Electrical Equipment and Machines: Finite Element Analysis
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Lecture 03
Revisiting EM Concepts: Vector Algebra and Coordinate Systems

Good morning and welcome to the third lecture, in the last two lectures we actually saw the introduction to the course outline, need for FEM analysis and also, we understood the differences between analytical and numerical techniques.

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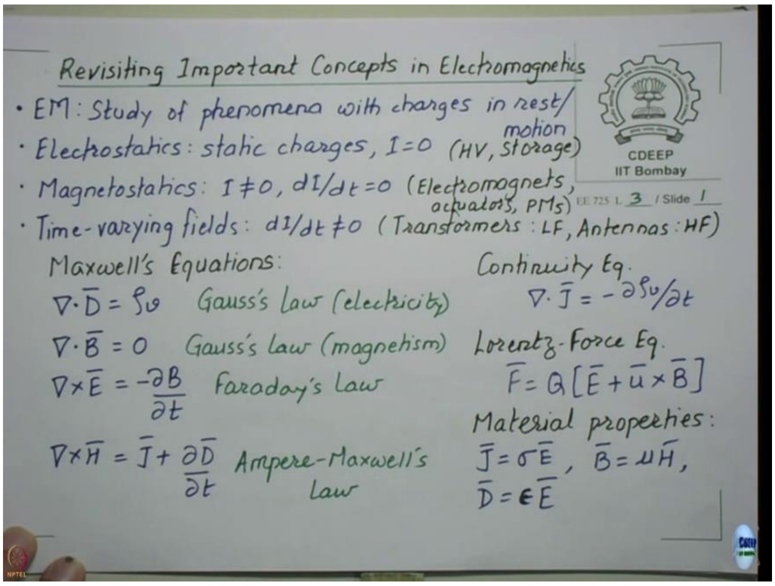
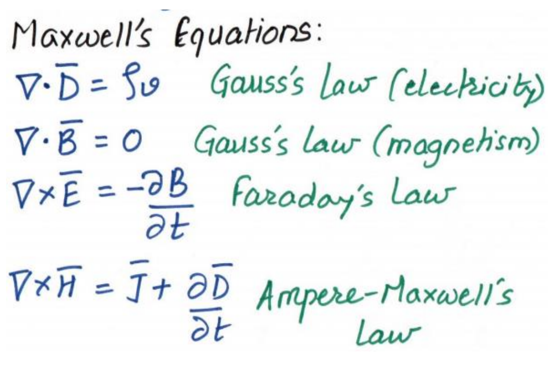


Today, as I was mentioning to you yesterday, we will actually do a revisit to important concepts in electromagnetics and only those concepts will be covered which are more relevant to FE analysis. So, the first and foremost thing that we always remember when we deal with electromagnetics is Maxwell's equations (given below).



So, basically electromagnetics is a study of phenomena of charges in either rest or motion. And then depending upon whether the charges are at rest or motion, you have either electrostatics in which charges are static which means current (dq/dt) is 0. And electrostatics has many applications like in high voltage engineering or energy storage.

When it comes to magnetostatics, current is not equal to 0, charges are in motion, but they are moving with uniform velocity and $\frac{di}{dt} = 0$ and that has many applications like in electromagnets, plungers, permanent magnets. Although in permanent magnets you may not have free currents, a permanent magnet itself acts as a source of magnetic field.

And next is the time varying fields, in which charges are accelerated and the rate of change of current is not 0. So, all these above phenomena, what I just described are formulated by Maxwell's equations. Now, the first law, $\nabla \cdot \mathbf{D} = \rho_v$ is Gauss's law for electrostatics, then the second one $\nabla \cdot \mathbf{B} = 0$ is Gauss's law for magnetics, $\nabla \times \mathbf{E} = -\frac{\partial B}{\partial t}$ is Faraday's law and finally $\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$ is Ampere-Maxwell's law.

Now, if you see the evolution of these laws, basically the equation $\nabla \cdot \mathbf{B} = 0$ represents the magnet kind of properties in terms of lodestones discovered many centuries ago, but only it was in 15^{th}, 16^{th} and 17^{th} century, more understanding came and finally around 1800, Faraday was the one who again postulated this Gauss's law of magnetism in the form $\nabla \cdot \mathbf{B} = 0$. Then this Faraday's law, basically as the name suggests, it was discovered after experimentations by Faraday, around 1831.

