Electrical Equipment and Machines: Finite Element Analysis
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Lecture No 32
Permanent Magnets: FEM Implementation

Welcome to lecture 32. In this lecture, we will derive the FEM formulation for permanent magnets.

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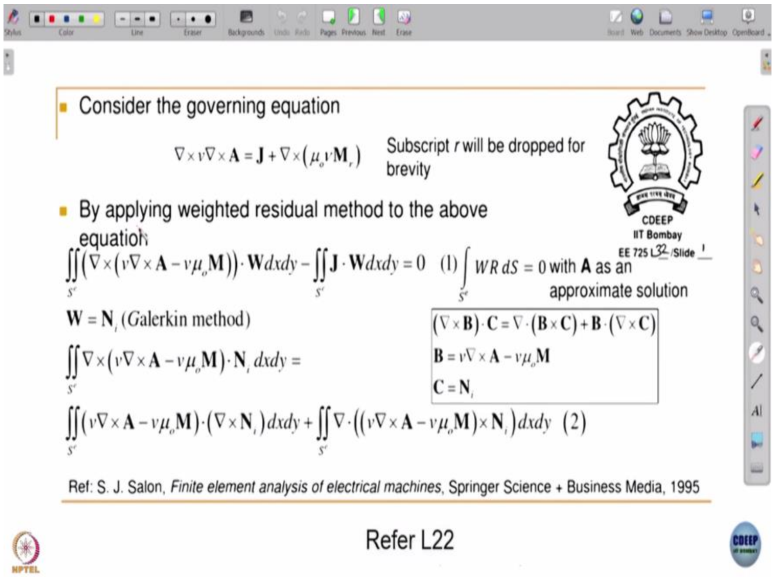

We start with the following governing equation which we have seen in the previous lecture.

$$\nabla \times \nu \nabla \times \mathbf{A} = \mathbf{J} + \nabla \times (\mu_o \nu \mathbf{M}_r)$$

In the above equation, $\mathbf{J}$ is due to free current and $\nabla \times (\mu_o \nu \mathbf{M}_r)$ is representing the source due to a permanent magnet. Now, in the above equation subscript r will be dropped. So, we should always remember that M in the equations given in the above slide is actually M_r and subscript r is dropped for simplicity.

So, now we will apply the weighted residual method to the above PDE. In weighted residual method, we integrate the weight into residue over the domain and equate it to 0. The residue will be there because we have substituted the approximate solution function and we minimize the residue in the weighted integral sense.

Now, as compared to the weighted residual approach that we saw in the previous lecture there is one difference here. In the previous lecture, it was a purely scalar formulation because it was two-dimensional formulation with all the vectors in z direction. Since, all the vectors in the governing equation are in z direction, so the direction was fixed and only magnitude had to be determined.

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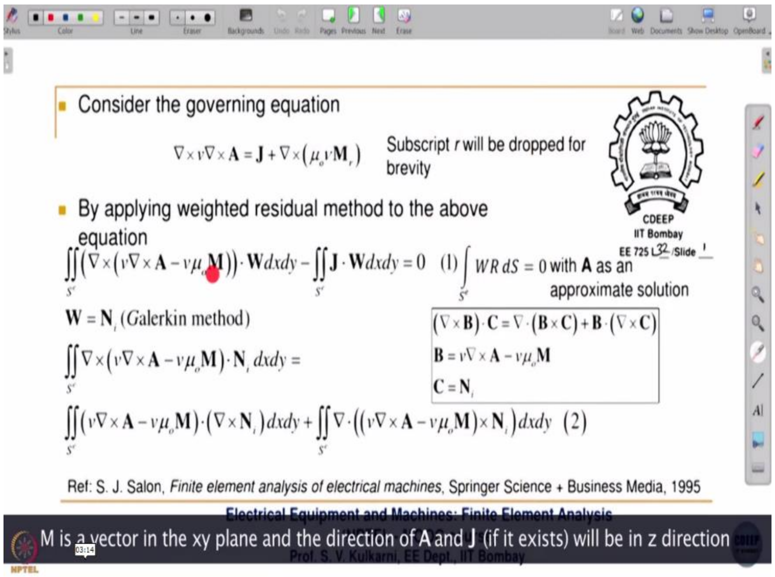

But here, M is a vector in xy plane and its direction is not along z direction, so we have to develop the formulation in vector notation. So, that is why the weighting function is also a vector because residue is a vector. So, now, we are considering Galerkin's approach and we took the weighting function as shape function and the only difference is both functions are vectors.

