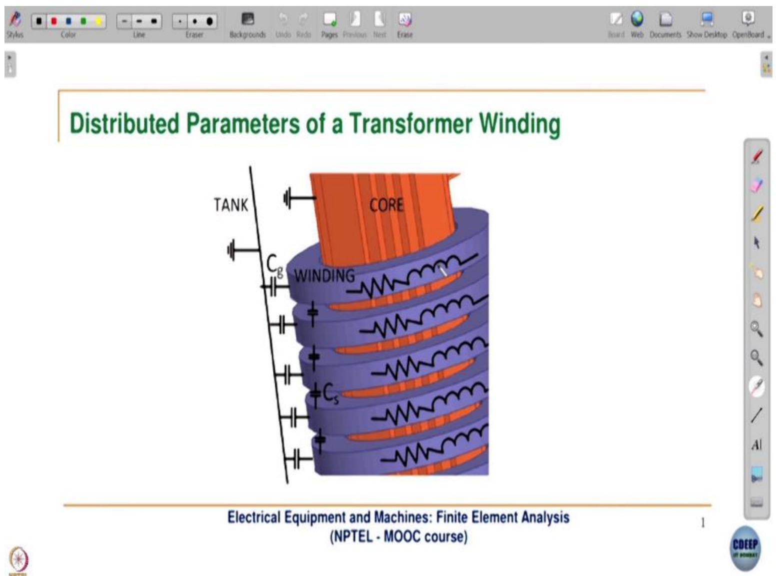
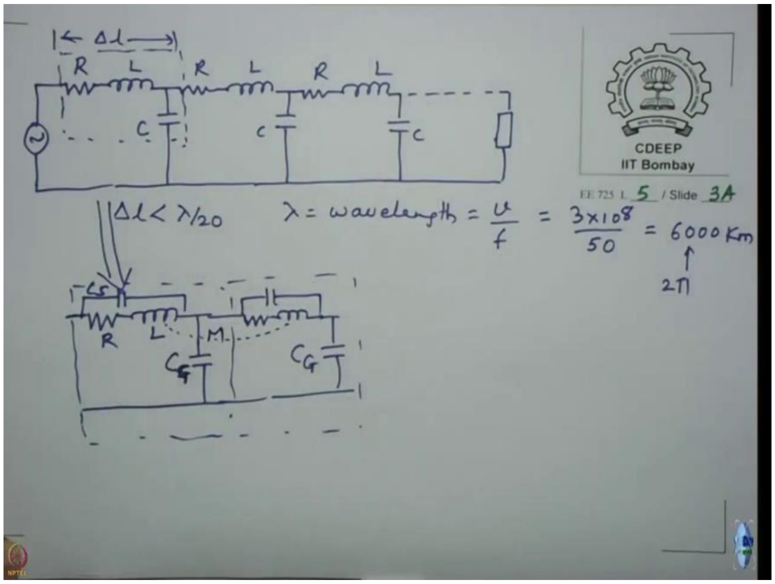
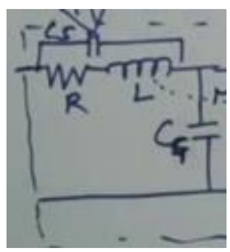


This slide shows core of a transformer, which is surrounded by a winding. Now, here you can see, various circuit components that will eventually appear in the distributed parameter model of this transformer. So, capacitances between winding and ground, winding and winding and these two discs, all those capacitances and these resistances and inductances of the winding will get represented in distributed parameter model as will be shown next.

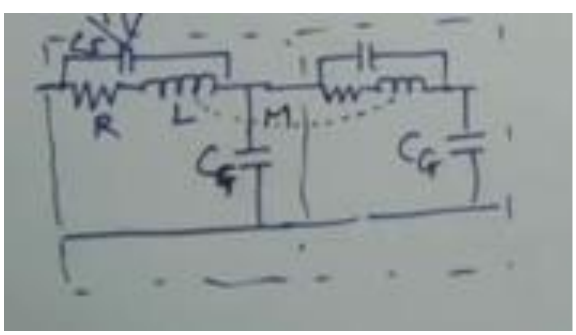
So, with reference to these series and shunt capacitances and the inductances that we saw in that slide for a transformer, this distributed line parameter model which was for transmission line has to be modified for the transformer in the following way.



So, I am showing just one entity. So, again, there will be resistance of winding, then inductance, then there will be series capacitance C_S and then ground capacitance C_G and one entity now is shown in the following figure.



There is one more difference here because you have this second section with the corresponding inductance, series capacitance and ground capacitance C_G. Now, apart from this self-inductance there will be mutual coupling also between these inductances as indicated in the following figure.