And the Ampere's law, $\nabla \times \mathbf{H} = \mathbf{J}$ was postulated by Ampere in somewhere around 1826, whereas this $\frac{\partial \mathbf{D}}{\partial t}$ term was introduced by Maxwell around 1861. That is why it is called as Ampere-Maxwell's law. And Maxwell's main contribution was this $\frac{\partial \mathbf{D}}{\partial t}$ term which explains current through the capacitors or wave propagation in free and any medium and because of these last two discoveries one is Faraday's law and Maxwell's contribution of $\frac{\partial \mathbf{D}}{\partial t}$, you see the tremendous progress that technology has made in many areas of Electrical and Electronics Engineering.

Apart from the four Maxwell's equations shown above, you have also continuity equation, which is $\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$. Now, this equation is not sort of independent of these Maxwell's equations because you can clearly see in Maxwell's equations and later, we will derive the correlation between this continuity equation and this $\frac{\partial \mathbf{D}}{\partial t}$ term.

In fact, Maxwell introduced this $\frac{\partial \mathbf{D}}{\partial t}$ term, so that continuity equation is not violated and is always valid under all conditions. Apart from these Maxwell's equations, you need, what is known as the Lorentz force equation (given below) for calculating forces on static and moving charges,

$$\bar{F} = Q[\bar{E} + \bar{u} \times \bar{B}]$$

where $\bar{u}$ is the velocity, $\bar{\mathrm{B}}$ is the flux density, Q is the charge, and $\bar{\mathrm{E}}$ is the electric field intensity.

Now, in order to calculate electromagnetic fields, you also need to know material properties, which are defined by these relations $\mathbf{J} = \sigma\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, and $\mathbf{D} = \epsilon\mathbf{E}$.

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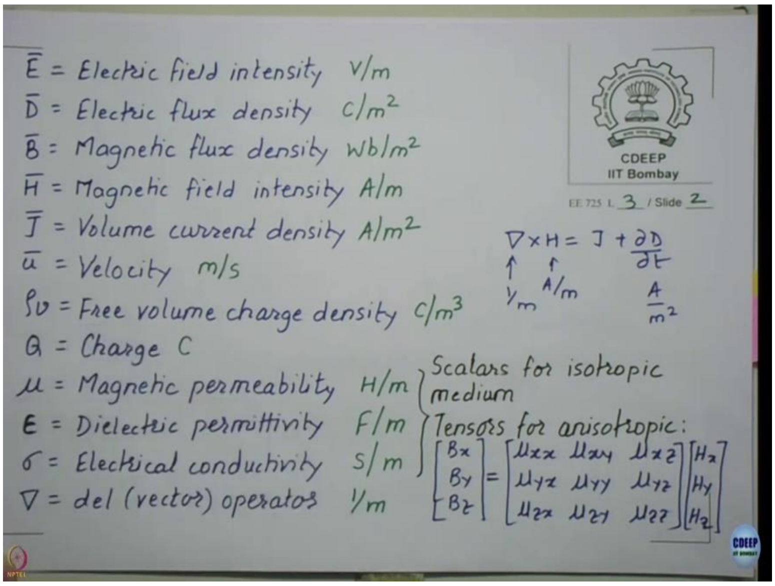

Now, let us quickly see what are all these vectors and scalars are and their units, understanding units and matching units on both sides of a given equation is important to understand electromagnetic fields in a better way. For example, consider $\nabla \times \mathbf{H} = \mathbf{J} + \frac{d\mathbf{D}}{dt}$, we know the unit of H is A/m, unit of ∇ is 1/m. So, the total unit of $\nabla \times \mathbf{H}$ becomes A/m^2, and we know the unit of J is A/m^2 and then for $\frac{\partial D}{\partial t}$, unit of D is C/m^2, C/s is again A and that is why you get again A/m^2 for this last term.

So, I suggest all those who want to learn an electromagnetic phenomenon whenever they see any equation, it is good to match units on both sides, so, the understanding becomes better. So, I think the fields in the above slide or figure are all well-known quantities, those quantities which are marked by a bar are the vectors, they have both magnitude and direction and there are some scalars also.

So, you have E, D, which are representing electric fields, B and H are describing the magnetic fields, I will tell you the subtle differences between D and E, B and H, little later as we progress in understanding more concepts.