So, we are using the following vector identity

$$(\nabla \times \mathbf{B}) \cdot \mathbf{C} = \nabla \cdot (\mathbf{B} \times \mathbf{C}) + \mathbf{B} \cdot (\nabla \times \mathbf{C})$$
$$\mathbf{B} = v\nabla \times \mathbf{A} - v\mu_o \mathbf{M}$$
$$\mathbf{C} = \mathbf{N}_i$$

Using this vector identity the first term of the weighted residual statement can be modified as given below.

$$\iint_{S^e} \left(\nabla\times\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\right)\cdot\mathbf{W}dxdy-\iint_{S^e}\mathbf{J}\cdot\mathbf{W}dxdy=0$$

$$\mathbf{W}=\mathbf{N}_i \text{ (Galerkin method)}$$

$$\iint_{S^e}\nabla\times\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\cdot\mathbf{N}_i\, dxdy=$$

$$\iint_{S^e}\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\cdot\left(\nabla\times\mathbf{N}_i\right)dxdy+\iint_{S^e}\nabla\cdot\left(\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\times\mathbf{N}_i\right)dxdy$$

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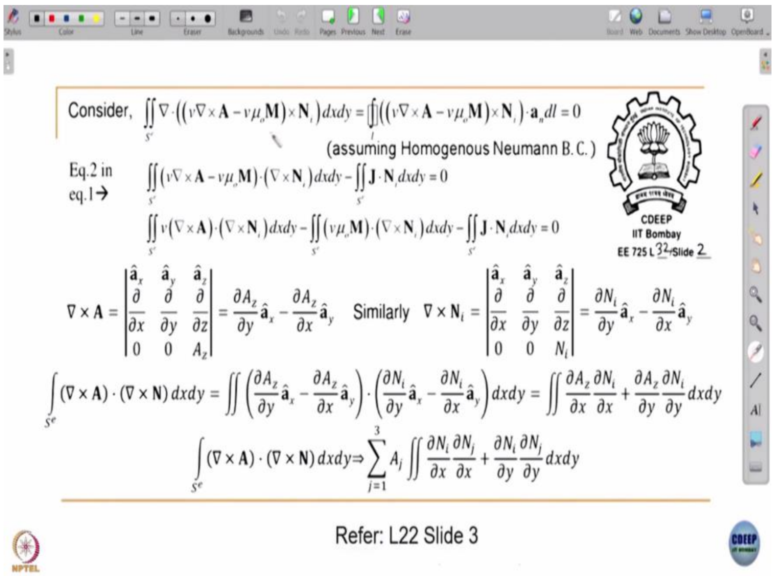

For the second term in the above equation we apply divergence theorem in 2D to get

$$\iint_{S^e}\nabla\cdot\left(\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\times\mathbf{N}_i\right)dxdy=\oint_l\left(\left(v\nabla\times\mathbf{A}-v\mu_o\mathbf{M}\right)\times\mathbf{N}_i\right)\cdot\mathbf{a}_n dl=0$$

In 2D approximation, dv becomes $dxdy$ and dS becomes $\mathbf{a}_n dl$. We have seen this earlier. The whole vector in the integral on the left hand side is going to be the vector in the integral on the right hand side multiplied with the normal vector to that surface.

In two dimensional formulation the normal to the surface becomes $\mathbf{a}_n dl$. Here, dl represents the contour. So, the vector in the integral in the right hand side of the above equation is multiplied with the normal vector and that will result in normal component to the surface. If there is a homogenous Neumann condition then this integral will become 0. If it is not then this integral will result into an additional term in the final FEM matrix equation. This term does not exist always. If it is homogeneous Neumann condition then the derivative of the field in the normal direction will be 0. If there is no homogeneous condition then this integral will not be 0 and there will be an additional term in our final matrix equation.

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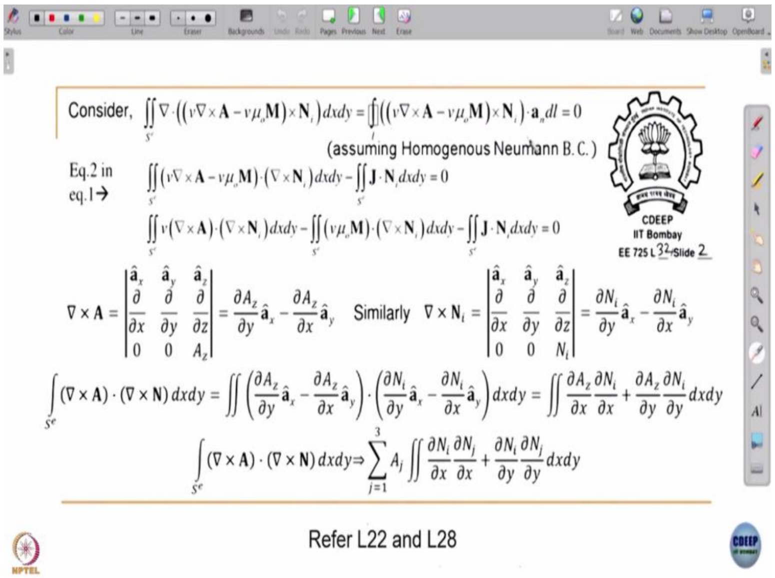

Remember, this formulation is at the element level and when we integrate the effect for all the elements the internal contributions over the common edges will get cancelled because for any common edge the $\mathbf{a}_n$ vectors will be opposite. The integral will contribute only to the outside boundary. This will be done when we assemble all the elements. So, that is why when we say homogeneous Neumann condition the above integral will contribute only to the outermost boundary because for the inter element segments the normal vectors are opposite and those contributions will get cancelled. We have seen the same theory earlier. The above integral is applicable only for the outermost boundary.

