Electrical Equipment and Machines: Finite Element Analysis
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Lecture No 36
Current Fed Coupled Circuit Field Analysis

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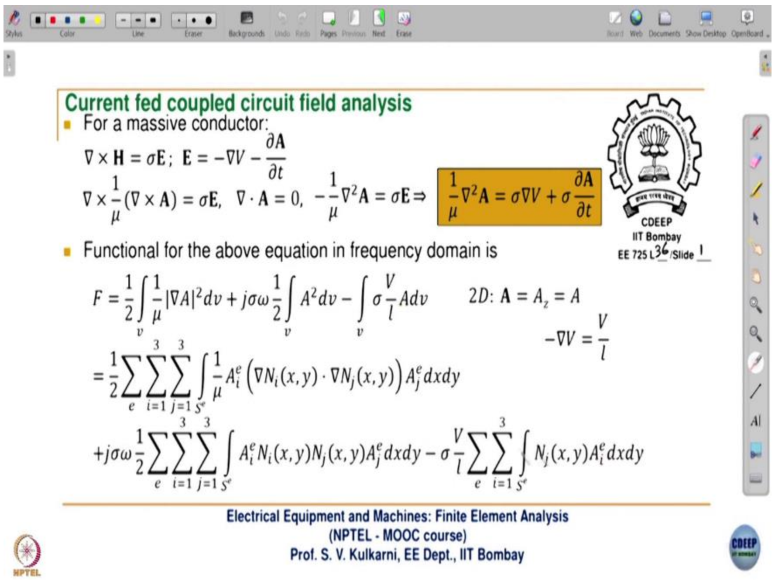

Welcome to 36th lecture. In this lecture, we will see current fed coupled circuit field analysis. In many of our problems related to electrical machines and equipment, we generally have current fed systems. But most of our devices are voltage fed. We make the analysis as a current fed FE analysis to simplify it because the current passing through a coil is generally known to us while formulating the analysis.

Generally, we should remember that all the devices are typically voltage fed. But for analysis purpose, we take as current fed cases for understanding in depth behavior of field distribution and the corresponding performance parameters, more about it when we see actual examples. So, when we say massive conductor we are considering eddy currents in that conductor.

That means if eddy current losses are considered then $\sigma \frac{\partial A}{\partial t}$ (eddy current term) has to be considered. Now, we start with $\nabla \times \mathbf{H} = \mathrm{J}$ which is one of the Maxwell's equations with displacement current density being neglected because we are modelling the low frequency fields.

In the Ampere's law, **J** is replaced by $\sigma\mathbf{E}$ and **E** is replaced by $-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}$. Remember that ∇V is not on account of charge accumulation, but it represents the voltage that is impressed across the terminals of the coil. Going further, **H** in the Ampere's law is replaced by $\frac{\mathbf{B}}{\mu}$ and **B** is replaced by $\nabla \times \mathbf{A}$. So, the curl equation will be modified as given below.

$$\nabla \times \frac{1}{\mu}(\nabla \times \mathbf{A}) = \sigma\mathbf{E}$$

We will invoke Coulomb gauge $\nabla \cdot \mathbf{A} = 0$ which we have seen in basics of electromagnetics.

Generally, $\nabla \cdot \mathbf{A} = -\mu\epsilon\frac{\partial V}{\partial t}$. This equation is called as Lorenz gauge. But if the frequency is small then $\nabla \cdot \mathbf{A} = 0$ has to be invoked, because $\nabla \times \nabla \times \mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) - \nabla^2\mathbf{A}$ which is a vector identity. After invoking $\nabla \cdot \mathbf{A} = 0$ we are left with the following equation,

$$-\frac{1}{\mu}\nabla^2 A = \sigma \mathrm{E}$$

We replace this **E** expression with $-\nabla V - \frac{\partial \boldsymbol{A}}{\partial t}$.

$$\frac{1}{\mu}\nabla^2 A = \sigma\nabla V + \sigma\frac{\partial A}{\partial t}$$

