Electrical Equipment and Machines: Finite Element Analysis
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Lecture No. 33
Periodic and Antiperiodic Boundary Conditions in Rotating Machines

Welcome to the 33rd lecture. In this lecture, we will see modelling of rotating machines. In time harmonic simulation when we analyzed torque speed characteristics of an induction motor we took one condition of the stator and motor position. So, that analysis was like a static condition.

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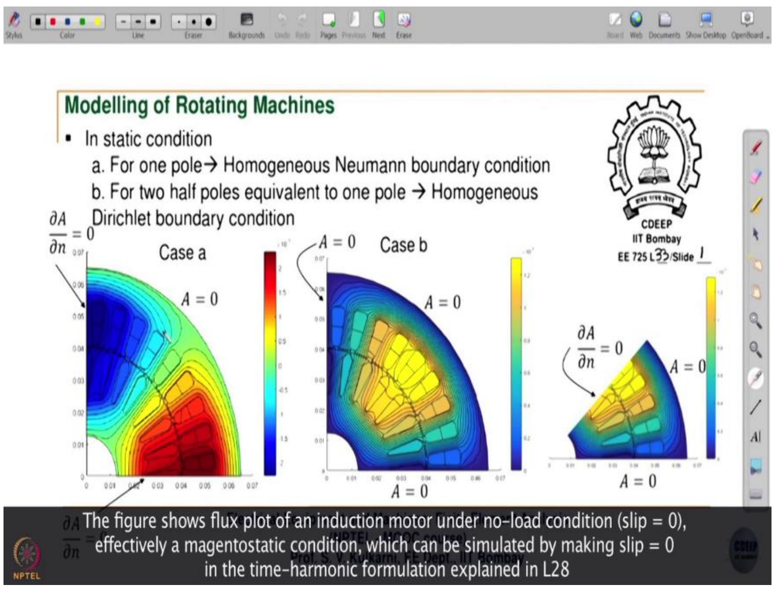

In that analysis, we varied the slip and then we calculated the torque from rotor losses. That analysis was like a static condition. So, if we are formulating simulation for a fixed rotor structure in our simulation then we have the option of modelling a section of the motor using either Neumann or Dirichlet conditions. For example, in case a of the above slide, one pole of the machine is modelled. In this case, A = 0 is always imposed on the outer and inner circumferential boundaries. Because this condition will contain all the flux within the stator structure. Now here on straight-line boundaries of the motor we have imposed homogenous Neumann condition because the flux contours are going normal to these two boundaries.

So, homogenous Neumann conditions are imposed on the two straight-line boundaries. But if the same section of the motor is modelled with two half poles instead of one full pole then the flux pattern in the motor section will be as shown in case b.

Here, for this case b we have to impose A = 0 for the circular and straight-line boundaries. This will make the flux parallel condition as shown in the figure whereas in case a the flux was normal on the two straight line boundaries. So, when we are modelling two half poles in the motor section then we have to impose flux parallel condition (A = 0) on these two straight-line boundaries.

The third case is if we want to further simplify the motor section by dividing it into half as shown in the figure that corresponds to case c. For this, we can cut the model further over the radial symmetry line. Now, we can see on one of the radial straight lines the flux is normal. So, then we can impose $\frac{\partial A}{\partial n} = 0$ (homogenous Neumann condition). Now, the model becomes more simpler. So, cases a and b correspond to one-fourth models and case c becomes one-eight model. So, in case c we have Dirichlet and Neumann conditions applied on the two straight line boundaries. Depending upon which section of the motor we consider, either we will have to impose homogenous Neumann condition or Dirichlet condition or combination of both conditions.

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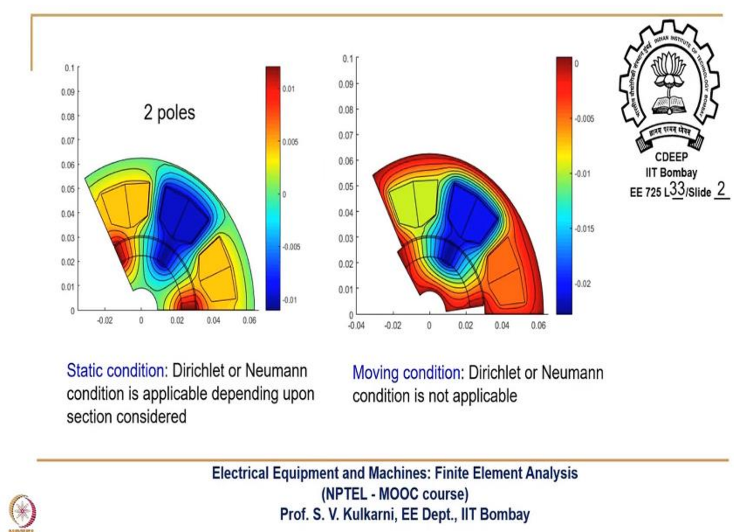

Going further, in the example shown in the above slide we have modelled two poles of a six-pole permanent magnet synchronous motor. The flux plot on the left hand side is for a section of the motor with magnets on its rotor surface and we have imposed homogenous Neumann boundary conditions on the two straight-line boundaries. We have imposed A = 0 on the outermost and innermost circular boundaries. It should be noted that, A = 0 is imposed on the innermost boundary which is the outer boundary of the shaft because we are considering that the shaft is made from a non-magnetic material and the flux plot shown on the left hand side is for a static condition.

If the rotor is moved by a certain angle as shown in the flux plot on the right hand side of the above slide we can see that the flux plot is distorted. On the outermost and the innermost circular boundaries we can still define A equal to 0. But on the other two straight line boundaries the flux is neither parallel nor perpendicular and they cross the boundary in arbitrary directions. Therefore, for the two straight-line boundaries, Dirichlet or Neumann boundary conditions are not applicable. However, we still want to simplify the model and we will see how to do that.