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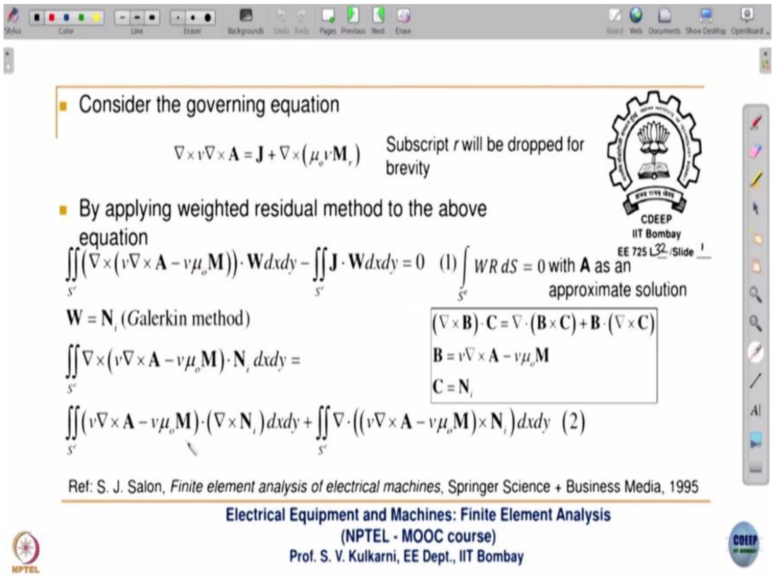

So, the above integral goes to 0 if there is a homogeneous Neumann condition on the outermost boundary. So, now we are left with the following term.

$$\iint_{S^e} (v\nabla \times \mathbf{A} - v\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy$$

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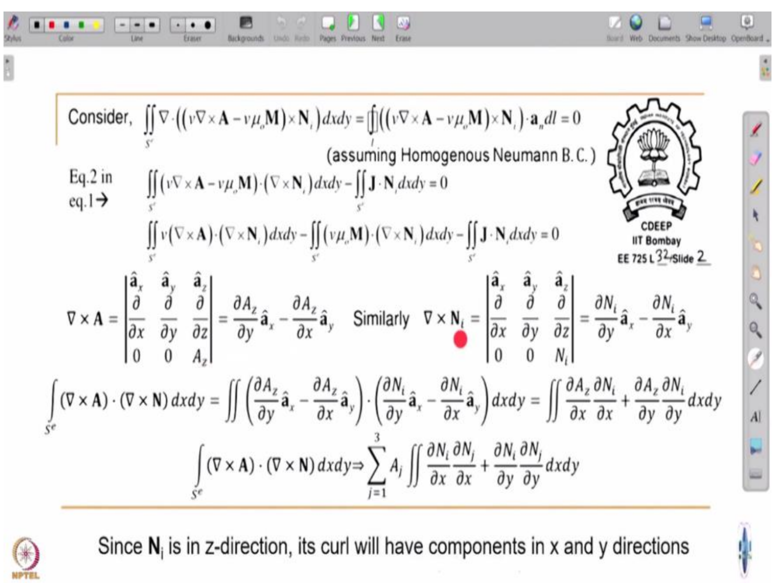

The modified weighted residual statement is given below.

$$\iint_{S^e} (v\nabla \times \mathbf{A} - v\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy - \iint_{S^e} \mathbf{J} \cdot \mathbf{N}_i dxdy = 0$$

Now, the first integral in the above equation is split into two terms as given in the following equation.

$$\iint_{S^e} v(\nabla \times \mathbf{A}) \cdot (\nabla \times \mathbf{N}_i) dxdy - \iint_{S^e} (v\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy - \iint_{S^e} \mathbf{J} \cdot \mathbf{N}_i dxdy = 0$$

With two dimensional formulation and A being only in z direction, $(\nabla \times \mathbf{A})$ will be reduced as given below.

$$\nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathbf{a}}x & \hat{\mathbf{a}}y & \hat{\mathbf{a}}z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{vmatrix} = \frac{\partial A_z}{\partial y} \hat{\mathbf{a}}_x - \frac{\partial A_z}{\partial x} \hat{\mathbf{a}}_y$$

