Electrical Equipment and Machines: Finite Element Analysis

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Lecture 30: Torque Speed Characteristics of an Induction Motor and FE Analysis of Axisymmetric Problem

Welcome to the 30th lecture, we will see computation of torque speed characteristic of an induction motor using time harmonic formulation that we saw in the last two lectures.

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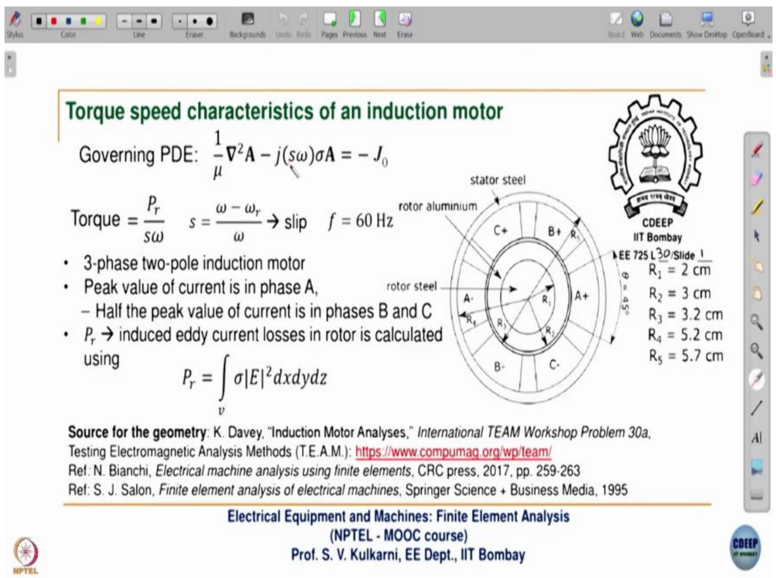

The governing partial differential equation for this problem is given below.

$$\frac{1}{\mu}\nabla^2\mathbf{A} - j(s\omega)\sigma\mathbf{A} = -\boldsymbol{J}_0$$

This equation is familiar to us, the only change here is there is an s term with $j\omega$. Here, s is the slip of the induction motor. Because depending upon the frequency of induced emf, we have to take $s\omega$ as slip frequency. Suppose, if it is rotor then we have to take the corresponding slip. If it is stator, then the value of slip s will be equal to 1. Depending upon the section of the motor we have to choose the value of s. So, the above equation is generalized. The other two terms remain the the same. We know from the induction motor theory, if we know the value of rotor losses (P_r) then torque is calculated by using the following equation.

$$\text{Torque} = \frac{P_r}{s\omega}$$

Here, in the geometry of induction motor the rotor structure is made of solid iron because the value of slip is very small and so the frequency of induced voltage and the value of eddy current losses will be quite small. So we can consider this rotor as solid and it is not laminated. Whereas the stator structure and the outer most part of the stator is laminated steel. But in the stator part we are not interested in the loss calculation. So, we will not be defining conductivity for the stator. There are 2 more simplifications in the model shown in the above slide, they are the stator part does not have teeth and there are only coils which are indicated as A+, A–, B+, B–, C+, and C–.

The outer path indicated as stator steel is just path for the containment of flux. So, the gap between stator and rotor is air gap. The rotor is in the form of an aluminum ring as indicated in the figure. So instead of having squirrel cage induction motor in which the cage having aluminum bars which are short circuited at both ends, in this model we are simplifying the rotor as a simple circular aluminum ring. The source for this geometry is https://www.compumag.org/wp/team/ which have Team Workshop problems. These are standard benchmark problems. Whenever we develop a complex FEM based analysis we have an option to verify our formulation by solving some standard benchmark problems given in this website.

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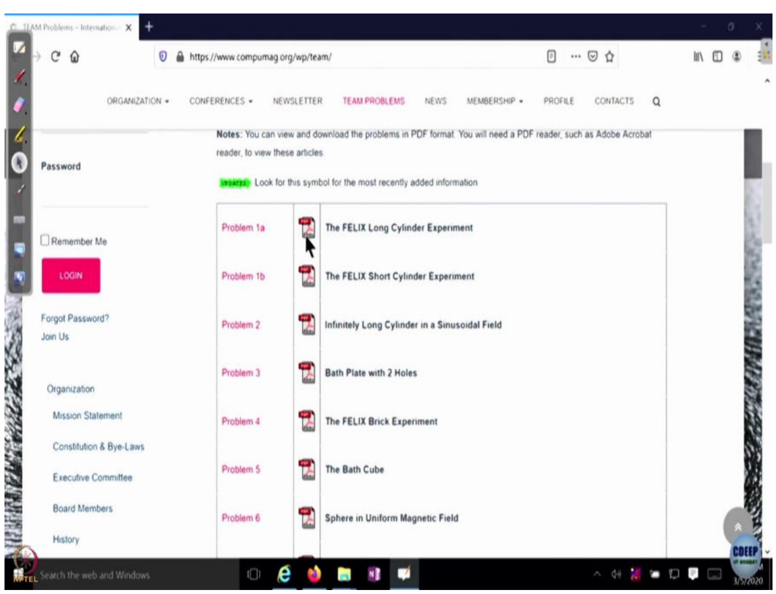
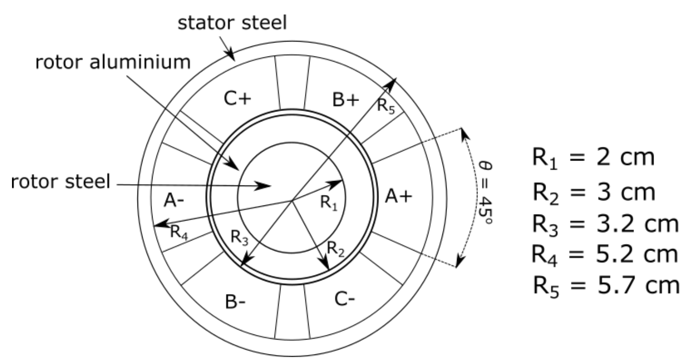