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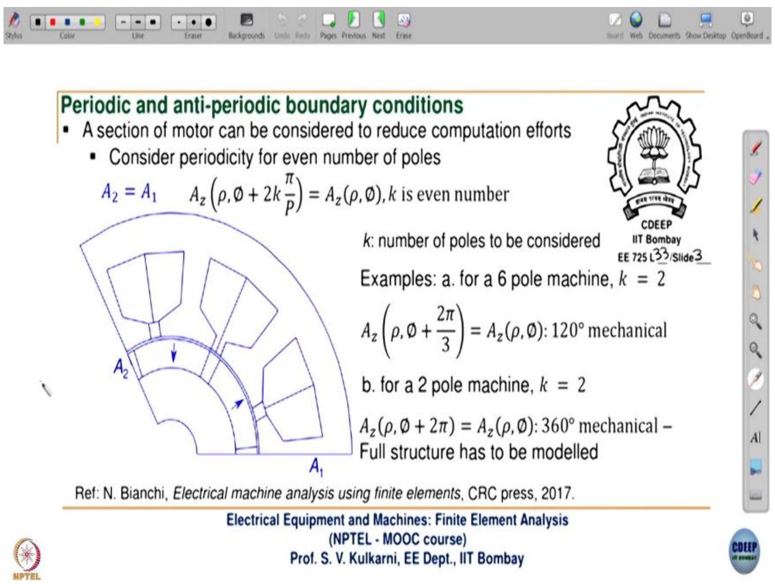

So, we can do that by using periodic and antiperiodic boundary conditions. First let us consider periodic boundary condition. Again, consider a section of a rotating machine which is shown in the figure given in the above slide. This section of motor corresponds to 120^{o} mechanical part of the motor. If we consider even number of poles then we have the following equation

$$A_z\left(\rho, \emptyset + 2k\frac{\pi}{P}\right) = A_z(\rho, \emptyset)$$

This equation represents that we are equating the potential values on the boundary at an angle $\emptyset$ with the boundary at $\emptyset + 2k\frac{\pi}{P}$. So, remember that on these two boundaries neither homogenous Neumann nor Dirichlet condition is imposed.

So, here in this expression k is the number of poles to be considered in the section that we are modelling. In this case, k is even number and then we are imposing periodic boundary condition. Here, on the two circular boundaries A = 0 is imposed. Whereas on the two straight-line boundaries which we are forcing $A_2 = A_1$. So, now let us see two examples.

For a six-pole machine, if we want to model a section with two poles as shown in the figure given in the above slide then we have to take $k = 2$ and $P = 6$. So, the term $2k\frac{\pi}{P}$ becomes $\frac{2\pi}{3}$. So, that means whatever is the value of A_z at ϕ is equal to A_z at $\phi + \frac{2\pi}{3}$. So, if $\phi = 0$ then $\phi + \frac{2\pi}{3}$ is equal to 120°. Now, let us consider a two pole machine if we have to model both the poles then k is even number and it is equal to 2 then obviously we have to model all 360°.

That is obvious because if $P = 2$ and $k = 2$ then $2k\frac{\pi}{P} = 2\pi$. So, that means effectively we have to model the whole geometry. That does not give any advantage but we discussed this for confirming the above equation is indeed correct for a two pole machine. But we can see the advantage if we are modelling only one third of the geometry. We will get the full flux plot by using the solution for the 120° part.

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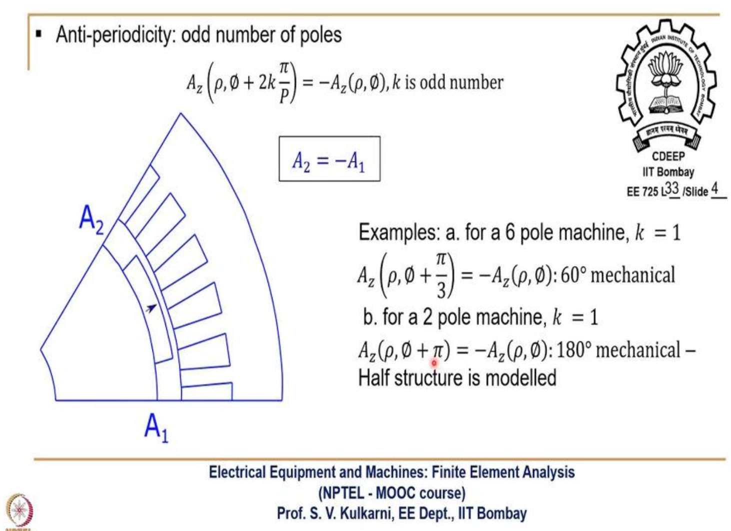

Now, in this slide, we will understand antiperiodic boundary conditions which are applicable when we are modelling odd numbers of poles. The corresponding equation for this boundary condition is given below.

$$A_z\left(\rho, \emptyset + 2k\frac{\pi}{P}\right) = -A_z(\rho, \emptyset)$$

In the above equation, we equate potentials on the two boundaries but with a negative sign. The section shown in the above figure corresponds to a permanent magnet synchronous motor. The magnetic vector potential on one of the straight line boundaries equals to the negative of magnetic vector potential on the other straight line boundary. Let us understand intuitively why the potentials on these two boundaries should be equal with an opposite sign. In this slide, a section with one pole is modelled and flux pattern will be such that half the flux will cross one straight line boundary and the other half will take the other path and cross the other boundary.

Fictitious current sources producing this pattern can be placed around the region of the permanent magnet with dot on one side of the magnet and cross on the other side. So, that means in one side of the magnet current is coming out of the paper and on the other side current is going into the paper as per the right hand rule. Since the directions of the fictitious current sources are opposite, the corresponding magnetic vector potential values on the two straight line boundaries will also have opposite signs.

Now let us see an example of a six-pole machine with one pole being modelled. For this case $P = 6$ and $k = 1$, so we will get the value of $2k\frac{\pi}{P}$ as $\frac{\pi}{3}$. Hence, we need to model 60° mechanical of the total motor geometry with antiperiodic boundary conditions specified on the two straight line boundaries. In another example of a two pole machine, if one pole is modelled then $P = 2$ and $k = 1$, so we will get $2k\frac{\pi}{P}$ as π. So we need to model 180° mechanical of the entire motor geometry with antiperiodic boundary conditions on the straight line boundaries.