Similarly, with $\mathbf{N}_i$ being in z direction, we can write $\nabla \times \mathbf{N}_i$ as given below.

$$\nabla \times \mathbf{N}_i = \begin{vmatrix} \hat{\mathbf{a}}x & \hat{\mathbf{a}}y & \hat{\mathbf{a}}z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & N_i \end{vmatrix} = \frac{\partial N_i}{\partial y} \hat{\mathbf{a}}_x - \frac{\partial N_i}{\partial x} \hat{\mathbf{a}}_y$$

If $\mathbf{A}$ is in z direction then $\mathbf{N}_i$ also will be in z direction because magnetic vector potential for any element can be written as

$$\mathbf{A} = \mathbf{N}_1 A_1 + \mathbf{N}_2 A_2 + \mathbf{N}_3 A_3$$

So, from the above equation, we can say that the direction to A will be given by the direction of N's and A_1, A_2, and A_3 are just scalars at the three nodes.

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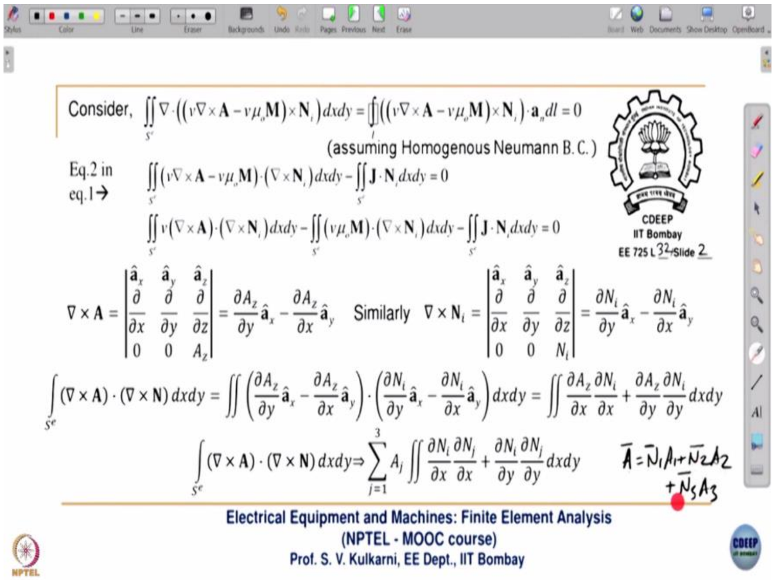

For $\mathbf{A} = \mathbf{N}_1A_1 + \mathbf{N}_2A_2 + \mathbf{N}_3A_3$, $\mathbf{A}$ on the left hand side is a vector so on the right-hand side also there has to be a vector, but A_1, A_2, and A_3 are just the scalar magnitudes. So, here $\mathbf{A}$ being a vector should get reflected by $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ being vectors.

If $\mathbf{A}$ has only z component then $\mathbf{N}_1$, $\mathbf{N}_2$, and $\mathbf{N}_3$ also will have z component only. So, that is why $\mathbf{N}_i$ is N_{iz}. But we have dropped the subscripts z in the above determinant for simplicity. The expressions of $\nabla\times\mathbf{A}$ and $\nabla\times\mathbf{N}_i$ are similar. Now, we require $(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{N}_i)$. It can be written as given below.

$$\int_{S^e}(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{N})\,dS = \iint\left(\frac{\partial A_z}{\partial y}\hat{\mathbf{a}}_x - \frac{\partial A_z}{\partial x}\hat{\mathbf{a}}_y\right)\cdot\left(\frac{\partial N_i}{\partial y}\hat{\mathbf{a}}_x - \frac{\partial N_i}{\partial x}\hat{\mathbf{a}}_y\right)dxdy$$

$$= \iint\frac{\partial A_z}{\partial x}\frac{\partial N_i}{\partial x}+\frac{\partial A_z}{\partial y}\frac{\partial N_i}{\partial y}dxdy$$

So, in the above equation we just have a dot product. Then $\mathbf{A}$ is getting replaced with $\mathbf{A} = \mathbf{N}_1A_1 + \mathbf{N}_2A_2 + \mathbf{N}_3A_3$. Then A_1, A_2, and A_3 will come out of the integral as given in the following equation

$$\int_{S^e}(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{N})\,dS \Rightarrow \sum_{j=1}^{3}A_j\iint\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x}+\frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy$$

Then, we will get this as just a product of the two terms as shown in the above equation.