It is mentioned earlier that the right hand side of the above equation $\sigma\nabla V$ is representing the source current density and $\sigma\frac{\partial A}{\partial t}$ is representing induced current density. So, the functional for the above equation in the frequency domain is

$$F = \frac{1}{2}\int_v \frac{1}{\mu}|\nabla A|^2 dv + j\sigma\omega\frac{1}{2}\int_v A^2 dv - \int_v \sigma\frac{V}{l}A dv$$

The first term in the above equation is given by the following equation

$$\frac{1}{2}\sum_e \sum_{i=1}^{3}\sum_{j=1}^{3}\int_{Se}\frac{1}{\mu}A_i^e\left(\nabla N_i(x,y)\cdot\nabla N_j(x,y)\right)A_j^e dxdy$$

We have seen this expression in the previous lecture and this will give us the global coefficient matrix. By combining all element coefficient matrices we will get the global coefficient matrix.

Then the second term of the functional is obtained by replacing $\sigma\frac{\partial A}{\partial t}$ by $j\omega A$. So, $j\omega A$ becomes $j\omega A^2$ in the functional. We have seen earlier that if we take the terms except $\frac{1}{\mu}\nabla^2 A$ on the right hand side then the sign of those terms gets reflected in the corresponding functional terms. This we have seen when we studied functionals. The ½ multiplicand comes with the first two terms of functional because there is a square term.

The ∇V term on the right hand side of the PDE is $-\frac{V}{l}$ because the gradient of voltage is from low to high and electric field (E) is from high to low. That is why $\nabla V = -\frac{V}{l}$ because $\frac{V}{l}$ is basically directed from positive to negative. Effectively, $\mathrm{E} = -\frac{V}{l}$ because E is directed from positive to negative.

So, $\nabla V = -\frac{V}{l}$ is substituted in the PDE that is why we have the minus sign for this term in the functional expression. Since there is no A with this term in the PDE then $\frac{V}{l}$ in the functional will get multiplied by A. After FEM discretization, A is replaced by $\sum_{i=1}^{3} N_i A_i^e$ and then after simplification we will get two summation terms for the first two integrals in the functional expression. For the third term, $\sigma\frac{V}{l}$ is anyway constant and A is replaced by $\sum_{i=1}^{3} N_i A_i^e$ and then summed over all the elements.

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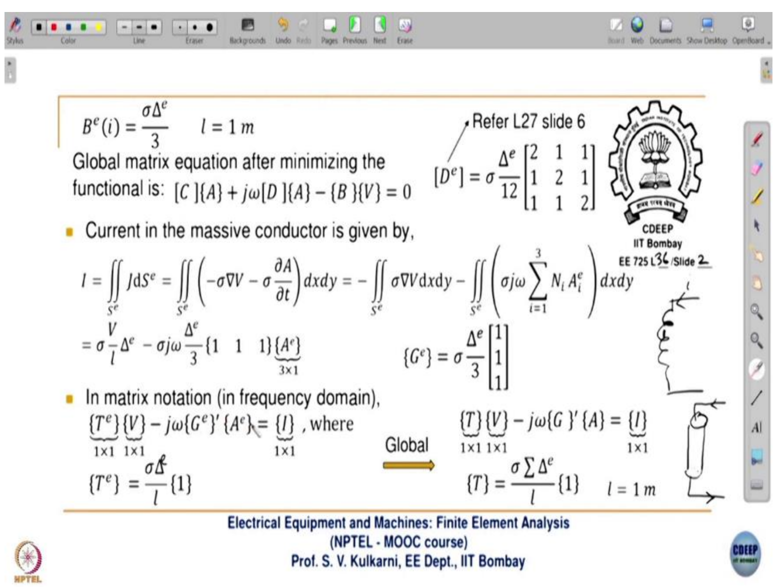
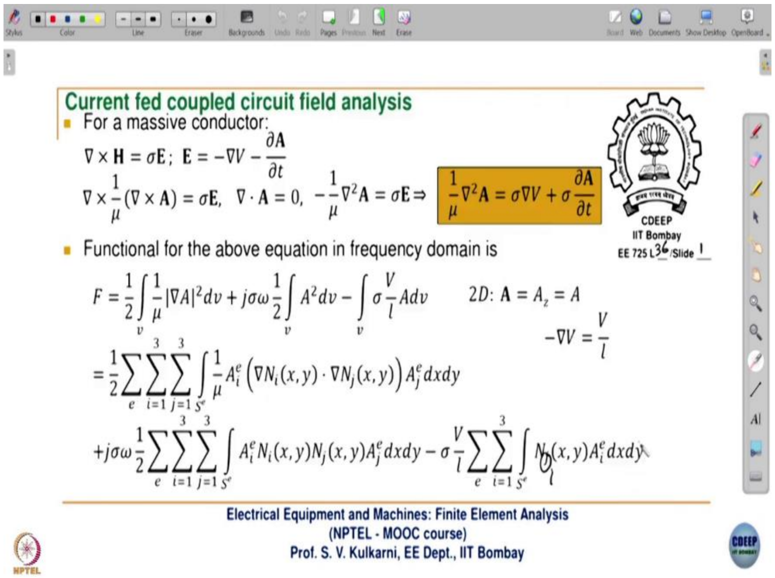