So, then of course, you have ρ_v free volume charge density, always remember, many people generally make mistake in understanding this entity. ρ_v is free volume charge density and not bound charge density. That is the first thing, more about it later. Also, these material properties for isotropic medium, they are just scalar numbers.

Whereas, when you have anisotropic medium that means properties of the medium are changing with direction i.e., with x y z, then they are represented as tensors, that means you have to write the corresponding material parameter whether it is dielectric constant, ϵ or permeability μ in terms of a tensor 3×3 matrix as shown below for magnetic fields.

$$\begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} = \begin{bmatrix} \mu_{xx} & \mu_{xy} & \mu_{xz} \\ \mu_{yx} & \mu_{yy} & \mu_{yz} \\ \mu_{zx} & \mu_{zy} & \mu_{zz} \end{bmatrix} \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix}$$

Now, the 3×3 matrix in the above equation has both diagonal and off diagonal elements, but the off diagonal elements $\mu_{xy}, \mu_{xz}, \mu_{yx}$, etc., will become 0 if material axes and coordinate axes are aligned. If they are not aligned then of course, you will have all the nine terms, I hope you understand. Suppose, you have a coordinate axis in one way, and material axes are not aligned along the coordinate axis, then you will have these off diagonal terms.

Now, what I am going to do is, in a typical book on basics of electromagnetics, you will have various chapters, on each chapter I am going to have one or two slides of explanation, because this is just revisiting important concepts, I am not going to go into depth. So, but the only thing what I am going to do is I am going to stress more on the concepts which are sometimes difficult to understand.

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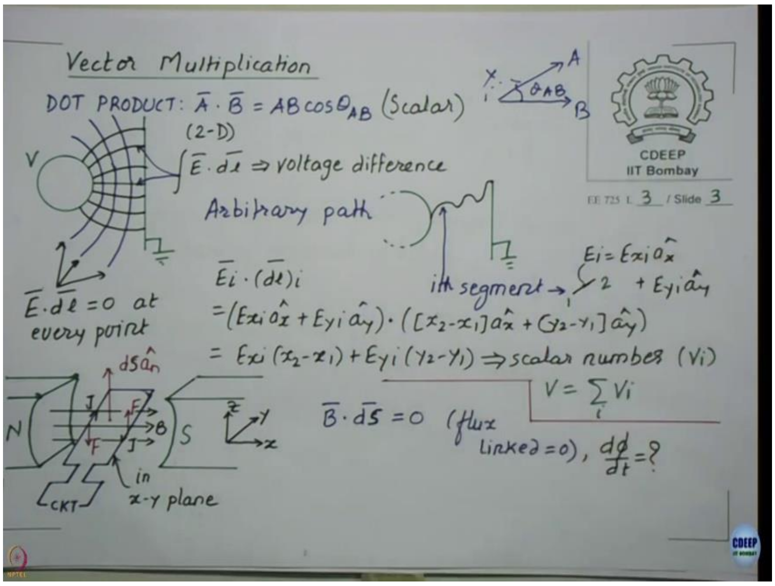
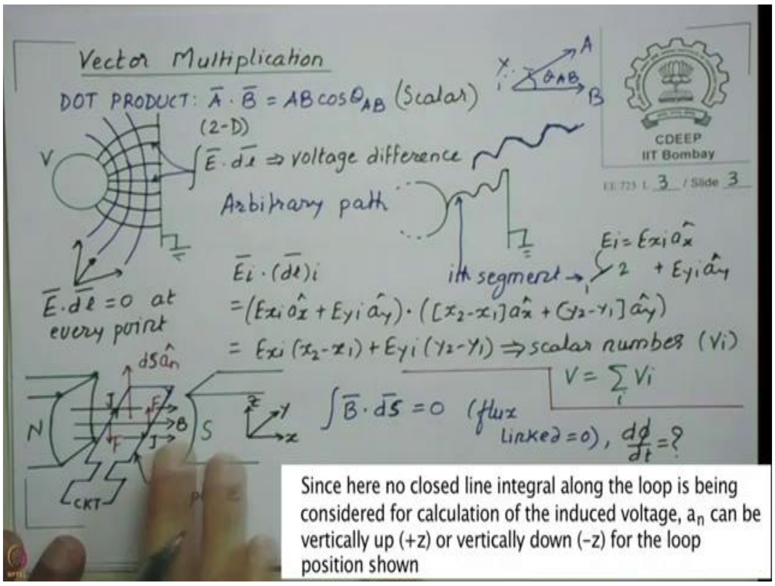
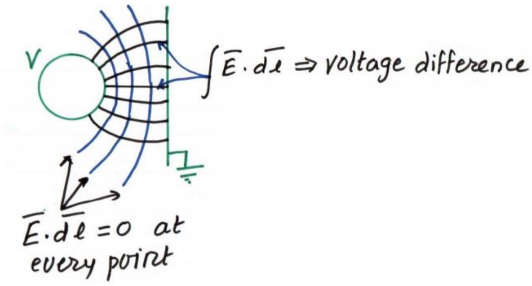