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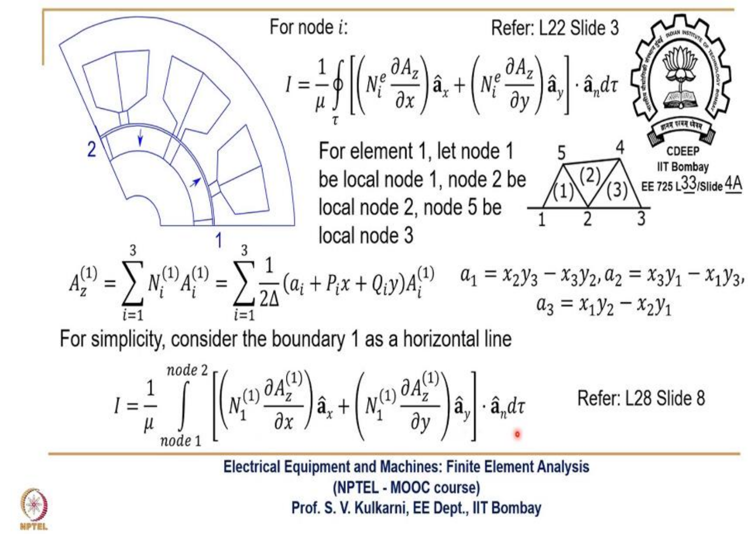
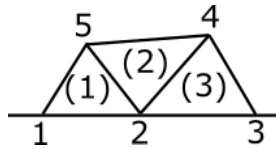


Going further, for the geometry given in the above slide, periodic boundary conditions are applied only to boundary 2. However, leaving boundary 1 without applying a boundary conditions leads to the automatic imposition of homogenous Neumann boundary conditions (Refer L12, slide 6 and L22, slide 7). It should be noted that the imposition of homogenous Neumann boundary condition means forcing $\frac{\partial A_z}{\partial n} = 0$. Forcing this boundary condition leads to flux crossing perpendicular to the boundary. But, this is not what we want, as explained earlier when the rotor is moving the flux is neither perpendicular nor parallel to the straight line boundaries. Therefore, we need to consider the following contour integral which is obtained after writing the weighted residual statement for an ith node of a finite element (Refer L22).

$$I = \frac{1}{\mu}\oint_{\tau}\left[\left(N_i^e \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{a}}_x + \left(N_i^e \frac{\partial A_z}{\partial y}\right)\hat{\mathbf{a}}_y\right]\cdot\hat{\mathbf{a}}_n d\tau$$

In this case of a rotating structure, the above integral does not lead to a boundary condition matrix ($[B_b]$) which we discussed in L28. Because in this problem a constant value of non-homogenous Neumann boundary condition is not imposed on the boundary.

This integral in fact leads to a square matrix whose number of rows and columns equals the number of nodes in the entire problem domain. This matrix will have non-zero entries only for the nodes on boundary 1. Let us consider a part of boundary 1 with 3 elements as shown in the following figure.



For element 1, let node 1 be the local node number 1, node 2 be the local node number 2, and node 5 be the local node number 3. Then, approximating the magnetic potential in element 1 with the following equation.

$$A_z^{(1)} = \sum_{i=1}^{3} N_i^{(1)} A_i^{(1)} = \sum_{i=1}^{3} \frac{1}{2\Delta} (a_i + P_i x + Q_i y) A_i^{(1)}$$

The expressions of a_1, a_2, and a_3 are given below.

$$a_1 = x_2 y_3 - x_3 y_2,\ a_2 = x_3 y_1 - x_1 y_3,\ a_3 = x_1 y_2 - x_2 y_1$$

We have seen these expression earlier.

For simplicity, to understand the mathematical formulation consider the boundary 1 as a horizontal line with $\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_y$. That means normal to the boundary will be in negative y direction and $d\tau$ will be *dx*. Let this case be the initial position of the rotor structure. It should be noted that the angle of the rotor part of the boundary 1 changes with the rotor angle and $\hat{\mathbf{a}}_n$ will be in an arbitrary direction. But the part of boundary that corresponds to the stator will always be horizontal.

So, the above closed contour integral for node 1 reduces to the following expression.

$$I = \frac{1}{\mu} \int_{node\ 1}^{node\ 2} \left[\left(N_1^{(1)} \frac{\partial A_z^{(1)}}{\partial x} \right) \hat{\mathbf{a}}_x + \left(N_1^{(1)} \frac{\partial A_z^{(1)}}{\partial y} \right) \hat{\mathbf{a}}_y \right] \cdot \hat{\mathbf{a}}_n d\tau$$

Because, the value of the integral for the two inner edges of the element 1 will get cancelled when we form the global level matrices (Refer L28). Hence, the integration will be only from node 1 to node 2. So, the integration will be non zero only for the edge joining nodes 1 and 2 and for these two edges of the element 1, the integration will be reduced to 0.

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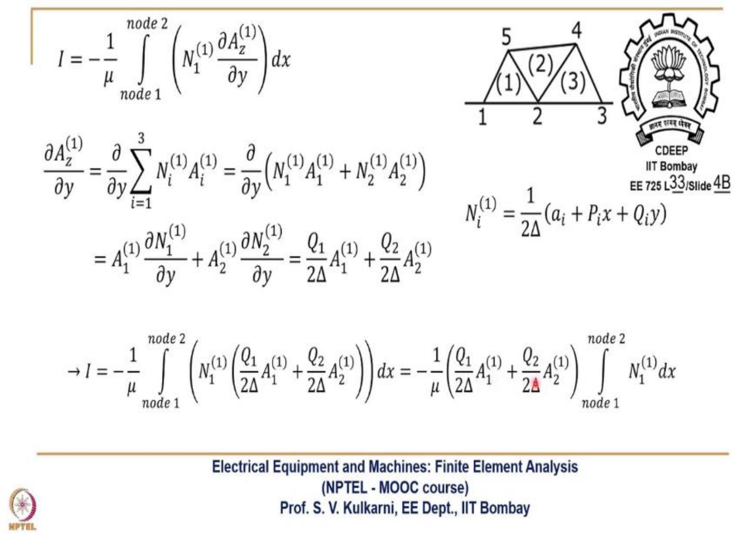