We will get the entries of element level B^e matrix as $\sigma \frac{\Delta}{3}$ because we have $\sigma \frac{V}{l}$ as constant. Now, if we are doing 2D approximation then $l = 1$. So, entries of B^ematrix are just $\sigma \frac{\Delta}{3}$. Here, V is the unknown variable because this is a current fed formulation system and voltage V is unknown. In the following term, σV is constant and $l = 1$

$$\sigma \frac{V}{l} \sum_e A_i^e \sum_{i=1}^{3} \int_{Se} N_i(x,y) dxdy$$

The $\int_{Se} N_j(x,y)dxdy$ term reduces to $\frac{\Delta}{3}$ and the term A_i^e will be cancelled in the process of minimization. Since we have considered $l = 1$, we are doing per meter depth calculation.

So, $\sigma \frac{V}{l}\frac{\Delta}{3}$ will go to element level B matrix and V goes to the unknown matrix when we form the global coefficient matrix.

In the following matrix equation, $[C]$ is our normal coefficient matrix which is multiplied with the corresponding unknown magnetic vector potentials which is column vector $\{A\}$ with all unknown nodal magnetic vector potentials.

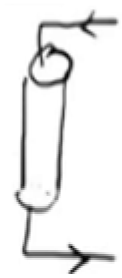
$$[C\ ]\{A\} + j\omega[D\ ]\{A\} - \{B\}\{V\} = 0$$

The $j\omega[D\ ]\{A\}$ is the same what we had seen in the diffusion equation solution (Refer L27 Slide 6). So, $[D]$ matrix represents the diffusion term in the PDE. The element level $[D^e]$ is appended to form global $[D]$ matrix as we have seen earlier.

The current in the massive conductor is defined by the following equation.

$$I = \iint_{S^e} J \mathrm{d}S^e$$

Suppose, we are modelling a massive conductor (shown in the following figure) which is fed by some current source I.



When the current enters the massive conductor then it will have eddy currents and source current defined by the above equation. As we are considering conductivity, then there will be eddy currents in the conductor. So the current inside this massive conductor will be the superimposition of the source current and the corresponding induced eddy currents. But the total current (I) will remain the same because KCL has to be satisfied.

The total current in any cross section of the massive conductor will be $I = \iint_{S^e} \mathbf{J} \cdot \mathrm{d}\mathbf{S}^e$. If we integrate the current density at any cross section of the conductor will give the terminal current. Remember that terminal current is due to combination of the source current and induced currents.

Now, $\mathbf{J}$ in $\iint_{S^e} \mathbf{J} \cdot \mathrm{d}\mathbf{S}^e$ is replaced by $-\sigma\nabla V - \sigma\frac{\partial A}{\partial t}$ and expression of current in the conductor gets modified as

$$I = \iint_{S^e} J \mathrm{d}S^e = \iint_{S^e} \left(-\sigma\nabla V - \sigma\frac{\partial A}{\partial t}\right) dxdy$$

So, the corresponding source current is $\sigma \frac{V}{l}$, l is the length of the conductor and V is the terminal voltage. Then, $\frac{V}{l}$ is voltage per unit length and remember the terminal voltage V is unknown. In this problem, current fed to the conductor is known.