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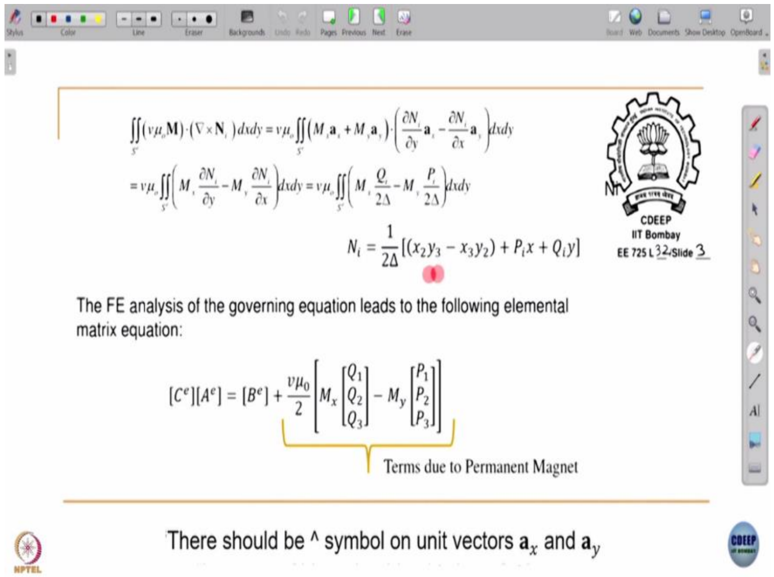

The function N_i is our standard shape function. As we discussed earlier the shape function was in z direction because A was in z direction. So, N_i was in z and it was a function of *x* and *y*.

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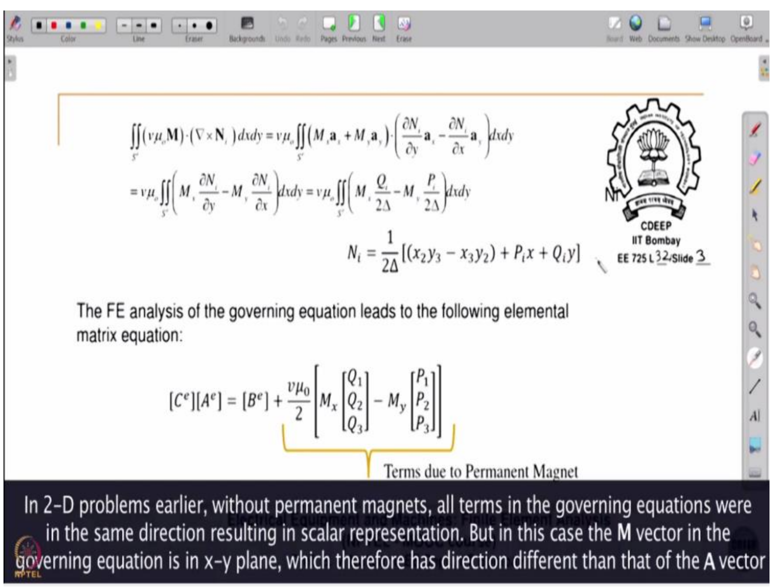

But here we are developing vector formulation because **M** will have two components (*x* and *y* components) as we see later.

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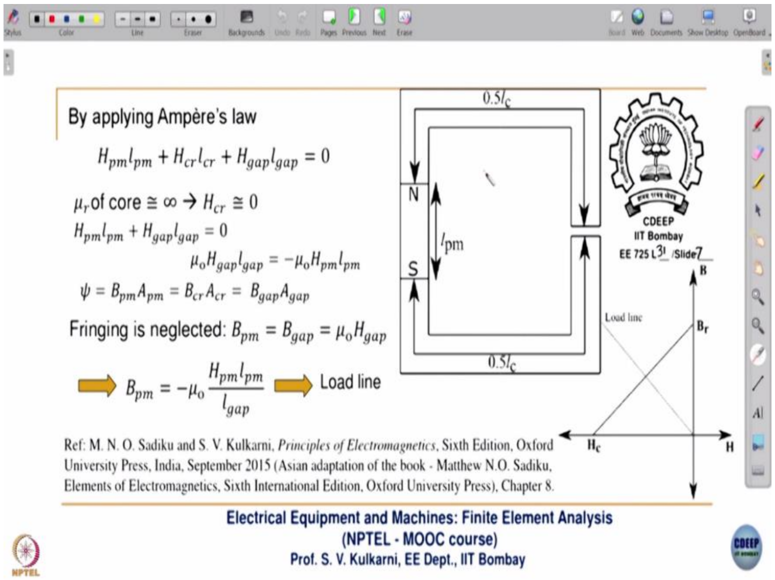

Generally, the magnetization vector of a permanent magnet is in xy plane along the line joining north and south poles. So, M vector can have in general x and y components depending upon the orientation of the permanent magnet. Suppose the permanent magnet is vertical, it will have only y component. If it is at an angle with respect to *x* or *y* axis then it will have both x and y components.