As seen on the previous slide, $\hat{\mathbf{a}}_n = -\hat{\mathbf{a}}_y$ and $d\tau$ is equal to dx, the integral I on the previous slide reduces to the following expression.

$$I = -\frac{1}{\mu}\int_{node\ 1}^{node\ 2}\left(N_1^{(1)}\frac{\partial A_z^{(1)}}{\partial y}\right)dx$$

Using the approximate function for $A_z^{(1)}$ and remembering that the shape function of node 3 of element 1 is equal to 0 on the edge joining nodes 1 and 2, the differential term in the integral I is simplified as given below.

$$\frac{\partial A_z^{(1)}}{\partial y} = \frac{\partial}{\partial y}\sum_{i=1}^{3} N_i^{(1)}A_i^{(1)} = \frac{\partial}{\partial y}\left(N_1^{(1)}A_1^{(1)} + N_2^{(1)}A_2^{(1)}\right)$$
$$= A_1^{(1)}\frac{\partial N_1^{(1)}}{\partial y} + A_2^{(1)}\frac{\partial N_2^{(1)}}{\partial y} = \frac{Q_1}{2\Delta}A_1^{(1)} + \frac{Q_2}{2\Delta}A_2^{(1)}$$

Also, $A_1^{(1)}$ and $A_2^{(1)}$ of element 1 are independent on *x* and *y* and N_i(shape functions) expressions are functions of *x* and *y*. Therefore, the differential term in the above integral reduces to the above expression. Then, substituting the above term in the integral we get,

$$I = -\frac{1}{\mu}\int_{node\ 1}^{node\ 2}\left(N_1^{(1)}\left(\frac{Q_1}{2\Delta}A_1^{(1)} + \frac{Q_2}{2\Delta}A_2^{(1)}\right)\right)dx$$

In the above integral only shape function N_1 is a function of x. So only $N_1^{(1)}$ will appears under the integral sign and the term $\left(\frac{Q_1}{2\Delta}A_1^{(1)} + \frac{Q_2}{2\Delta}A_2^{(1)}\right)$ is taken outside the integral.

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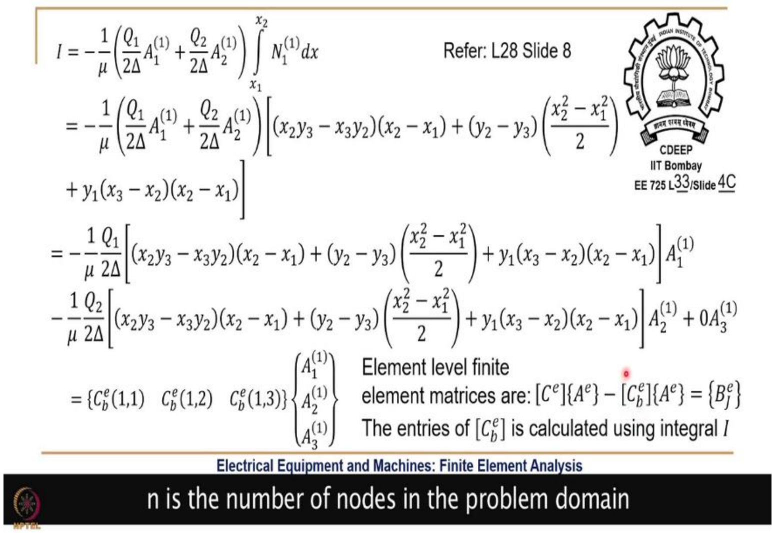

Going further, the integral I can be simplified by substituting the expression for $N_1^{(1)}$ and then evaluate the integral to get the following expression.

$$I = -\frac{1}{\mu}\left(\frac{Q_1}{2\Delta}A_1^{(1)} + \frac{Q_2}{2\Delta}A_2^{(1)}\right)\int_{x_1}^{x_2} N_1^{(1)}dx$$
$$= -\frac{1}{\mu}\left(\frac{Q_1}{2\Delta}A_1^{(1)} + \frac{Q_2}{2\Delta}A_2^{(1)}\right)\left[(x_2y_3 - x_3y_2)(x_2 - x_1) + (y_2 - y_3)\left(\frac{x_2^2 - x_1^2}{2}\right)\right.$$
$$\left. + y_1(x_3 - x_2)(x_2 - x_1)\right]$$

We can verify the derivation by substituting the expression of $N_1^{(1)}$ and then integrating with respect to dx. Now, this whole expression is rearranged to write it in matrix multiplication form as given below.

$$-\frac{1}{\mu}\frac{Q_1}{2\Delta}\left[(x_2y_3 - x_3y_2)(x_2 - x_1) + (y_2 - y_3)\left(\frac{x_2^2 - x_1^2}{2}\right) + y_1(x_3 - x_2)(x_2 - x_1)\right]A_1^{(1)}$$
$$-\frac{1}{\mu}\frac{Q_2}{2\Delta}\left[(x_2y_3 - x_3y_2)(x_2 - x_1) + (y_2 - y_3)\left(\frac{x_2^2 - x_1^2}{2}\right) + y_1(x_3 - x_2)(x_2 - x_1)\right]A_2^{(1)} + 0A_3^{(1)}$$
$$= \{C_b^e(1,1) \quad C_b^e(1,2) \quad C_b^e(1,3)\}\begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \\ A_3^{(1)} \end{Bmatrix}$$

The row matrix $\{C_b^e(1,1) \quad C_b^e(1,2) \quad C_b^e(1,3)\}$ is multiplied to the column matrix $\begin{Bmatrix} A_1^{(1)} \\ A_2^{(1)} \\ A_3^{(1)} \end{Bmatrix}$. Here, $C_b^e(1,3) = 0$ because the node 3 which is global node number 5 has no contribution to the integral as explained on the previous slide. Thus, at the element level we get the following equation when we apply the FEM procedure for Poisson's equation.

$$[C^e]\{A^e\} - [C_b^e]\{A^e\} = \{B_J^e\}$$

When the Poisson's equation is modelled using FEM the formulation leads to the equation $[C^e]\{A^e\} = \{B_J^e\}$. Now here, because of periodic boundary conditions, we have one more term $[C_b^e]\{A^e\}$ at the element level. As explained on slide 4A, $[C_b^e]$ will be a 3×3 matrix and when we form the global system of equations this matrix will be $n \times n$ matrix. So, periodic boundary conditions are taken into account by adding the term $[C_b^e]\{A^e\}$ at the element level and the corresponding term at the global level matrix equation. Then we can solve the problem by using the finite element procedure to get the field solution.