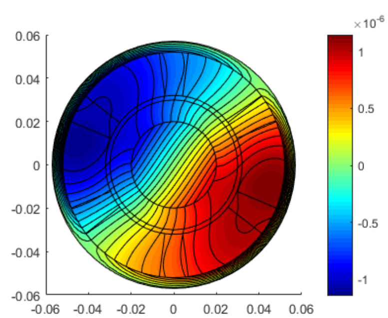
For example, in this website (or slide), there are a number of benchmark problems. We can just open any of those problems (pdf files) in which some theory, problem definition , and results are given. So, we can model any of these problems which are of interest to us and verify our FEM formulation. This a very useful website with a number of problems like, static, time harmonic, transient, coupled circuit field problems, etc.

We have taken this geometry from the above website and the problem is 30a. Of course, the geometry is a simplified one.



So, the above geometry is a 3 phase two pole induction motor. Here, we are actually going to define peak current for A phase. So A+ is peak current, B+ and C+ coils will have negative half peak current. So that means, B– and C– will have positive half peak currents.

The positive currents in A+, B–, and C– together will create flux contours that enclose the three coils. Of course, the currents in these coils are displaced in time. Similarly here, A– will have negative peak current, C+ and B+ will carry negative half peaks. There will be another set of flux contours which will be enclosing the 3 coils carrying negative current. Let us study the flux plot shown in the following figure that we have obtained by solving the time harmonic problem.



The flux contours in red colour are enclosing the coils that carry positive current and the another set of contours in blue colour encloses the coils carrying negative current. After understanding the geometry and ampere turn definition, we will solve the PDE that we saw in the previous slide by discretizing the geometry and following the FE procedure. In the PDE, $\frac{1}{\mu}\nabla^2\mathbf{A}$ term will give the global coefficient matrix having geometry and material information, $j(s\omega)\sigma\mathbf{A}$ will give the D matrix that we saw in the previous lecture and $-\boldsymbol{J}_0$ will give the BJ matrix.

We already saw the expression for torque which is given by rotor losses divided by $s\omega$. Here, ω is synchronous speed in radians per second. So, slip s is given by the following equation.

$$s = \frac{\omega - \omega_r}{\omega}$$

Here, ω is synchronous speed and ω_r is the rotor speed. Here, we are taking frequency as 60 Hz, because the frequency defined in problem 30a is 60 Hz. So, now having got the solution A (magnetic vector potential) we can then find out the induced electric field intensity which is given by the following expression

$$E = -\frac{\partial A}{\partial t}$$

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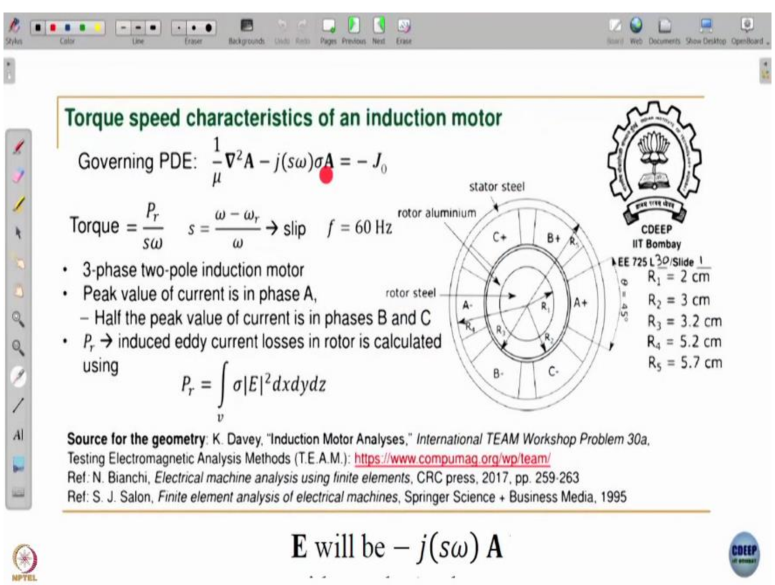


