Electrical Equipment and Machines: Finite Element Analysis
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Lecture No. 37
Transient FE Analysis

Welcome to lecture 37. Now, we will go to the next level of complexity by considering transient analysis.

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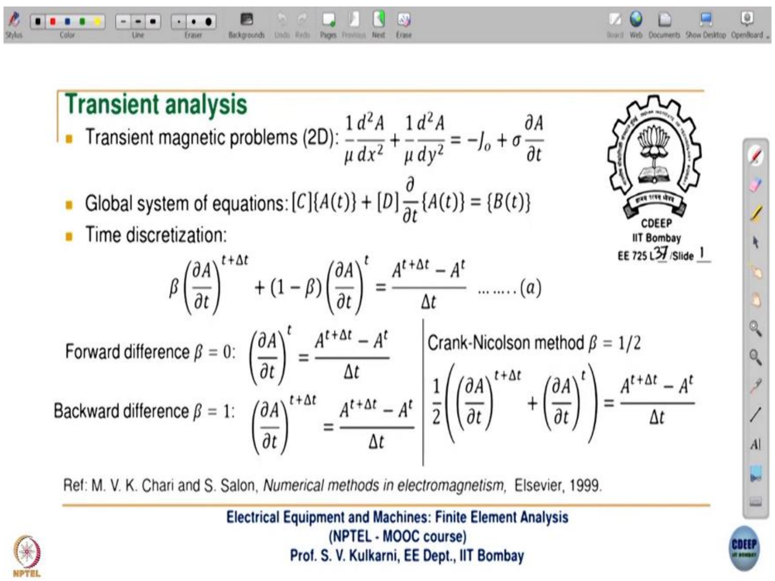

The governing partial differential equation is given below

$$\frac{1}{\mu}\frac{d^2A}{dx^2}+\frac{1}{\mu}\frac{d^2A}{dy^2}=-J_o+\sigma\frac{\partial A}{\partial t}$$

In this problem, we are considering conductivity and the corresponding diffusion term. So, the global system of equation after applying FE formulation will be

$$[C]\{A(t)\}+[D]\frac{\partial}{\partial t}\{A(t)\}=\{B(t)\}$$

So, the term $[D]\frac{\partial}{\partial t}\{A(t)\}$ represents the diffusion term and σ in PDE is directly reflecting into the D matrix. The Laplacian operator gets reflected into the global coefficient matrix (C matrix) which is a function of geometry and material properties. The column vector {A} is a column matrix of magnetic vector potentials and this matrix is a function of time t.

The most popular time discretization technique is given below.

$$\beta\left(\frac{\partial A}{\partial t}\right)^{t+\Delta t} + (1-\beta)\left(\frac{\partial A}{\partial t}\right)^{t} = \frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

In this equation, $A^{t+\Delta t}$ means magnetic vector potential at $t + \Delta t$, A^{t} is magnetic vector potential at time t and Δt is the time step. β is just a number between 0 and 1. So, if $\beta = 0$ then the first term on the left hand side is zero and then the second term $\left(\frac{\partial A}{\partial t}\right)^{t}$ will remain. So the above equation reduces to the following expression.

$$\left(\frac{\partial A}{\partial t}\right)^{t} = \frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

So, with $\beta = 0$, we have expressed the partial derivative with respect to time as a function of forward difference. Because we are representing the derivative of A with respect to time as a function of forward difference of $A^{t+\Delta t}$ and A^{t}. The backward difference will be for $\beta = 1$ and it can be expressed as

$$\left(\frac{\partial A}{\partial t}\right)^{t+\Delta t} = \frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

In the above equation, the partial derivative at $t + \Delta t$ is a function of this backward difference. Because we are representing the derivative as a function of the backward difference of $A^{t+\Delta t}$ and A^{t}.

Then, the third approach is Crank Nicolson method for $\beta = \frac{1}{2}$ which is represented as

$$\frac{1}{2}\left(\left(\frac{\partial A}{\partial t}\right)^{t+\Delta t} + \left(\frac{\partial A}{\partial t}\right)^{t}\right) = \frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

In the above equation, we represent the average of partial derivatives of A with respect to time at $t + \Delta t$ and t as $\frac{A^{t+\Delta t} - A^{t}}{\Delta t}$.