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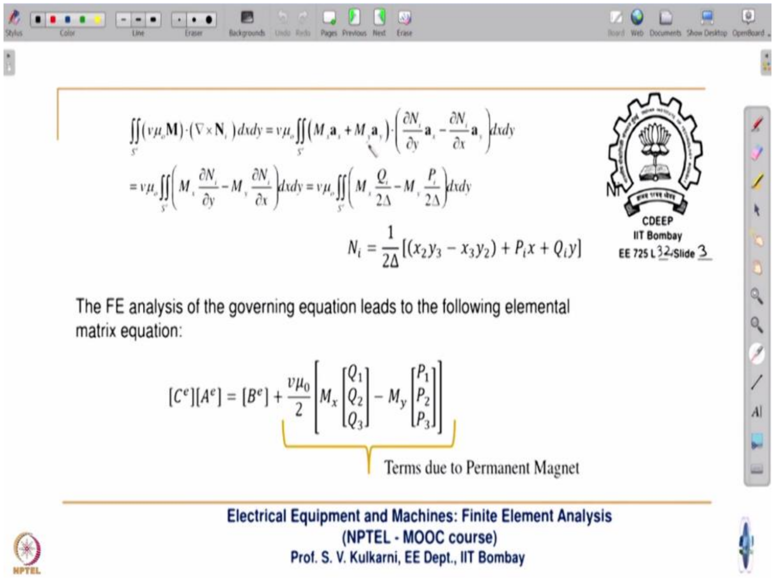

In the previous 2D formulations, each vector in the governing PDE, current density and A were in z direction and there was no other vector in any other direction. So, we were always solving in terms of A using a scalar formulation. Now, we are in a position to write the contribution from the permanent magnet as given below

$$\iint_{S^e} (\nu\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy$$

So, the M vector in the above integral have *x* and *y* components.

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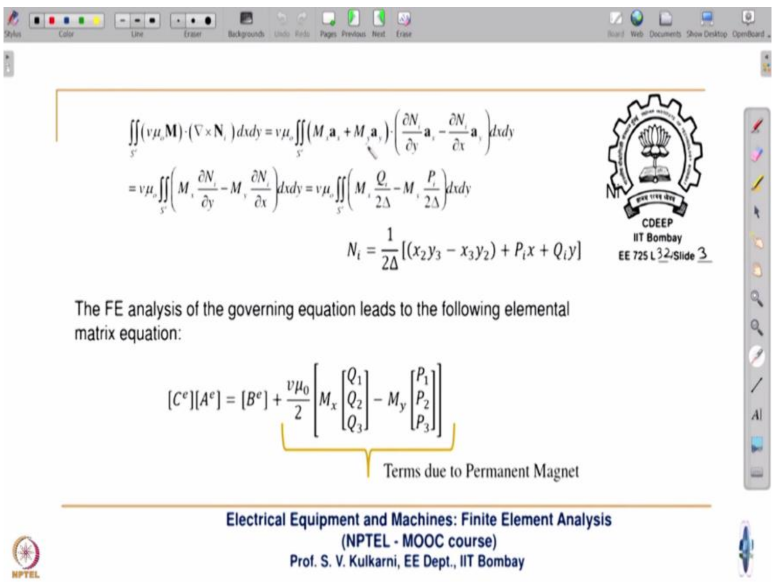

By substituting the expression of $\nabla \times \mathbf{N}_i$ and then simplifying the dot product in the above integral we get,

$$\iint_{S^e} (\nu\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy = \nu\mu_o \iint_{S^e} \left(M_x \mathbf{a}_x + M_y \mathbf{a}_y\right) \cdot \left(\frac{\partial N_i}{\partial y} \mathbf{a}_x - \frac{\partial N_i}{\partial x} \mathbf{a}_y\right) dxdy$$

$$= \nu\mu_o \iint_{S^e} \left(M_x \frac{\partial N_i}{\partial y} - M_y \frac{\partial N_i}{\partial x}\right) dxdy$$

Now, by taking the derivatives of the shape function N_i we get

$$\iint_{S^e} (\nu\mu_o \mathbf{M}) \cdot (\nabla \times \mathbf{N}_i) dxdy = \nu\mu_o \iint_{S^e} \left(M_x \frac{Q_i}{2\Delta} - M_y \frac{P_i}{2\Delta}\right) dxdy$$

So, standard derivative of N_i with respect to y will result only in Q_i and derivative with respect to x will result in P_i only .

Now, remember that the above integral will be for one node. Similarly, we will get two equations for the other two nodes. So, we will get three equations.