Because coils are wound very tightly, there will be mutual coupling between turns and between discs of the winding. So, there will be some mutual inductance also between the coils. So, all these parameters L, M and this C_S series capacitance and ground capacitance C_G, etc., can be calculated by using finite element analysis. And this is one of the very popular applications for doing FE analysis for transformers.

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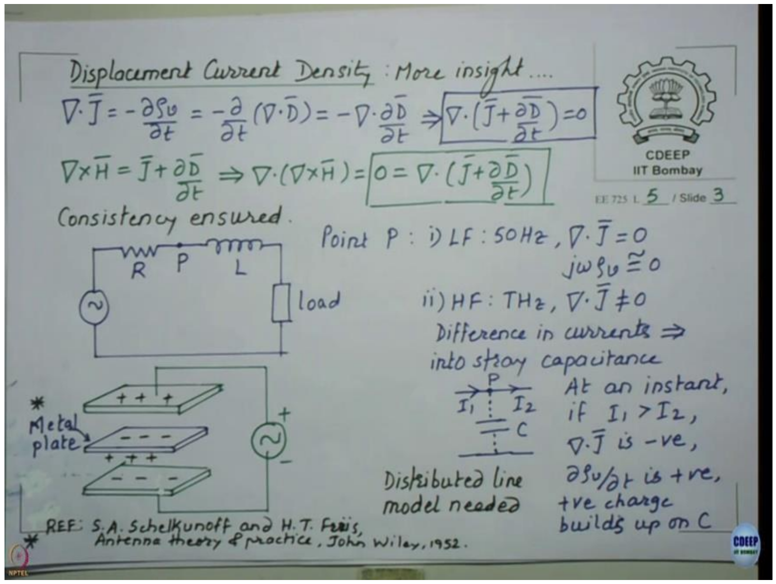

So, the last point on displacement current is explained using a parallel plate capacitor. These two plates are supplied by an alternating source in this case and you keep a floating metallic plate in between these two capacitor plates. Floating metallic plate means it is not electrically connected to any potential or circuit. Now in the positive half cycle, this upper plate will be positively charged, and the lower plate will be negatively charged. Now in this metallic plate the positive charges will go down, negative charges will get pulled up by columbic force of attraction.

So, there will be charge separation and what are these charges here in the floating plate? They are free charges. Now in the second half cycle what is going to happen, this condition will become reverse with negative charges on the lower surface and positive charges on the upper surface of the plate. So, now actually you zoom on that metallic plate and you can observe the movement of those free charges. And that will constitute to which type of current and current density? Is it conduction current density, convection current density or displacement current density? It constitutes to conduction current.

Because of free charges, it is conduction current density. There is a conduction current here (in the metallic plate). There is a displacement current here (in the dielectric between metal plate and capacitor plates). So, continuity is there, so this gives a very clear visualization of what is the meaning of displacement current density and the corresponding current.

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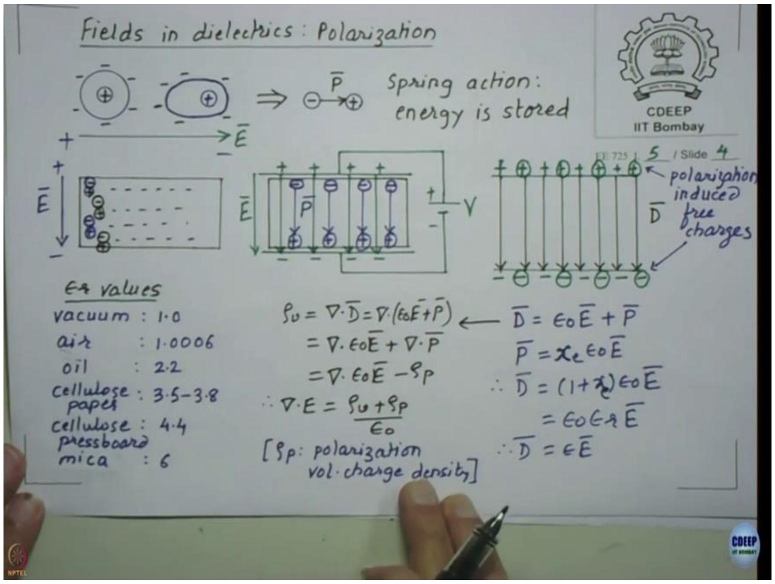
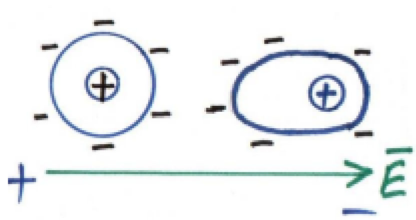
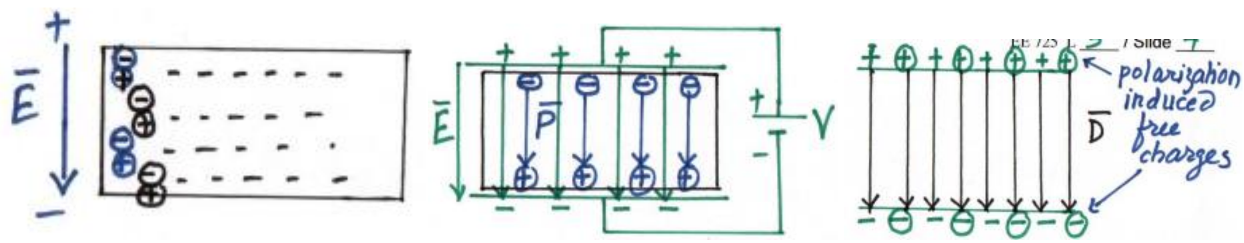