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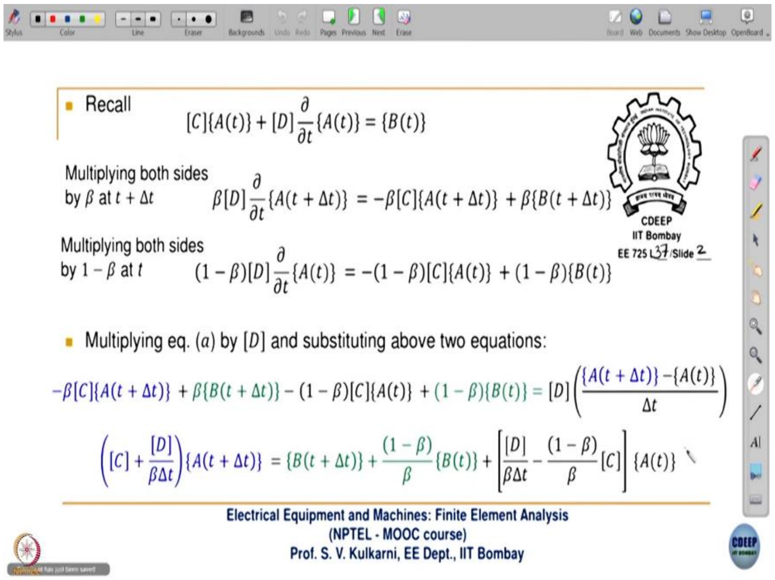

Now we start with the following equation which we have seen on the last slide.

$$[C]\{A(t)\} + [D]\frac{\partial}{\partial t}\{A(t)\} = \{B(t)\}$$

If we multiply the both sides of the above equation by β at time instant $t + \Delta t$ then we will get the following expression.

$$\beta[D]\frac{\partial}{\partial t}\{A(t+\Delta t)\} = -\beta[C]\{A(t+\Delta t)\} + \beta\{B(t+\Delta t)\}$$

Then, multiplying the previous equation on both sides by $1-\beta$ at t we will get the following equation.

$$(1-\beta)[D]\frac{\partial}{\partial t}\{A(t)\} = -(1-\beta)[C]\{A(t)\} + (1-\beta)\{B(t)\}$$

So, we multiply the time discretization formulation by D matrix so that we will get,

$$\beta[D]\left(\frac{\partial A}{\partial t}\right)^{t+\Delta t} + (1-\beta)[D]\left(\frac{\partial A}{\partial t}\right)^{t} = [D]\frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

We have to do that because in the previous two expressions, we have $\beta\frac{\partial}{\partial t}\{A(t+\Delta t)\}$ and $(1-\beta)[D]\frac{\partial}{\partial t}\{A(t)\}$ which can be eliminated using the above equation. So, by substituting the expressions of these two terms in the above equation we will get

$$-\beta[C]\{A(t+\Delta t)\} + \beta\{B(t+\Delta t)\} - (1-\beta)[C]\{A(t)\} + (1-\beta)\{B(t)\} = [D]\left(\frac{\{A(t+\Delta t)\} - \{A(t)\}}{\Delta t}\right)$$

Now, we will combine the terms which are having common factors in the whole equation. The terms marked in blue are combined because $A(t+\Delta t)$ factors are common in both terms and terms marked in green ($B(t)$ and $B(t+\Delta t)$) are taken together. Then the third set of terms ($A(t)$ and $A(t+\Delta t)$) which are in black color are combined together. We have also divided the equation by β and then the whole equation reduces to

$$\left([C] + \frac{[D]}{\beta\Delta t}\right)\{A(t+\Delta t)\} = \{B(t+\Delta t)\} + \frac{(1-\beta)}{\beta}\{B(t)\} + \left[\frac{[D]}{\beta\Delta t} - \frac{(1-\beta)}{\beta}[C]\right]\{A(t)\}$$