That means $E = -j\omega A$ in frequency domain. Then, we can find out the amplitude of E, and substitute it in the following equation to calculate eddy current losses in the rotor.

$$P_r = \int_v \sigma |E|^2 dxdydz$$

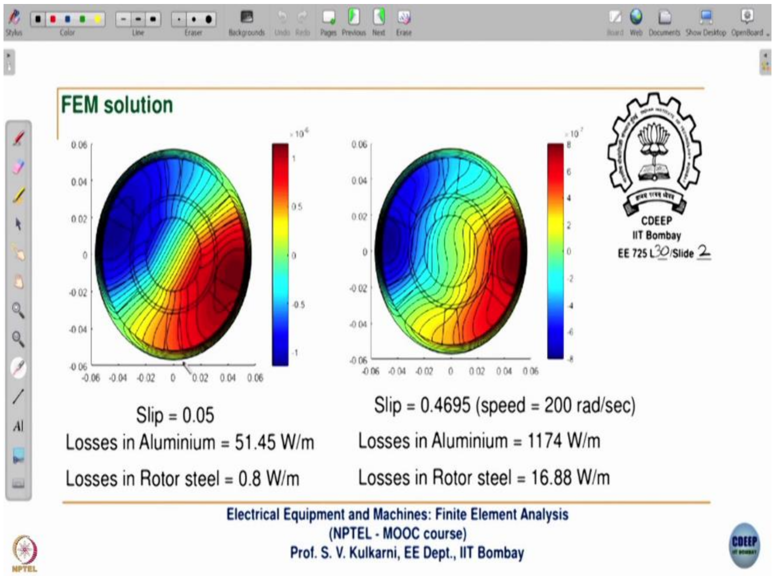
Here, σ is the conductivity. We can calculate the loss in terms of J by using the following expression.

$$P_r = \int_v \rho |J|^2 dxdydz$$

Here ρ is the resistivity. Because $E = \frac{J}{\sigma}$.

In some commercial softwares, they will code loss either in terms of $\sigma|E|^2 dv$ or $\rho|J|^2 dv$. As mentioned previously, the units of both these terms are W/m^3. The dimensions of the induction motor geometry are also given in the above slide. So, remember this is a very simplified geometry just for the verification purpose.

Earlier, we have discussed the geometry independent nature of the finite element methods as an advantage. So, if our finite element formulation works for a problem with trivial or simplified geometry, then it will definitely work for complicated geometry or real geometry of a motor with all the details considered.



So, we have already seen the flux plot on the right hand side of the above slide. Now this plot is for a slip of 0.05 which is a practical slip value.

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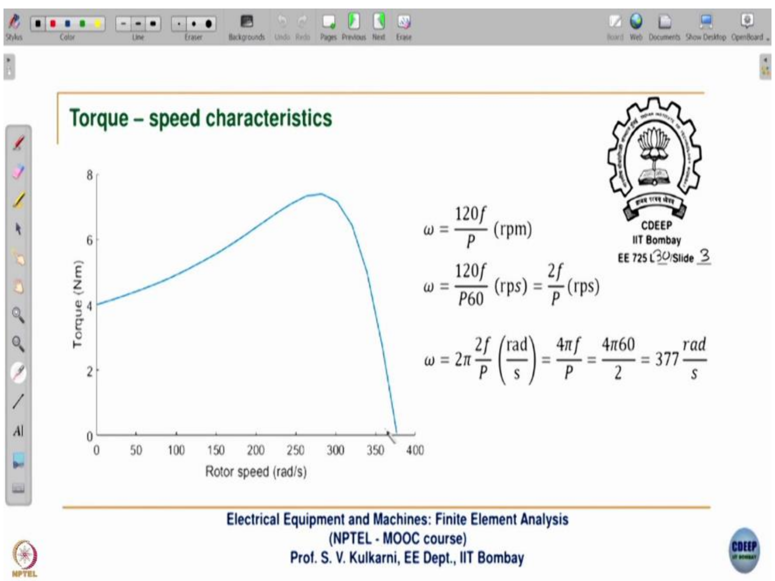
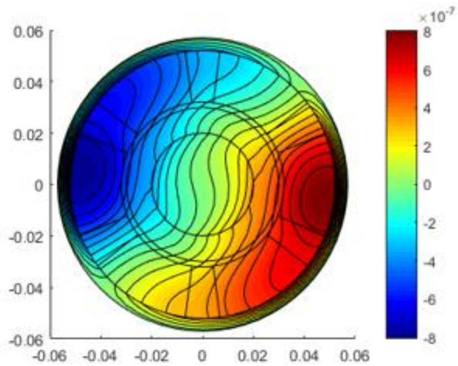


The slip of 0.47 is not practical. Because if we see the torque speed characteristics given in the above slide, the operating region or the stable region is in the linear region. The non linear region between 0 to 300 rad/s is unstable. So, $s = 0.5$ will come somewhere in the unstable region. But just for comparison purpose, we have plotted the flux for $s = 0.5$ which corresponds to the speed of 200 rad/s for 60 Hz case.