Next topic is fields in dielectrics and polarization. So, here we will analyse first the atomic model of a dielectric material and there is a central nucleus with positive charge and then there are negative charges around the nucleus. And now if you apply E field as shown in the above slide the positive charges will get pulled on this side (in the direction of E), negative charges will get pulled on this side (in the opposite direction of E) and this circular configuration becomes little bit elliptical and you can see negative charges are more concentrated on this side as compared to this side (as shown in the following figure).



And this is equivalent to a stretched spring and this electric field has done some work and stored potential energy in this what is known as electric dipole which is shown in the following figure.

Now there are millions of such atoms in an insulating material and the corresponding dipoles will get formed if the electric field is along this direction as shown in the part a of the following figure. Then you have all these dipoles getting formed and for simplified representation we can consider that these positive and negative charges get cancelled and effectively we can show the dielectric as polarized with polarization charges only on the surface as shown in the part b of the figure.



(a) (b) (c)

But remember the polarization happens throughout the volume of that dielectric. But this part b of the figure is just a simplified representation. Now, we are placing the dielectric inside a capacitor and you have the polarization vector which is from negative charge to positive charge as shown in part b of the figure.

And because of this dielectric being polarized in such a way, you have the corresponding positive and negative induced charges on this top and bottom surfaces of the plate. Moment this polarization occurs here, these charges are going to further push some electrons form this plate into the source. So, this plate will get further positively charged. That is what is shown by additional positive charges in circle (in part c of the figure) on the conductor and we are calling them as polarization induced free charges.

And that is how you get, additional positive and negative polarization induced free charges on these plates and then you have additional field lines in the dielectric region and when this dielectric was not there, the governing equation was only $\mathbf{D} = \epsilon_0 \mathbf{E}$. Now, because of this polarization phenomena, you have got additional field lines which are represented by this P vector.

So, effectively this additionally induced free charges on the top and bottom plates due to polarization and the corresponding field lines are representing this polarization phenomenon. And then polarization P is proportional to the applied E, the constant of proportionality is this electric susceptibility χ_e and then we know $1 + \chi_e = \epsilon_r$, here, ϵ_r is the relative permittivity and then finally you get $D = \epsilon_0 \epsilon_r E = \epsilon E$, where $\epsilon = \epsilon_0 \epsilon_r$.

Now, starting with this equation ($\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}$), you have, $\nabla \cdot \mathbf{D} = \rho_v$ and substituting the above expression of $\mathbf{D}$ in this equation, we get, $\nabla \cdot (\epsilon_0 \mathbf{E} + \mathbf{P}) = \rho_v$. So, if you now expand this divergence term on the LHS and then if you write $\nabla \cdot \mathbf{P} = -\rho_P$ (minus sign is because P vector is directed from negative charge to positive charge), and ρ_P is polarization volume charge density. So, then, by rearranging you get $\nabla \cdot E = \frac{(\rho_v + \rho_P)}{\epsilon_0}$ where, ρ_v is free volume charge density and ρ_P is the polarization volume charge density. So, that is the difference, divergence of D is always related to only free volume charge density, but divergence of E is related to free as well as polarization volume charge density, if both exist.

These ϵ_r values are of importance for electrical machines and equipment and for this course, and typical values of dielectric constants are mentioned here. The value of ϵ_r for vacuum is 1, that of air is very close to 1, for mineral oil used in oil filled transformers is 2.2, for cellulose paper, solid insulation, again used in transformers and bushings and some other high voltage equipment, it is 3.5 to 3.8, for cellulose pressboard (for thick pressboard) it is 4.4, and that of mica is 6. So, I think we will stop here and continue the rest of the topics in next lecture. Thank you.

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