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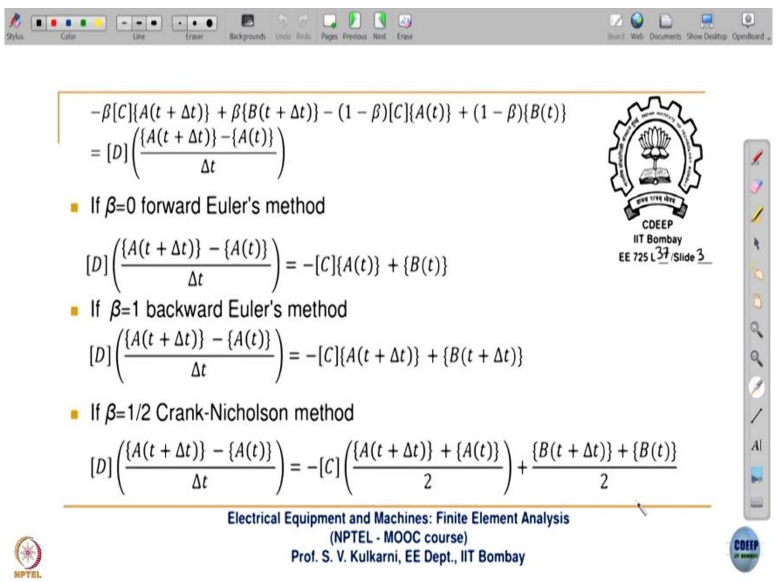

Now, starting with the following expression which is rewritten from the equation that we have seen in the previous slide

$$-\beta[C]\{A(t+\Delta t)\} + \beta\{B(t+\Delta t)\} - (1-\beta)[C]\{A(t)\} + (1-\beta)\{B(t)\}$$
$$= [D]\left(\frac{\{A(t+\Delta t)\} - \{A(t)\}}{\Delta t}\right)$$

From this equation we can derive the following three schemes that emerge for $\beta = 0$, $\beta = 1$, and $\beta = \frac{1}{2}$. So if $\beta = 0$ is substituted in the above equation then we can obtain forward Euler's method which is governed by the following equation.

$$[D]\left(\frac{\{A(t+\Delta t)\} - \{A(t)\}}{\Delta t}\right) = -[C]\{A(t)\} + \{B(t)\}$$

If $\beta = 1$ is substituted in the generalized expression then we will get backward Euler's method governed by the following equation.

$$[D]\left(\frac{\{A(t+\Delta t)\} - \{A(t)\}}{\Delta t}\right) = -[C]\{A(t+\Delta t)\} + \{B(t+\Delta t)\}$$

If $\beta = \frac{1}{2}$ is substituted then it leads to Crank Nicolson method governed by the following expression.

$$[D]\left(\frac{\{A(t+\Delta t)\} - \{A(t)\}}{\Delta t}\right) = -[C]\left(\frac{\{A(t+\Delta t)\} + \{A(t)\}}{2}\right) + \frac{\{B(t+\Delta t)\} + \{B(t)\}}{2}$$

For $\beta = \frac{1}{2}$, we have the average of potentials at $A(t+\Delta t)$ and $A(t)$.

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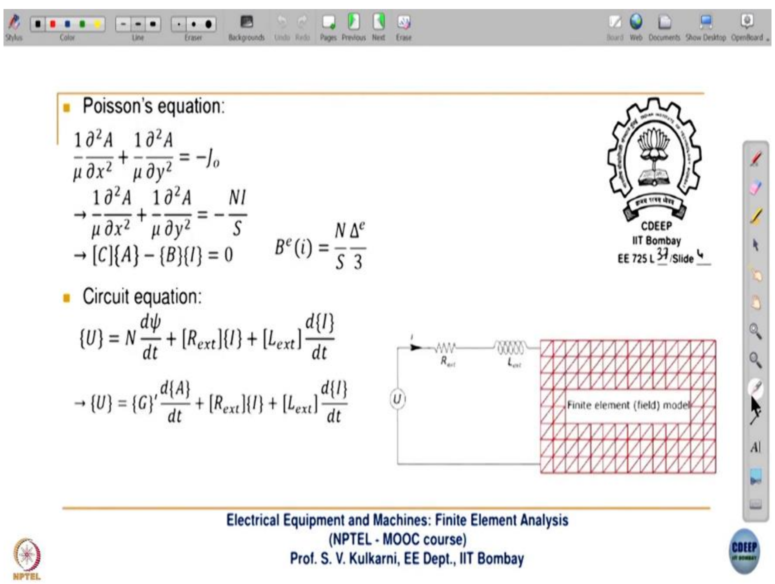