In the previous case, some known voltage was applied to one of the windings and the other winding was short-circuited. Also, currents in both windings were unknown. In the case of a massive conductor fed by a current source in which eddy currents are induced the terminal voltage across the conductor is unknown.

Going further, after separating the terms and replacing A by $\sum_{i=1}^{3} N_i A_i^e$ and following usual FEM discretization procedure we get the following equation.

$$I = -\iint_{S^e} \sigma \nabla V \mathrm{d}x\mathrm{d}y - \iint_{S^e} \left(\sigma j\omega \sum_{i=1}^{3} N_i \, A_i\right) dxdy = \iint_{S^e} \sigma \frac{V}{l} \mathrm{d}x\mathrm{d}y - \iint_{S^e} \left(\sigma j\omega \sum_{i=1}^{3} N_i \, A_i\right) dxdy$$

We are substituting $\nabla V = -\frac{V}{l}$. Now, $\sigma \frac{V}{l}$ is constant so $\iint_{S^e} \sigma \frac{V}{l} dxdy$ will simply give the area of the triangle. The second integral in the above equation will give $j\omega \frac{\Delta}{3}$. The unknown A_i will go into a column vector. The rest of the term gives $j\omega\sigma \frac{\Delta}{3}$. Then the above equation reduces to the following matrix equation.

$$I = \sigma \frac{V}{l} \Delta^e - \sigma j\omega \frac{\Delta^e}{3} \{1 \quad 1 \quad 1\} \underbrace{\{A_e\}}_{3\times1}$$

The above matrix equation is in frequency domain. In this equation, the term $\sigma j\omega \frac{\Delta^e}{3} \{1 \quad 1 \quad 1\}$ represents element level G' matrix that we saw in the previous lecture. Again here, since A is a column vector at the element level, transpose of G is taken so that the matrix multiplications become row vector into column vector and gives 1×1 matrix. So, the size of element level G' matrix 1×3 and element level A matrix is 3×1 so the product of two matrices will be 1×1 which matches with the right hand side variable whose size is 1×1.

The element level T matrix will have one entry whose value is $\sigma\frac{\Delta^e}{l}$ and V goes in to the variable matrix. When we change the T matrix to global matrix then Δ^e of all the elements in the conductor will get added and result into the cross sectional area of the conductor.

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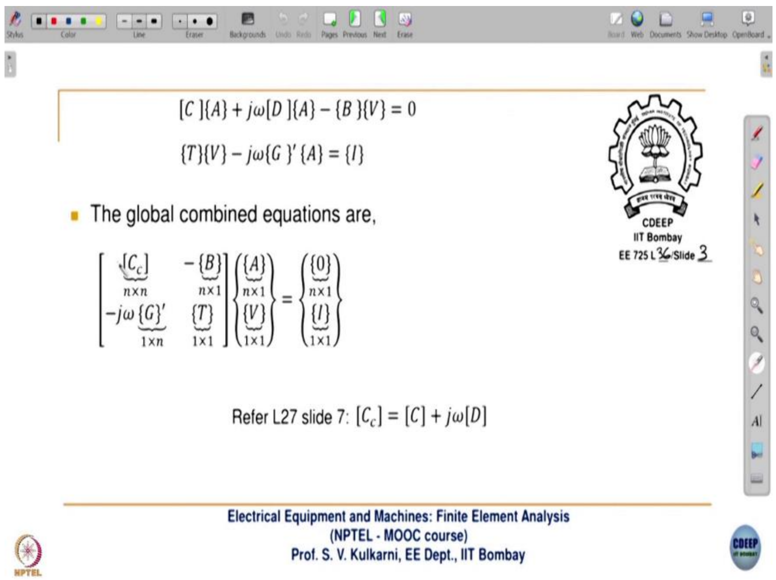

The field and circuit equations can be written as

$$[C]\{A\} + j\omega[D]\{A\} - \{B\}\{V\} = 0$$

$$\{T\}\{V\} - j\omega\{G\}'\{A\} = \{I\}$$

The first equation is field equation and the second equation is circuit equation. So we have to remember that in this coupled system of equations, there are coupling parameters. In the circuit equation, the coupling parameter is A and it represents field domain in the circuit equation whereas, in the field domain equation, the coupling parameter is the terminal voltage (V).