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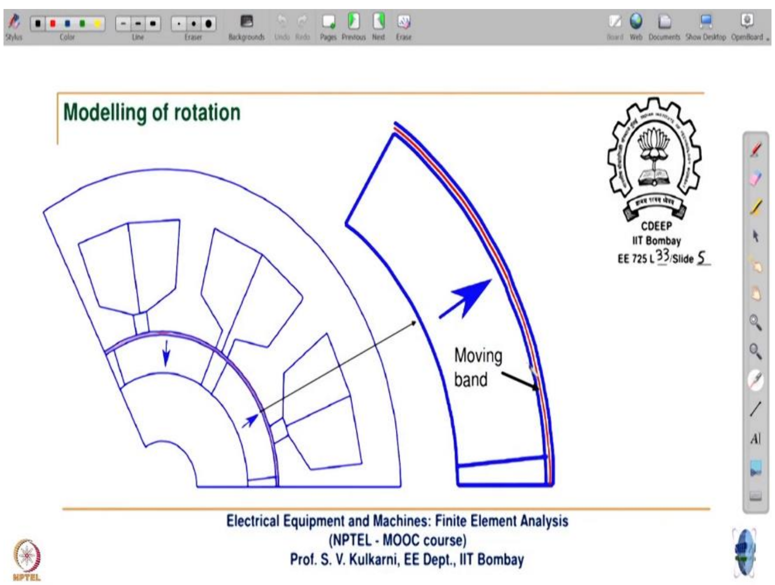

Now, in this slide, we will understand how to model rotation of a rotor structure. In the air gap (zoomed portion in the above slide), we can see that there is a red curve which separates a fixed mesh and a moving mesh. So, on the right hand side of the red curve, there will be a fixed mesh (stator part) and on the left side of the red curve in the gap, we will have one layer of mesh which will be attached to the rotor and this portion will move along with the rotor. So, we will have at least two layers of mesh in the air gap to model the rotation effectively.

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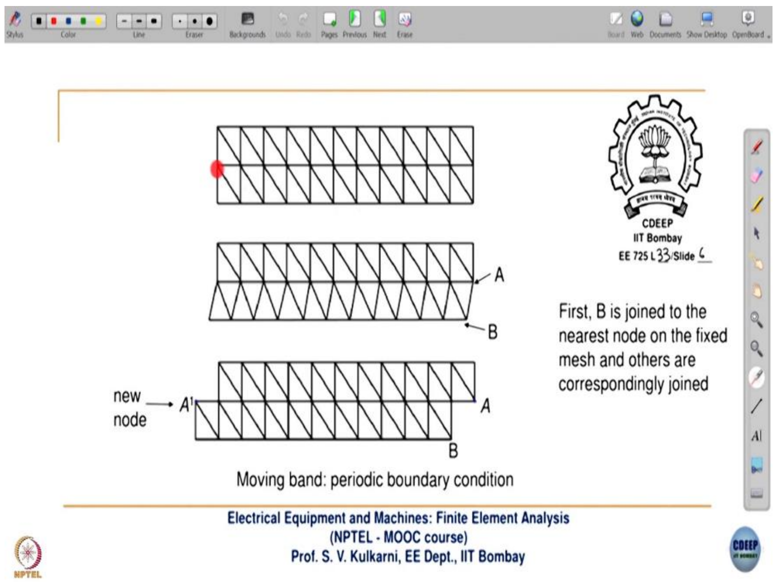
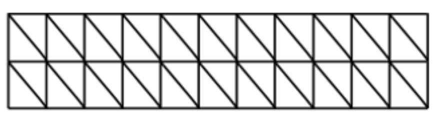
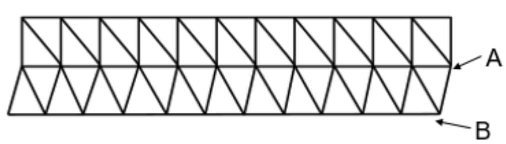


What happens in the air gap region? In the two layer mesh shown in the following figure, the center line between the two layers is the red curve that we saw in the earlier slide.



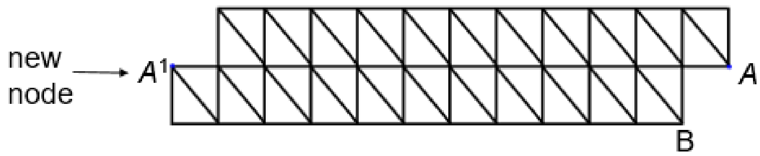
The upper part of the above mesh is attached to the stator, and it is a fixed mesh. The lower part of the above mesh is attached to the rotor and it moves along with the rotor. The above mesh corresponds to the initial instant. The following mesh corresponds to the next instant at which the rotor has moved



Now, the point B has moved and the point A of fixed mesh has not moved.

So, when the position point B is changed it is actually joined to the nearest point (point A) on the fixed mesh. For the above configuration, the nearest point is still A. The point next to point A is still farther from B. So, that is why B is joined to A and then the other nodes are respectively joined to other respective nodes.

Now, the moving part of the mesh is moved further as shown in the following figure.



The node next to point A will be closer as compared to node A. So, the node B is joined to the point next to node A and similarly, for all the other nodes. Effectively, one more node (A^1) gets added to the mesh. So, the main thing in this moving band method is that the number of elements are fixed but the number of nodes goes on increasing as the lower part moves. This will continue upto the instant when the B point comes to the end of the stationary part. So, moment we solve from the starting point to the ending point of the stationary part, we would have completed one set of simulations.