Let us see how to derive finite element formulation for Poisson's equation coupled to a circuit equation in time domain. The governing PDE for the field domain is given in the following equation.

$$\frac{1}{\mu}\frac{\partial^2 A}{\partial x^2} + \frac{1}{\mu}\frac{\partial^2 A}{\partial y^2} = -J_o$$

In the above equation, J is replaced by $\frac{NI}{S}$ as we had done in previous lectures and the equation leads to the following matrix equation.

$$[C]\{A\} - \{B\}\, I = 0$$

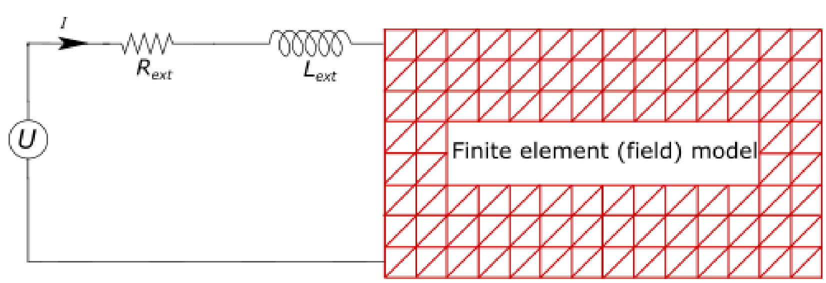
The i^{th} entry of element level B matrix is

$$B^e(i) = \frac{N}{S}\frac{\Delta}{3}$$

In the previous lectures, we have seen formulations for current driven and voltage driven systems. In this slide, we will see voltage driven system in which I is unknown. So, the corresponding circuit equation is

$$\{U\} = N\frac{d\psi}{dt} + [R_{ext}]\{I\} + [L_{ext}]\frac{d\{I\}}{dt}$$

We have seen this equation earlier. Consider the problem domain shown in the following figure



Now the above equation governs the circuit domain and $N\frac{d\psi}{dt}$ is the corresponding terminal voltage of the field domain. Then if we apply FE formulation, we will get the following matrix equation which we have seen earlier

$$\{U\} = \{G\}'\frac{d\{A\}}{dt} + [R_{ext}]\{I\} + [L_{ext}]\frac{d\{I\}}{dt}$$

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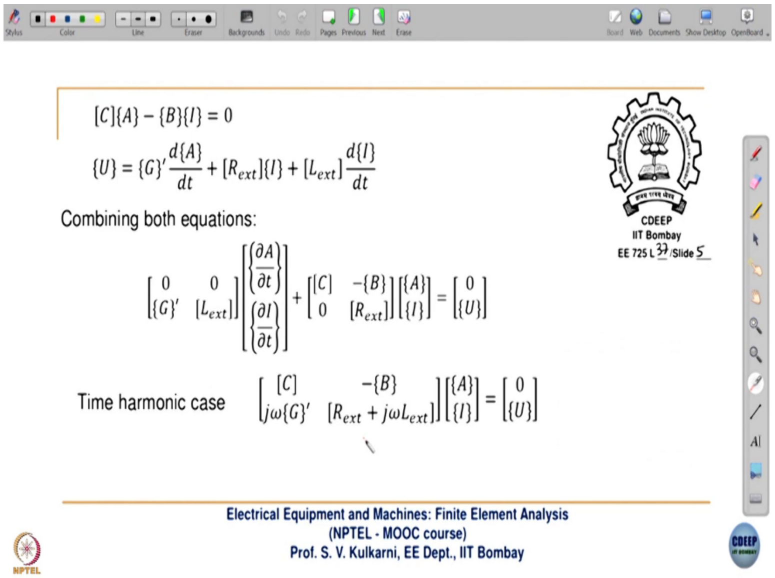