Because problem 30a has the corresponding value of torque using which we can verify. So, then we get the following values of losses in aluminium and rotor steel for the two cases.

Slip = 0.05	Slip = 0.4695 (speed = 200 rad/sec)
Losses in Aluminium = 51.45 W/m	Losses in Aluminium = 1174 W/m
Losses in Rotor steel = 0.8 W/m	Losses in Rotor steel = 16.88 W/m

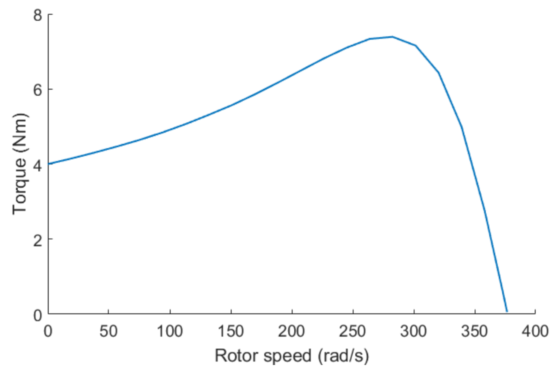
These losses can be calculated using the formula that we have seen in slide 1. Practically slip = 0.47 is not possible. In the following figure we can see that as the value of slip is high, the induced eddy currents and the corresponding distortion will be high.



Slip = 0.4695 (speed = 200 rad/sec)

The flux pattern shown in the above figure is considerably distorted as compared to the flux plot for s = 0.05.

Our objective is to plot torque speed characteristics. For that we have to vary slip in the FEM formulation. For every slip, we solve the governing PDE and we can eventually calculate the value of P_r using which we can calculate the torque. Then we can plot the characteristics given in the following figure.



The x axis of the above figure is nothing but slip which is calculated using the following equation.

$$s = \frac{\omega - \omega_r}{\omega}$$

So, for every slip or rotor speed, we can get one torque value, and that is how we can plot the torque speed characteristics. For this as many FEM simulations are required as many points we want to plot on the torque speed characteristics.

This procedure is fairly easy, because this is a static formulation and it is not a transient case. Once we develop our working code, we have to set up an outer do loop with s varying form 0 to 1. Using this, we can get all the torque values in one go. So, this procedure is fairly straightforward. Here the following calculation corresponds to the synchronous speed.

$$\omega = 2\pi\frac{2f}{P}\left(\frac{\mathrm{rad}}{\mathrm{s}}\right) = \frac{4\pi f}{P} = \frac{4\pi 60}{2} = 377\frac{rad}{s}$$

If rotor speed is equal to synchronous speed then the value of $s = 0$. If $\omega\left(=\frac{120f}{P}\right)$ is divided by 60 we get the speed in revolutions per second (rps). So the speed in rps is $\frac{2f}{P}$. So, if we want to find the speed in radians per second then we have to multiply the speed in rps with 2π because one revolution has 2π radians. So, the speed in radians per second is $\frac{4\pi f}{P}$. So, f = 60 Hz, the value of speed is 377 radians per second. So, this is how we use FE methodology to plot torque speed characteristics of an induction motor.

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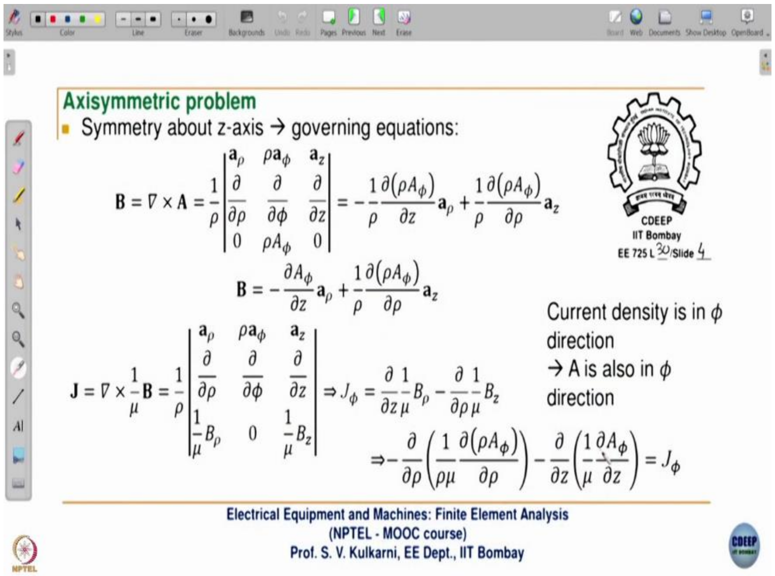

We go to the next topic, axisymmetric problem which are quite often encountered in our electrical engineering when we deal with coils which are mostly circular. If we can simplify some geometrical details and make it symmetrical about axis then the axisymmetric formulation is good because by doing a 2D simulation we are effectively getting 3D fields since the fields are independent in phi direction.