The above two equations are not independent, because we are solving a coupled circuit field system. So, there have to be coupling quantities in the two equations. A and V are the coupling quantities. In the previous lecture, A and I were coupling quantities and since we had two windings, so I was representing I_1 and I_2.

Then, I_1, I_2 and A were the coupling quantities in voltage fed coupled circuit field case which we have seen in the previous lecture. In this formulation, V and A are the coupling quantities. So, now the global combined equations are in the following matrix equation

$$\begin{bmatrix} \underbrace{[C_c]}_{n\times n} & -\underbrace{\{B\}}_{n\times 1} \\ -j\omega\underbrace{\{G\}'}_{1\times n} & \underbrace{\{T\}}_{1\times 1} \end{bmatrix} \begin{Bmatrix} \underbrace{\{A\}}_{n\times 1} \\ \underbrace{\{V\}}_{1\times 1} \end{Bmatrix} = \begin{Bmatrix} \underbrace{\{0\}}_{n\times 1} \\ \underbrace{\{I\}}_{1\times 1} \end{Bmatrix}$$

In the above equation, there is just one modification, we are combining $[C\ ]$ and $j\omega[D\ ]$ as we did in the diffusion case and we are calling it complex global coefficient matrix.

So, $[C_c] = [C] + j\omega[D]$, refer lecture 27 slide 7 for this. $[C_c]$ is the global system of equations. The dimension of $[C_c]$ is $n \times n$. The dimensions of other matrices are indicated in the above matrix equation. The size of the square matrix on the left hand side will be $(n+1) \times (n+1)$, unknown matrix is $(n+1) \times 1$. Also, the size of right hand side matrix is $(n+1) \times 1$.

Then we solve the above matrix equation. Here, the right hand side column vector is known. The square matrix on the left hand side is also known. The T matrix is function of σ, and G and B matrices are $\sigma\frac{\Delta}{3}$. So, all the quantities in the square matrix are known. $[C_c]$ matrix is a function of only geometry and material properties and this matrix is our usual global coefficient matrix. So, we take inverse of the square matrix and then we will get the unknown quantities which are magnetic vector potential values in whole field domain and the voltage across the terminals.

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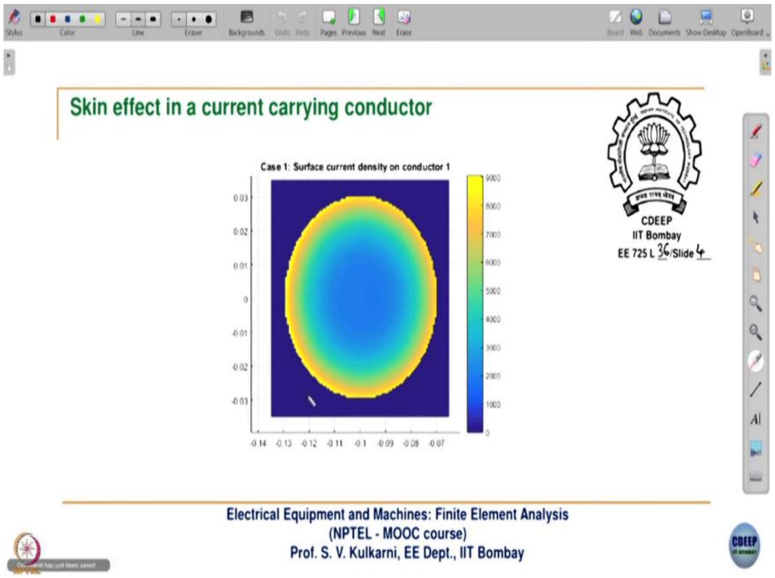

Let us see some examples. The above slide shows the skin effect of a current-carrying conductor. Let us assume that this conductor is same as the conductor that we saw in the previous slide. Suppose if we want to analyze a conductor excited with a voltage source and if the current is not known then we have to go back to use the formulation that we have discussed in the lecture. So if we excite the conductor by a voltage source then voltage will be known and the current will be unknown.