The two governing equations of the coupled field system are given below.

$$[C]\{A\}-\{B\}\{I\}=0$$

$$\{U\}=\{G\}'\frac{d\{A\}}{dt}+[R_{ext}]\{I\}+[L_{ext}]\frac{d\{I\}}{dt}$$

These two equations are identical to the equations that we had seen earlier for a voltage fed circuit coupled to field model in frequency domain. Combining these two equations, we get the following matrix equation

$$\begin{bmatrix}0 & 0\\ \{G\}' & [L_{ext}]\end{bmatrix}\begin{bmatrix}\left\{\frac{\partial A}{\partial t}\right\}\\ \left\{\frac{\partial I}{\partial t}\right\}\end{bmatrix}+\begin{bmatrix}[C] & -\{B\}\\ 0 & [R_{ext}]\end{bmatrix}\begin{bmatrix}\{A\}\\ \{I\}\end{bmatrix}=\begin{bmatrix}0\\ \{U\}\end{bmatrix}$$

In the first equation there are no $\frac{\partial A}{\partial t}$ and $\frac{\partial I}{\partial t}$ terms so that is why we have 0 and 0 in the first row of the above equation.

In the second equation, we have $\{G\}'$ getting multiplied by $\frac{\partial A}{\partial t}$. Similarly, $[L_{ext}]$ is getting multiplied to $\frac{\partial I}{\partial t}$. The first row of the second matrix equation represents field equation $[C]\{A\}-\{B\}\,I=0$. So, the first row of the matrix has $[C]$ and $-\{B\}$ which are multiplied by $\{A\}$ and I. The second row of the second matrix equation represents resistance drop $[R_{ext}]I$. Then in the right hand side of the above equation, we have $\{U\}$ from the second (circuit) equation and $\{0\}$ from the first (field) equation.

If we write the above equation in frequency domain, we will get the following equation seen earlier.

$$\begin{bmatrix} [C] & -\{B\} \\ j\omega\{G\}' & [R_{ext} + j\omega L_{ext}] \end{bmatrix} \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix} = \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}$$

The previous equation was in time domain but the same equation is converted into time harmonic case by replacing $\frac{\partial}{\partial t}$ by $j\omega$, which is given in the above equation.

In the time domain equation, if $\frac{\partial}{\partial t}$ is replaced by $j\omega$ then $\{G\}'$ gets multiplied with $j\omega$. Since there is no $\frac{\partial}{\partial t}$ term we can take $\begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}$ as common and $[L_{ext}]$ gets multiplied by $j\omega$. So, then we get the above matrix equation which we had seen earlier. Since we are dealing with time domain or transient simulation, we will be working with the previous equation.

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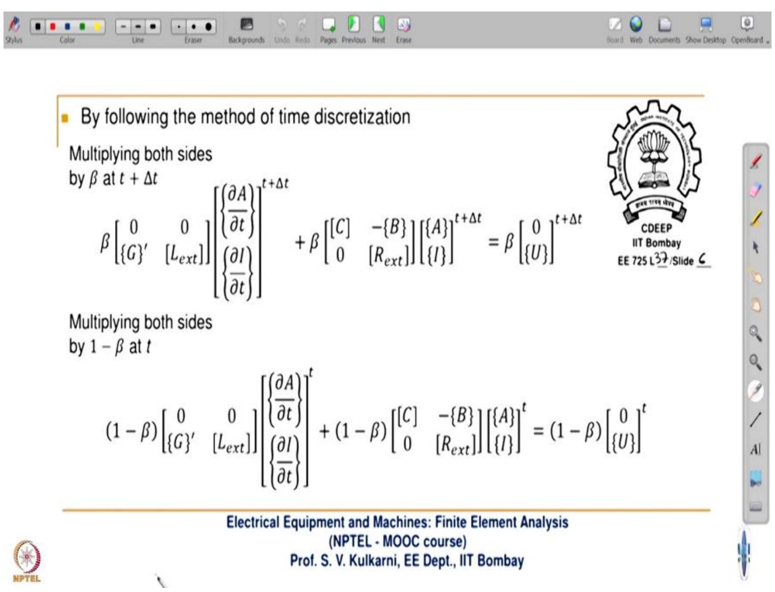