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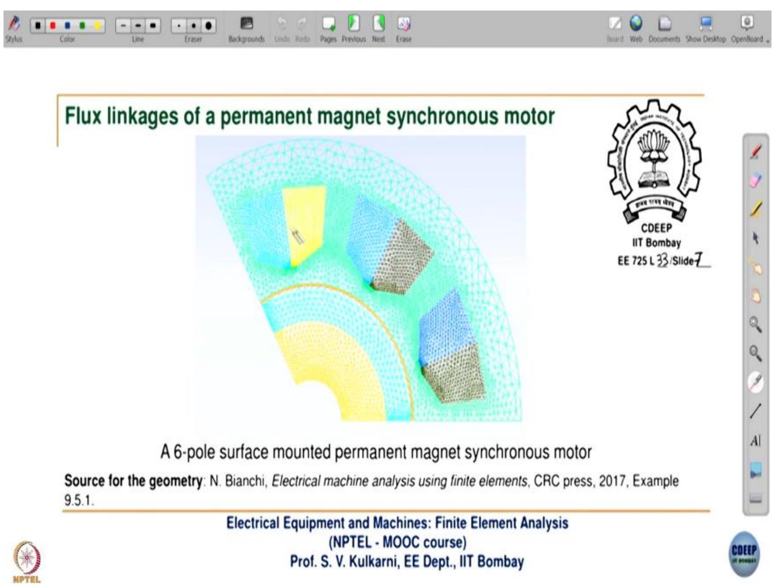

Here, at t=0 we have created the mesh shown in the above slide and then we will see what happens. In the above figure, there is an air gap and slots and we can see that each of the slots has two coil sides.

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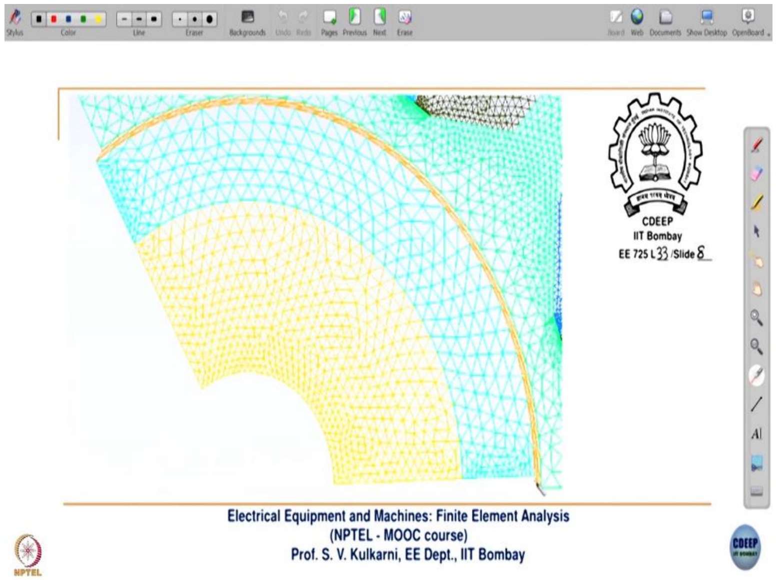

Now in the above slide, we can see that the rotor has moved so the associated moving mesh in the air gap has also moved. In this position of the rotor no new node is added yet but the mesh has got tilted or distorted.

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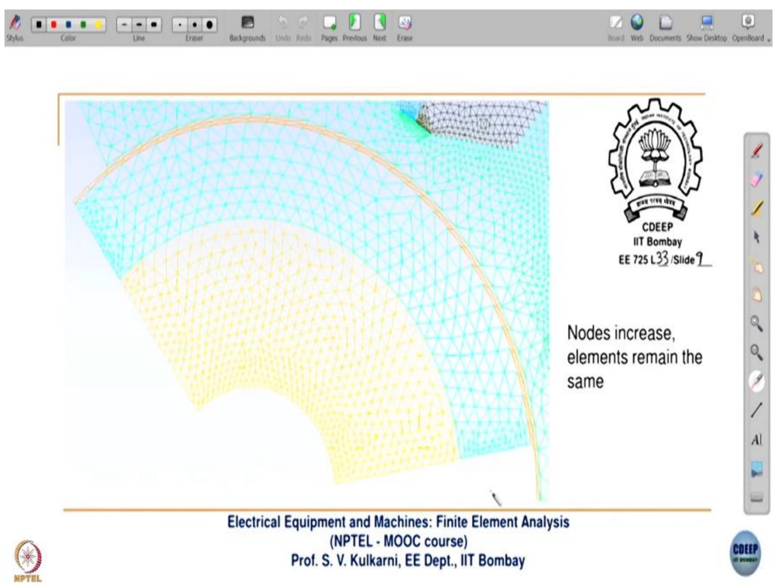

In the above slide, we can see that the rotor has moved further and two new nodes are added. When the rotor has moved further, the number of nodes increase but the number of elements remains the same.

The boundary conditions are assigned, based on the number of poles that are modelled.

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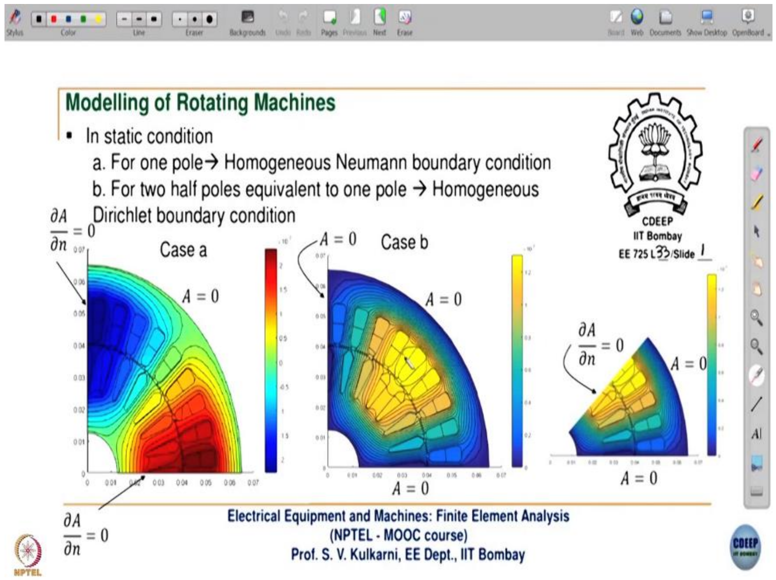

Depending on the number of poles or fraction of poles modelled, we can reduce the geometry and exploit the boundary conditions. So, if we have modelled even number of poles then we have to apply periodic boundary conditions or if odd number of poles are modelled then anti-periodic boundary conditions are imposed.