But we can simplify the analysis by exciting the massive conductor with a current source with some frequency of excitation and the currents get redistributed due to skin effect. For analysis purpose we feed the conductor with a current source operating at some frequency, because current fed systems are easier to analyze as compared to voltage fed systems. That is why we have used current fed FE formulation.

The current distribution shown in the above slide represents skin effect. This distribution can be obtained by using the code developed using the discussed formulation. In the figure, we can see the current is trying to be more at the surface of the conductor and less amount of current is in inner parts of the conductor. This effect can be simply understood if we analyze the conductor as made up of small annular conductors then we will find that the inside elements of the whole conductor are linking more flux as compared to the outside elements. So the corresponding inductance of the inner part of the conductor is more as compared to that of the outer part and hence current tries to be at the surface.

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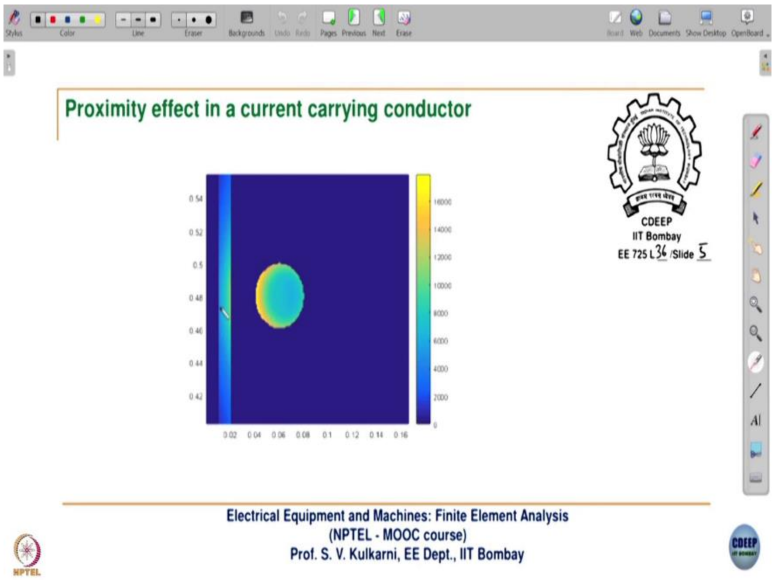

Now, if conductor is brought near a conducting part or a plate which is shown in the above figure. If the plate is made of mild steel in which eddy currents are induced then the skin effect gets skewed by the induced currents.

This phenomenon can be understood by considering the two conductors as two windings (say LV and HV windings) because the source current will induced currents in the conducting plate. The flux linkages will be maximum in the part between the two windings (or conductors). The flux linkages and so the inductance will be less in the parts of the conductors, which are facing each other.

That is why the current tries to concentrate more on the parts that are facing each other as shown in the above field plot. In basics also we have seen that the proximity effect further increases the AC resistance as the effective area decreases further because the current is now trying to concentrate more on the parts that are facing each other.

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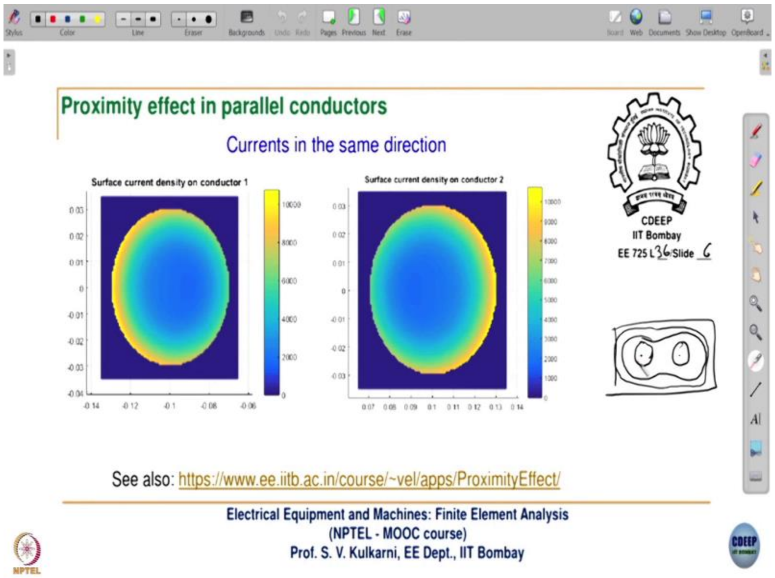
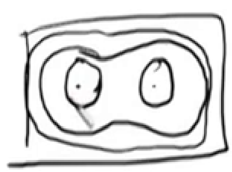