Now, by following the method of time discretization procedure we multiply the time domain equation at $t + \Delta t$ with β then we get

$$\beta \begin{bmatrix} 0 & 0 \\ \{G\}' & [L_{ext}] \end{bmatrix} \begin{bmatrix} \left\{\dfrac{\partial A}{\partial t}\right\} \\ \left\{\dfrac{\partial I}{\partial t}\right\} \end{bmatrix}^{t+\Delta t} + \beta \begin{bmatrix} [C] & -\{B\} \\ 0 & [R_{ext}] \end{bmatrix} \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t+\Delta t} = \beta \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t+\Delta t}$$

Again we multiply the equation at time t by $1 - \beta$ we get

$$(1-\beta) \begin{bmatrix} 0 & 0 \\ \{G\}' & [L_{ext}] \end{bmatrix} \begin{bmatrix} \left\{\dfrac{\partial A}{\partial t}\right\} \\ \left\{\dfrac{\partial I}{\partial t}\right\} \end{bmatrix}^{t} + (1-\beta) \begin{bmatrix} [C] & -\{B\} \\ 0 & [R_{ext}] \end{bmatrix} \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t} = (1-\beta) \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t}$$

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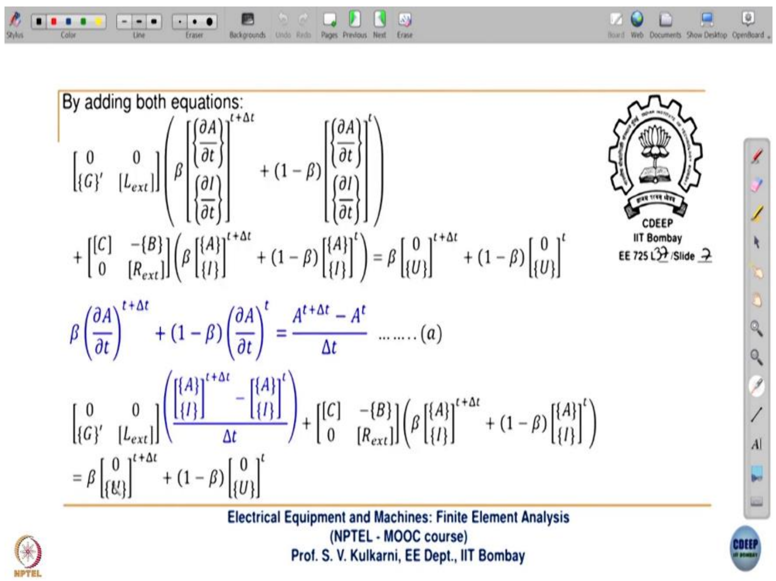

So now, we add both the above equations and we will get the following equation

$$\begin{bmatrix} 0 & 0 \\ \{G\}' & [L_{ext}] \end{bmatrix} \left(\beta \begin{bmatrix} \left\{\dfrac{\partial A}{\partial t}\right\} \\ \left\{\dfrac{\partial I}{\partial t}\right\} \end{bmatrix}^{t+\Delta t} + (1-\beta) \begin{bmatrix} \left\{\dfrac{\partial A}{\partial t}\right\} \\ \left\{\dfrac{\partial I}{\partial t}\right\} \end{bmatrix}^{t} \right)$$
$$+ \begin{bmatrix} [C] & -\{B\} \\ 0 & [R_{ext}] \end{bmatrix} \left(\beta \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t+\Delta t} + (1-\beta) \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t} \right) = \beta \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t+\Delta t} + (1-\beta) \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t}$$