So that is why axisymmetric formulation can be quite handy in the analysis of electrical machines and equipments. Now in this formulation, we are assuming symmetry about the z axis, which is quite logical. Let us see some theory. We know that $\mathbf{B} = \nabla \times \mathbf{A}$. In this case J (current density) is in phi direction. If the axis is in z direction then the current will be in ϕ direction. So, J also will be in ϕ direction.

If current density is in ϕ direction, then A also will be in ϕ direction as we have seen in the basics of electromagnetics. So that is why $\mathbf{B} = \nabla \times \mathbf{A}$ can be expanded as given below

$$\mathbf{B} = \nabla \times \mathbf{A} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \dfrac{\partial}{\partial \rho} & \dfrac{\partial}{\partial \phi} & \dfrac{\partial}{\partial z} \\ 0 & \rho A_\phi & 0 \end{vmatrix} = -\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho}\mathbf{a}_z$$

$$\mathbf{B} = -\frac{\partial A_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial \rho}\mathbf{a}_z$$

The determinant in the above derivation is the basic definition of curl in cylindrical coordinate system. If we expand the determinant, we will get the two components of B that are given in the above equation, which is obvious because J in ϕ direction will lead to B with two components in ρ and z directions. In earlier case in the Cartesian system we used to take J in z direction and then B will have x and y components. Similarly, if J is in ϕ direction then B will have ρ and z components. Now, ρ from $\frac{1}{\rho}\frac{\partial(\rho A_\phi)}{\partial z}$ can be taken out because ρ is constant when we are taking derivative with respect to z. We can calculate J using $\mathbf{J} = \nabla \times \mathbf{H}$. Also, H is $\frac{B}{\mu}$. So, then we can expand the curl expression as given below.

$$\mathbf{J} = \nabla \times \frac{1}{\mu}\mathbf{B} = \frac{1}{\rho}\begin{vmatrix} \mathbf{a}_\rho & \rho\mathbf{a}_\phi & \mathbf{a}_z \\ \frac{\partial}{\partial \rho} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ \frac{1}{\mu}B_\rho & 0 & \frac{1}{\mu}B_z \end{vmatrix} \Rightarrow J_\phi = \frac{\partial}{\partial z}\frac{1}{\mu}B_\rho - \frac{\partial}{\partial \rho}\frac{1}{\mu}B_z$$

By substituting the two components of B, the above equation can be written as

$$-\frac{\partial}{\partial \rho}\left(\frac{1}{\rho\mu}\frac{\partial(\rho A_\phi)}{\partial \rho}\right) - \frac{\partial}{\partial z}\left(\frac{1}{\mu}\frac{\partial A_\phi}{\partial z}\right) = J_\phi$$

Here J will be in only ϕ direction and that can be verified using the above equation.

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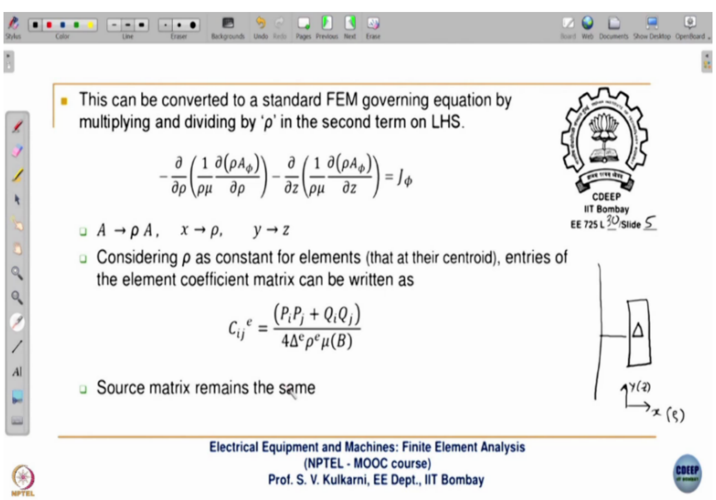


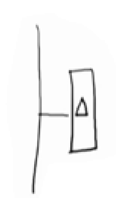
Going further, if we have to use the same formulation that we developed for Cartesian systems then we have to make the above equation as symmetrical. Then we can apply the same formulation that we developed in *xy* system for this ρz system. For that, we have to multiply the numerator and denominator of $\frac{1}{\mu}\frac{\partial A_\phi}{\partial z}$ by ρ. Because ρ is constant and it can be taken inside $\frac{\partial}{\partial z}$. After this modification the equation will become symmetrical as given below.