Suppose if the above equation is for an ith node then we will get additional two such statements for the other two nodes. That is the reason our final matrix equation at the element level source matrix due to permanent magnet will be as given below

$$\frac{v\mu_0}{2}\left[M_x\begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix} - M_y\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}\right]$$

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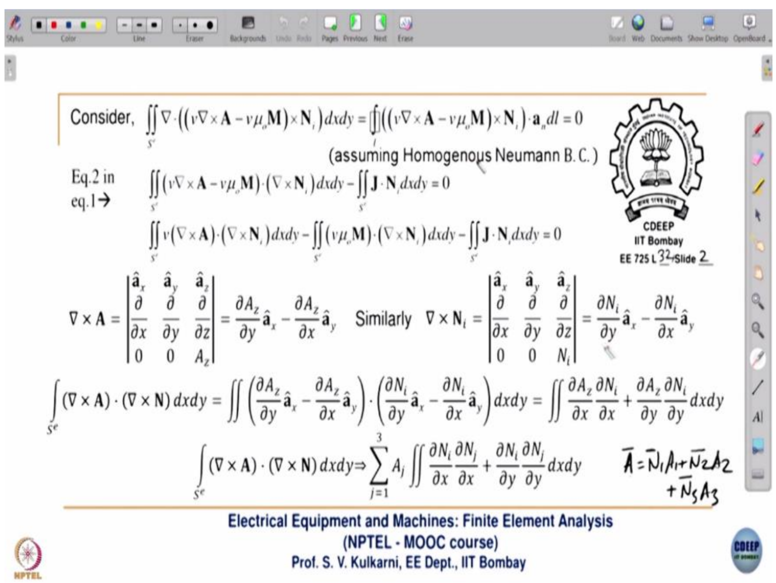

$$\int_{S^e}(\nabla\times\mathbf{A})\cdot(\nabla\times\mathbf{N})\,dS \Rightarrow \sum_{j=1}^{3} A_j\iint\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y}dxdy$$

Now, the above integral will give the 3×3 element coefficient matrix (C matrix) as we have seen in the previous lecture.

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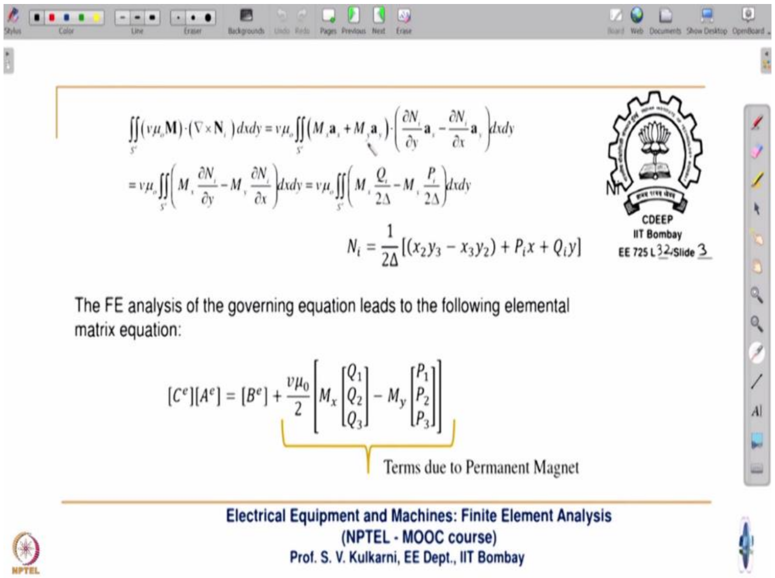

$$[C^e][A^e] = [B^e] + \frac{v\mu_0}{2}\left[M_x \begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \end{bmatrix} - M_y \begin{bmatrix} P_1 \\ P_2 \\ P_3 \end{bmatrix}\right]$$

The $[B^e]$ matrix in the above equation is our standard matrix which will come from a current source condition. That means if there is J in the problem domain then the entries of $[B^e]$ matrix will be $J\Delta/3$. When we combine all the element coefficient matrices then at the end if there is a non homogeneous condition, it will lead to a boundary condition matrix which will come on account of the following integral

$$\oint_l \left((v\nabla \times \boldsymbol{A} - v\mu_o \boldsymbol{M}) \times \boldsymbol{N}_i\right) \cdot \boldsymbol{a}_n dl$$

If there is an outer most boundary with $A = 0$ or some Dirichlet condition then that has to be imposed in the final global system of equations.