Consider two conductors carrying currents in the same direction as shown in the following figure.



So, if the currents are in the same direction then the corresponding flux will be as shown in the above figure. In the figure, we can observe that the part of the two conductors, which are facing each other will link most of the flux. So, the corresponding impedance offered is more and that is why current distribution is less in this region.

Whereas the flux linking the parts which are not facing each other is less and the corresponding impedance will be lower and hence the current distribution will be more.

To see the corresponding flux plot and understand more about these flux linkages and corresponding current distribution, see an experiment in the following link of the virtual lab.

https://www.ee.iitb.ac.in/course/~vel/apps/ProximityEffect/

In the above experiment, one can verify that the two parts where more flux linkage is there, there current will reduce.

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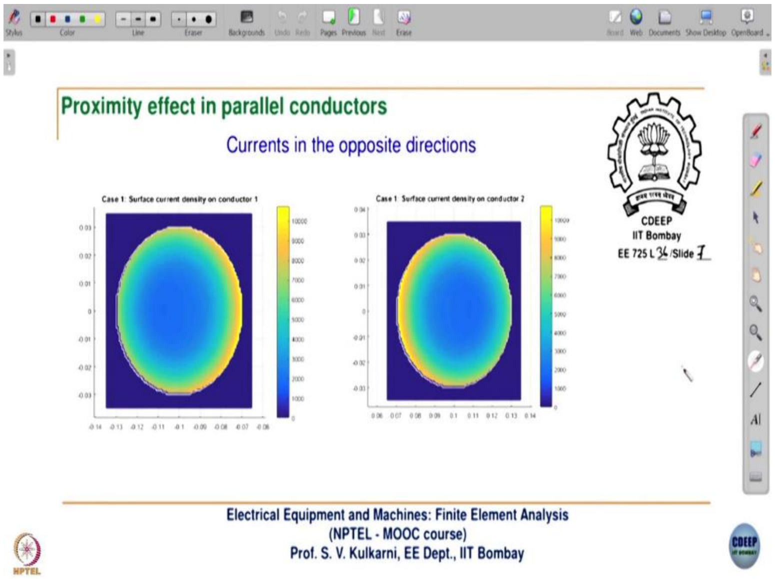
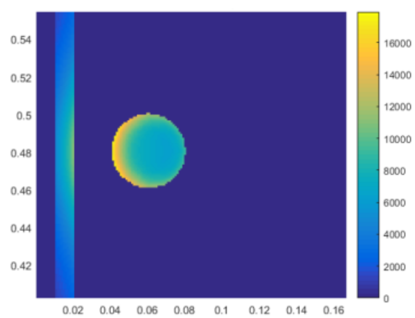


Now, consider the opposite case where the direction of currents are opposite. This case is similar to the problem that we have seen in one of the previous slide.



So, in the case given in the above slide also the currents are concentrating in the faces which are facing each other. Again, the directions of currents in the above figure are opposite.

This way we can understand skin and proximity effects. The examples that we have seen in this lecture are very representative simple examples. But, many of our devices like transformers and motors are fed by power electronic circuits with frequencies greater than 50 Hz and excitations with a lot of harmonics then the skin and proximity effects will increase substantially.

For such problems also the formulation that we have seen in this lecture can be used to find out the increase in losses and temperature rise in such windings. So, always remember that the complexity of the analysis further goes up if we model individual conductors in the winding but the formulation remains the same. However, complexity in terms of geometry and meshing will go up and the efforts in terms of formulation remain the same, so that is the advantage of FE analysis.

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