$$-\frac{\partial}{\partial \rho}\left(\frac{1}{\rho\mu}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho}\right)-\frac{\partial}{\partial z}\left(\frac{1}{\rho\mu}\frac{\partial\left(\rho A_\phi\right)}{\partial z}\right)=J_\phi$$

Here, we have to replace ρ with x and z with y to compare with respect to our formulation in Cartesian system. Suppose in *xy* system, the horizontal axis is x and the vertical axis is y. In cylindrical system, the horizontal axis is ρ and vertical axis is z. So, x is nothing but ρ and y is nothing but z. So that is why, ρ is replacing x, z is replacing y and ρA is replacing A_z.

In two dimensional approximation in xy system we always took current in the z direction, so A was in the z direction. A_Z is now replaced by ρA_ϕ. Now, with this manipulation we can use the code developed for the Cartesian system for the axisymmetric case with no change. The only thing that we have to remember is the solution that we get is not $A_\phi$but it is ρA_ϕ. Then if we want to find the magnitude of A at any point then we have to divide the solution value by ρ at that point.

After having solution, suppose if we want to find the value of A_ϕ at a point then we have to divide the solution by the corresponding ρ at that point. So that $\frac{\rho A_\phi}{\rho}$ will give A_ϕ at that point. Considering ρ as constant for the element is another approximation we do for simplifying our calculation. That means suppose we have a triangular element as shown in the following figure then the value of ρ is changing at every point within the element.



But if the element is fairly small, we can assume that ρ is constant for every element and we can consider its value as equal to the value at centroid of the element. Then if we do all these simplifications and manipulations as explained earlier then the formula for entries of element coefficient matrix will remain the same as given below.

$$C_{ij}{}^{e} = \frac{\left(P_i P_j + Q_i Q_j\right)}{4\Delta^{e} \rho^{e} \mu(B)}$$

In this formula the only change is in the denominator, we use one ρ at the element in which calculations are being made. Since we are assuming ρ for the element as constant we can just write here one ρ value in the above expression. Earlier in the Cartesian system, we had only μ in the denominator. Now here in this formulation, we have $\rho\mu$ in the denominator.

In the above expression, we have μ as a function of B, this is not valid yet because we have not formulated any non-linearity. Just consider this $\mu(B)$ as just μ.

μ has to be considered as a function of B when we actually formulate non-linearity which we will see in a later lecture. So now, we consider this as μ which is constant over each element if the properties of all materials as linear. Source matrix remains the same which we have calculated earlier, that means J_{ϕ} in the PDE will result into $\frac{J\Delta}{3}$.

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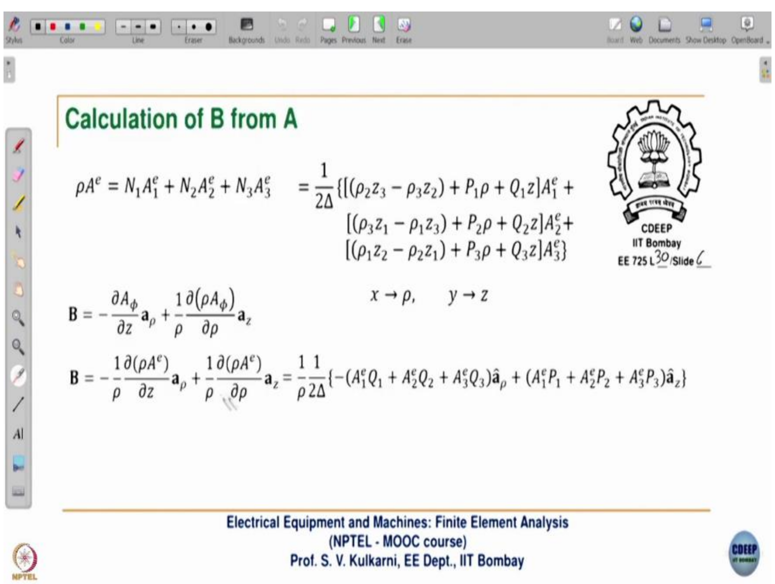

Now, we will see how to calculate B from A. In this formulation, the approximation of unknown potential function is given below

$$\rho A^e = N_1 A_1^e + N_2 A_2^e + N_3 A_3^e \qquad = \frac{1}{2\Delta}\{[(\rho_2 z_3 - \rho_3 z_2) + P_1\rho + Q_1 z]A_1^e + [(\rho_3 z_1 - \rho_1 z_3) + P_2\rho + Q_2 z]A_2^e + [(\rho_1 z_2 - \rho_2 z_1) + P_3\rho + Q_3 z]A_3^e\}$$

Here, the expressions for N_1, N_2, and N_3 are functions of ρ and z as given in the above equation with *x* and *y* replaced by ρ and z. Now in one of the previous slides we have already calculated the following expression of B.