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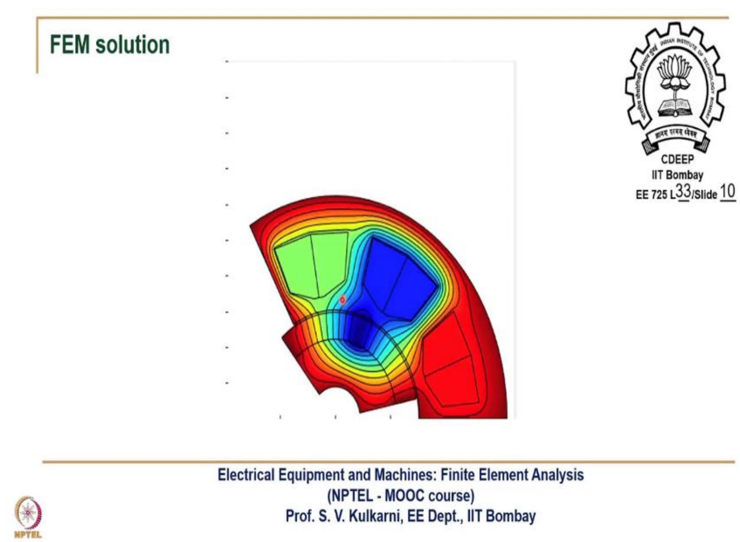

In this slide, the flux plot is for a section of six pole permanent magnet synchronous motor and rotation of the rotor is modelled. Magnets are placed on its rotor and windings are on stator. In the considered section, two poles of the motor are modeled. The field solution is obtained after imposing periodic boundary conditions that are formulated in the earlier slides. Because of periodic boundary conditions whatever flux we see on one side of the rotor and stator parts appears on the other side also. In the flux plot, in both the parts we can see that continuity of flux is maintained and this will be demonstrated on the next slide.

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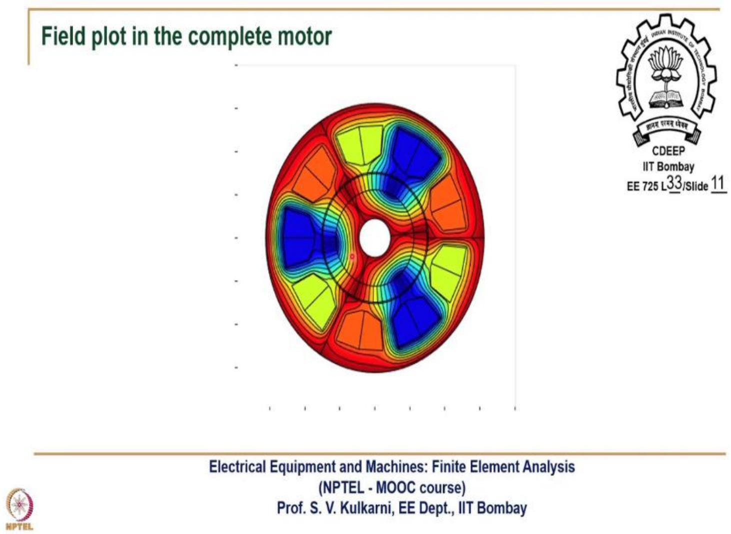

Now, in the above slide, the flux plot in the complete motor structure is shown. Assuming the periodic boundary conditions, we have solved for 120^o mechanical of the motor, but using that solution we have obtained flux plot for the entire 360^o structure. The flux pattern in the region of 120^o structure is repeated twice in the remaining 240^o. Also, we can see that the flux continuity is maintained at the interfaces.

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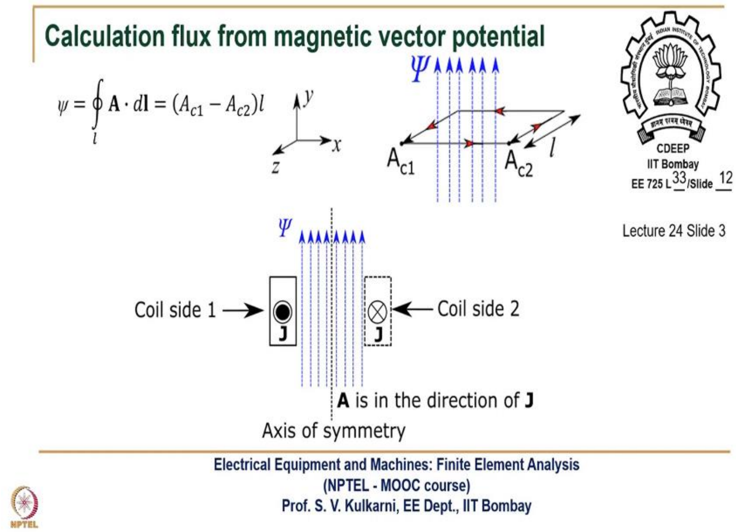
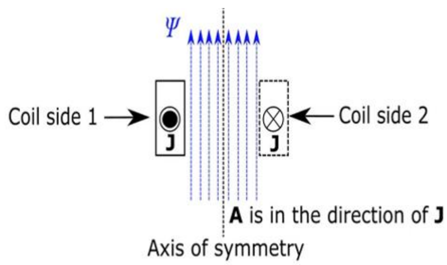


Let us refresh ourselves with the procedure for calculating flux linkages of a coil we studied in L24. Consider a coil with two coil sides 1 and 2 as shown in the following figure.