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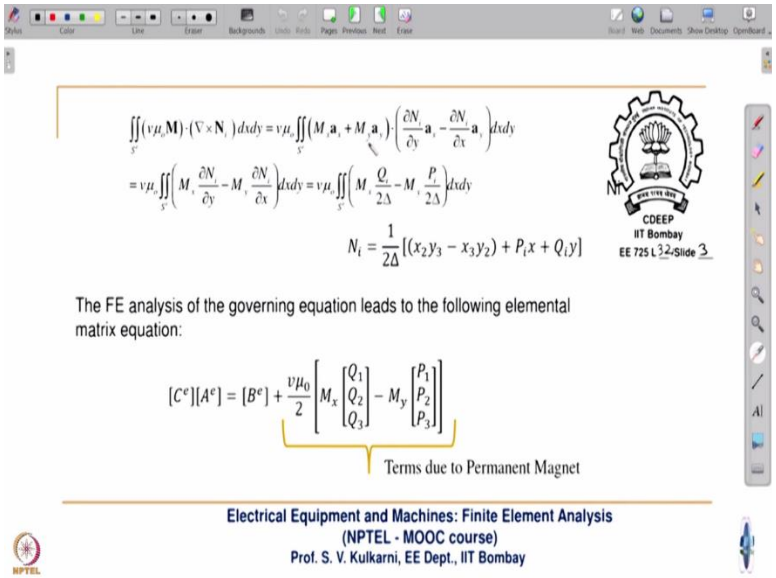

$$[C^e][A^e] = [B^e] + \underbrace{\frac{\upsilon\mu_0}{2}\left[M_x\begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix} - M_y\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}\right]}_{\text{Terms due to Permanent Magnet}}$$

So, in the above FEM equation, the source matrix due to permanent magnet is the new term. The complete derivation that we have witnessed corresponds to this term. In this formulation there are M_x and M_y due to the magnetization vector of the permanent magnet. If there is no permanent magnet then $M_x = M_y = 0$ ant the this whole term becomes 0, and we will get the matrix equation that corresponds to Poisson’s equation without permanent magnet.

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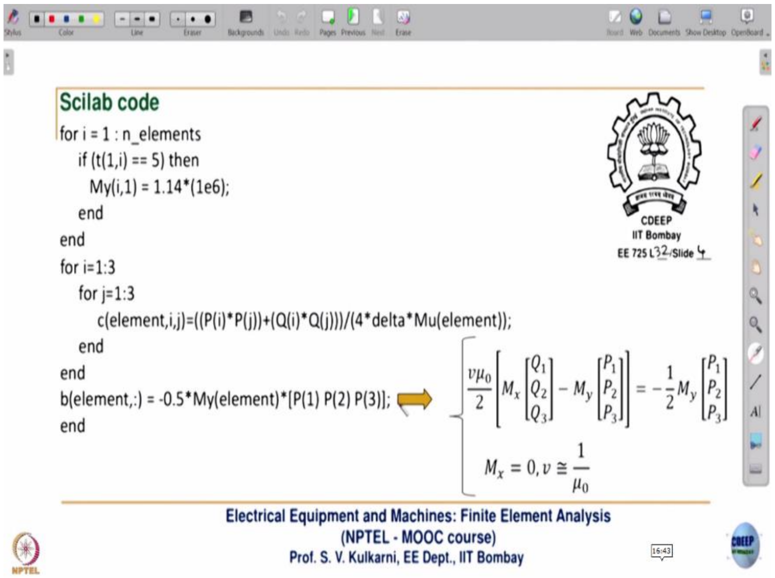

Now, we will quickly see change required in the standard Scilab code. The change will only correspond to the following term and the rest will remain the same.

$$\frac{\upsilon\mu_0}{2}\left[M_x\begin{bmatrix}Q_1\\Q_2\\Q_3\end{bmatrix} - M_y\begin{bmatrix}P_1\\P_2\\P_3\end{bmatrix}\right]$$

So, wherever there is a permanent magnet in the problem domain we have to form the above element level matrix using the code given in the above slide. The value of the magnetization vector is assigned using the following part of the code.

```
for i = 1 : n_elements
    if (t(1,i) == 5) then
      My(i,1) = 1.14*(1e6);
    end
end
```

Here we are considering that the sub domain number of permanent magnet is 5. Remember that the first entry of a column of the t matrix is the sub domain number. We assume that the permanent magnet is in y direction and M will have y component only. So, we are defining the value of M_y as 1.14×10^6.

Now, this value of M_y is quite high as compared to non magnetic materials because we are modelling a permanent magnet. Remember for a ferromagnetic material also M_y will be high.This M_y is actually residual magnetization.

Going further the following part of the code that correspond to element level coefficient matrix is exactly identical to the code that we have seen in the previous lectures.

```
for i=1:3
   for j=1:3
      c(element,i,j)=((P(i)*P(j))+(Q(i)*Q(j)))/(4*delta*Mu(element));
   end
end
```

Now, the additional term that corresponds to the permanent magnet is coded as given below.

```
b(element,:) = -0.5*My(element)*[P(1) P(2) P(3)];
```

Here, $M_x = 0$ because the magnet is oriented in the vertical direction. We are assuming that μ_r for the permanent magnet is equal to 1. That is why reluctivity ν is $1/\mu_0$. But assuming $\mu_r = 1$ means that $\chi_m = 0$. Then $\nu = 1/\mu_0$ and $\nu\mu_0 = 1$. So, that is why only –0.5 remains in the above code. Rest of the code remains the same.

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