$$\boldsymbol{B} = -\frac{\partial A_\phi}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial\left(\rho A_\phi\right)}{\partial \rho}\mathbf{a}_z = -\frac{1}{\rho}\frac{\partial(\rho A^e)}{\partial z}\mathbf{a}_\rho + \frac{1}{\rho}\frac{\partial(\rho A^e)}{\partial \rho}\mathbf{a}_z$$

Then we take the derivatives of ρA_ϕ with respect to z and ρ. This leads to the following expression.

$$\mathbf{B} = \frac{1}{\rho}\frac{1}{2\Delta}\{-(A_1^e Q_1 + A_2^e Q_2 + A_3^e Q_3)\hat{\mathbf{a}}_{\boldsymbol{\rho}} + (A_1^e P_1 + A_2^e P_2 + A_3^e P_3)\hat{\mathbf{a}}_z\}$$

This expression is similar to the one that we have seen in the Cartesian system except that we have $\frac{1}{\rho}$ which is logical. The terms Q_1, Q_2, and Q_3 and P_1, P_2, and P_3 are the standard ones as earlier which are in terms of ρ and z co-ordinates.

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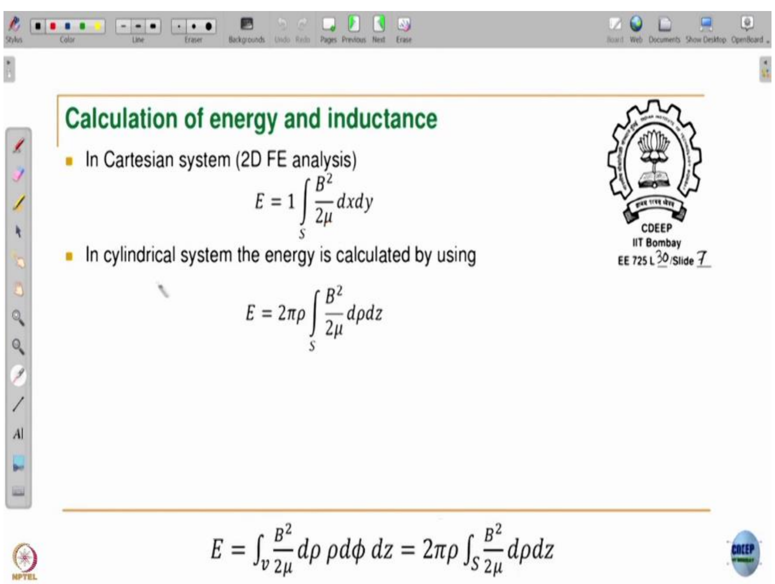

After calculating the value of B we have to evaluate the values of energy and inductance, which is straightforward. In the Cartesian system, we had multiplied the energy density term $\frac{B^2}{2\mu}$ with the differential volume $dxdydz$. But here we are taking per meter depth that is why 1 appears in the following equation.

$$E = 1\int_S \frac{B^2}{2\mu} dxdy$$

The above expression represents the total energy. In cylindrical system, the total energy can be written as

$$E = 2\pi\rho \int_S \frac{B^2}{2\mu} d\rho dz$$

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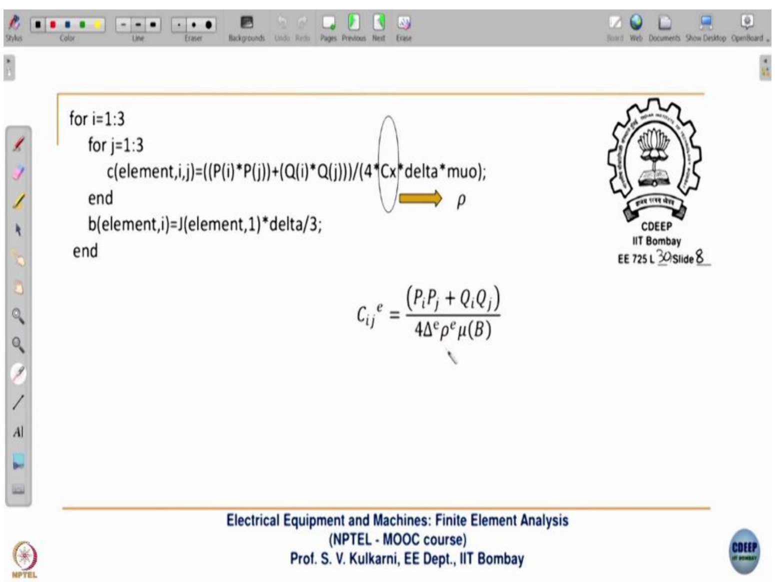

In terms of code that corresponds to element coefficient matrix we should have μ accompanied with ρ as given in the above slide. So this is the only change because of ρ in the denominator. The source matrix remains the same as $\frac{J\Delta}{3}$.