The time discretization procedure is governed by the following equation.

$$\beta \left(\frac{\partial A}{\partial t} \right)^{t+\Delta t} + (1-\beta) \left(\frac{\partial A}{\partial t} \right)^{t} = \frac{A^{t+\Delta t} - A^{t}}{\Delta t}$$

By using the above equations we get

$$\begin{bmatrix} 0 & 0 \\ \{G\}' & [L_{ext}] \end{bmatrix} \left(\frac{\begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t+\Delta t} - \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t}}{\Delta t} \right) + \begin{bmatrix} [C] & -\{B\} \\ 0 & [R_{ext}] \end{bmatrix} \left(\beta \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t+\Delta t} + (1-\beta) \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t} \right)$$
$$= \beta \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t+\Delta t} + (1-\beta) \begin{bmatrix} 0 \\ \{U\} \end{bmatrix}^{t}$$

In the above equation, we can see that there are some common terms and multiplicands.

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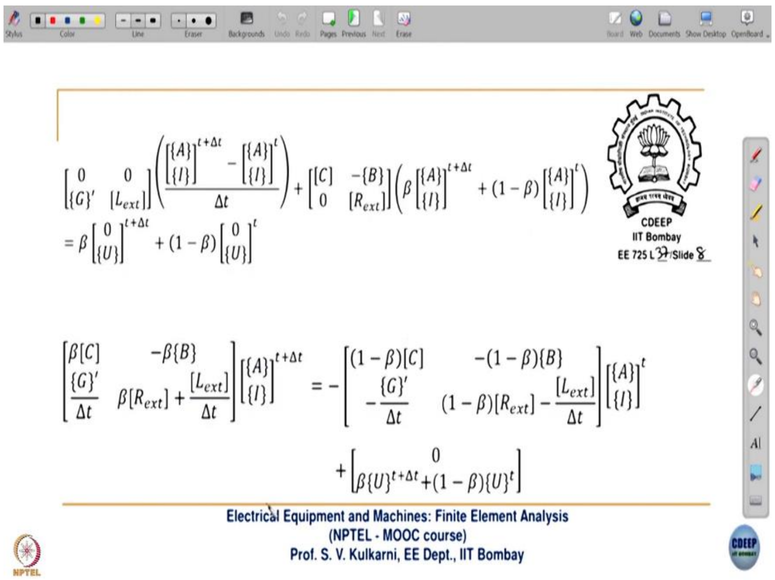

The same equation is written again in the above slide. Since this is a transient formulation we have to bring $t + \Delta t$ terms on one side and all t terms on the other side as given in the following equation.

$$\begin{bmatrix} \beta[C] & -\beta\{B\} \\ \frac{\{G\}'}{\Delta t} & \beta[R_{ext}] + \frac{[L_{ext}]}{\Delta t} \end{bmatrix} \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t+\Delta t} = - \begin{bmatrix} (1-\beta)[C] & -(1-\beta)\{B\} \\ -\frac{\{G\}'}{\Delta t} & (1-\beta)[R_{ext}] - \frac{[L_{ext}]}{\Delta t} \end{bmatrix} \begin{bmatrix} \{A\} \\ \{I\} \end{bmatrix}^{t}$$
$$+ \begin{bmatrix} 0 \\ \beta\{U\}^{t+\Delta t} + (1-\beta)\{U\}^{t} \end{bmatrix}$$

In a transient formulation, as we are marching in time, we already have the values of potential variables at time t and calculate the values of variables at $t + \Delta t$.

In the above equation, we are just rearranging the variables or terms. So, the above equation is our final equation for implementation in finite element method. When we discretize the time and field domain by using finite element procedure, all the matrices in the above equation can be determined by following our usual FE procedure. So, global coefficient matrix C, matrices B and G, R_{ext} and L_{ext} are known.

Depending upon the time discretization procedure, we select the value of β and it can be either 0 or 1 or $\frac{1}{2}$. So, depending upon the β value the corresponding time discretization will be chosen. So, with this, we complete the theory of transient FE formulation. In next lecture, we will see non-linear formulation. Thank you.

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