These coils are one pole pitch apart. The total flux linkage of the coils is given by the following formula,

$$\psi = \oint_l \mathbf{A} \cdot d\mathbf{l} = (A_{c1} - A_{c2})l$$

In the above expression, A_{c1} and A_{c2} are the average values of magnetic vector potentials of the two coil sides. Because of symmetry $A_{c1} = A_{c2}$ and after some mathematical manipulations we get the above formula.

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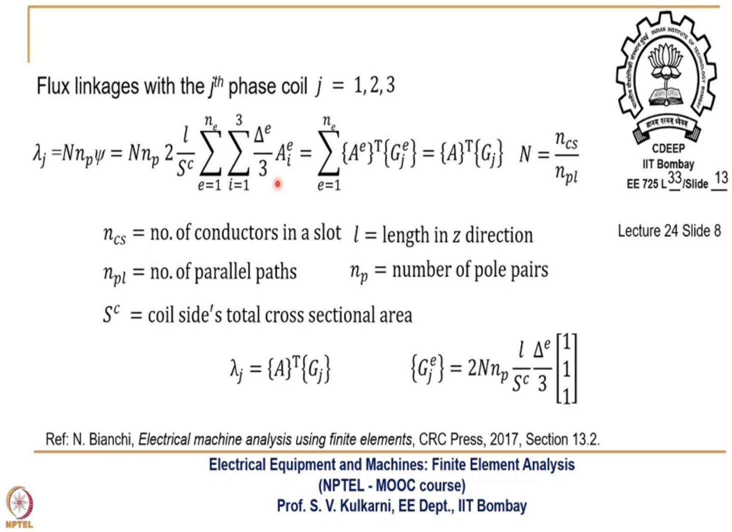

The flux linkages for a j^{th} phase coil of the considered machine is given below.

$$\lambda_j = N n_p \psi = N n_p \, 2 \frac{l}{S^c} \sum_{e=1}^{n_e} \sum_{i=1}^{3} \frac{\Delta^e}{3} A_i^e = \sum_{e=1}^{n_e} \{A^e\}^{\mathrm{T}} \{G_j^e\} = \{A\}^{\mathrm{T}} \{G_j\} \quad N = \frac{n_{cs}}{n_{pl}}$$

In the above expression, factor 2 appears if we are modelling only one coil side. If both coil sides are modelled, then factor 2 will not appear.

We know that flux is defined by $\oint_l \mathbf{A} \cdot d\mathbf{l}$ and summation i goes from 1 to 3. $A_i^e/3$ gives the average value of A over the triangular element under consideration. Then, summation of the average A multiplied by the element area divided by the coil side's cross sectional area gives the average value of A over the coil side containing n_e triangular elements. The overall average A multiplied by length l gives the flux and when it is multiplied by number of turns N and the number of pole pairs n_P gives the total flux linkages of the coil.

The element level G_j^e matrix is given by the following expression.

$$\{G_j^e\} = 2 N n_p \frac{l}{S^c} \frac{\Delta^e}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

When we assemble element level matrices by executing the summation over all the elements of the coil sides, we get the flux linkages in terms of global A and G matrices.

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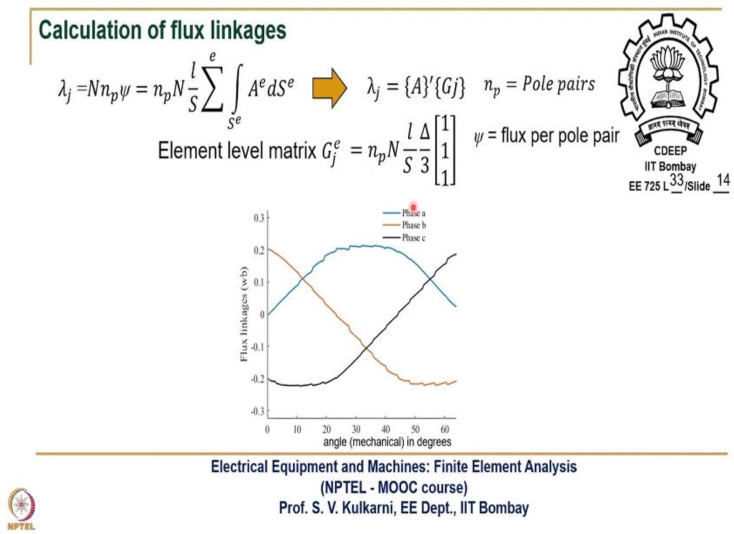


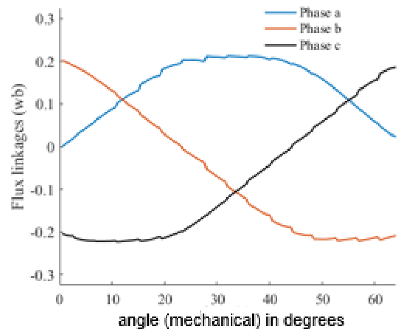
Now, the flux linkage expression in the previous slide is modified as given below by dropping the factor 2 because in this simulation result shown on slide 10 of the lecture, we have modeled only one pole pair.

$$\lambda_j = N n_p \psi = n_p N \frac{l}{S} \sum^{e} \int_{S^e} A^e dS^e$$

If we have model only a single pole then we have to use factor 2 in the above expression. Hence dropping the factor 2 and following the same procedure discussed in the previous slide we get the above expression of the flux linkage and element level matrices as given below.

$$\text{Element level matrix } G_j^e = n_p N \frac{l}{S}\frac{\Delta}{3}\begin{bmatrix}1\\1\\1\end{bmatrix}$$

The discussed formulation is then applied to calculate flux linkages with stator coils for the PMSM geometry, shown in slide 10. The flux linkages of phases a, b, and c are shown in the following figure.



It should be noted that the flux linkage are calculated for different rotor positions which are indicated on the x axis of the above figure. This means at every rotor position we calculate flux linkages using the magnetostatic formulation. Thank you.

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