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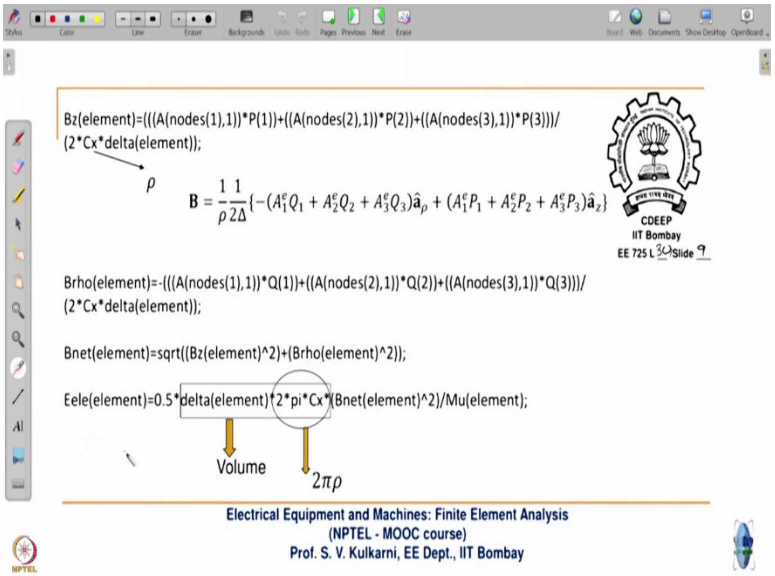

Similarly in the code that corresponds to calculation of magnetic flux density we have B_z and B_ρ instead of B_y and B_x. We have already seen the expressions of B_z and B_ρ that are given in the above slide. Compared to the code for the Cartesian system, we have additional ρ term (which is variable Cx in the above code) as indicated in the above slide. Then $B_{net} = \sqrt{B_\rho^2 + B_z^2}$. We have to calculate the energy for each element using the expression $2\pi\rho \frac{B^2}{2\mu}\Delta$. We have to integrate to calculate the energy for the whole domain.

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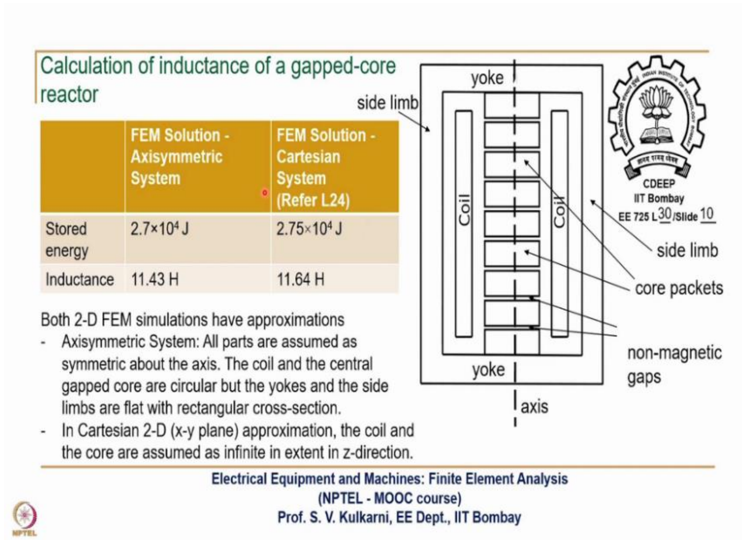

In the above slide, we will analyze inductance of a gapped-core reactor which was discussed in L24 using axisymmetric FE formulation. As shown in the figure given in the above slide, there is a central core with non-magnetic gaps whose total length is designed to have a particular value of inductance. Most of the energy is stored in these gaps because of their high reluctance and hence the energy stored in these gaps predominantly decides the inductance value.

Here, the central core and the coil are circular and they are symmetric about the axis. The top and bottom yokes and two side limbs are flat structures with rectangular cross section. Hence truly speaking, the entire geometry is not axisymmetric. Earlier we have analyzed the same problem in 2D Cartesian system and the computation is done per meter depth in z direction. So, in that analysis it is essentially assumed that the entire geometry is infinite in extent in the z direction, which is also not true.

Axisymmetric approximation is not bad if the coil and the core have large diameters. Thus we have to understand that both Cartesian and axisymmetric 2D approaches are approximate as summarized in the above slide. But these approximations are good enough for inductance calculation. For more accurate computation, one has to do 3D FEM analysis. The values of inductance calculated by using both methods are close to each other as given in the above slide.

In the method based on Cartesian coordinates, we had calculated energies per meter depth in various parts and we have multiplied them with the corresponding mean diameters to obtain the total energy. In the axisymmetric approach, we multiplied the elemental energies by corresponding mean diameters at the element level using the FEM code. This is another difference between the two models. In the next lecture, we will see permanent magnets and corresponding FE formulation. Thank you.

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