So, first is, when you come to vector algebra, you have what are called as vector multiplications in terms of dot and cross products. Now, we all know the dot product is simply defined as $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta_{AB}$, now you need to remember that this θ_{AB} is the smaller angle between the vectors **A** and **B**. Second is you can call this dot product as a 2D product, two dimensional, because you can always make a plane pass through any two vectors. And basically, dot product is telling you the interaction of these two vectors and when you take a dot product it gives the magnitude of one vector along the direction of the other. So, it is basically the interaction between the two vectors and since those two vectors are in one plane, this is called a sort of 2D product. Whereas later on we will see cross product which is a 3-dimensional product because it requires all the three dimensions.

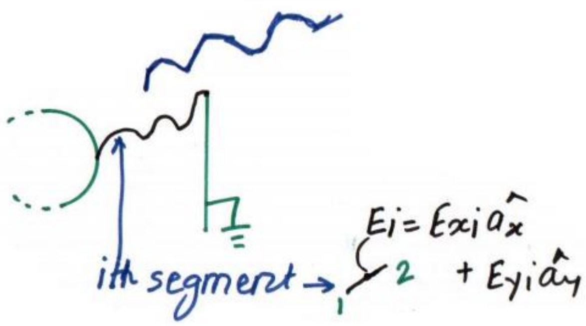
Now, a little bit more understanding on this dot product, suppose we take again this high voltage lead and ground configuration (shown in the following figure) which we have seen in the previous lecture.



You have these equipotential lines as shown by the blue contours. And then you have the locus of electric field intensity vectors given by the black contours. The $\int \mathbf{E} \cdot d\mathbf{l}$, along any of the field contours gives the voltage difference between the two electrodes.

Now, if you take the line integral along the black colored contours that means along the electric field intensity contours, then the line integral reduces to a simple scalar product because E and dl are along the same direction. So, then it becomes a very simple scalar product to calculate, if you take both the vectors along the same direction. So, you can also observe that although the length is more for the two extreme contours in the above figure, the integral $\int \mathbf{E} \cdot d\mathbf{l}$ is same, why? Because although dl is more what is going to be less along this is E. So, $\int \mathbf{E} \cdot d\mathbf{l}$ is going to be same along this contour as well as along this shortest contour.

Then if I take some arbitrary path between these two electrodes as shown below.



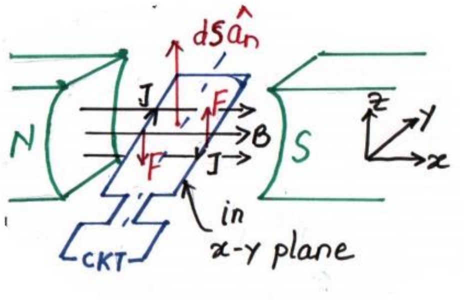
Since I do not have an easy expression of E as a function of space, I need to have a numerical integration procedure. So, suppose you do FEM analysis and get electric field intensity values in the whole domain. You know electric field intensity at more or less every point in the domain, why I am saying more or less, because since you are using a numerical procedure as we will see later, generally field values are calculated only at few points and then you do interpolation to calculate fields at other points. So, what we have to do is, in this case, for arbitrary contour, we have to divide this into small segments as shown with blue line in the above figure. So, effectively we are doing sort of linearization, over each segment. Suppose for one of those segments, which is between nodes 1 and 2 which is called as an i^{th} segment (indicated in the figure). You know that electric field intensity is given by $E_{xi}\hat{\mathbf{a}}_x + E_{yi}\hat{\mathbf{a}}_y$ and then you calculate the dot product $\mathbf{E} \cdot d\mathbf{l}$ for that segment, now here again there is some assumption that E is taken to be same all along this segment and at the center of that segment.

So, that again is an approximation, if somebody says oh!, this approximation may lead to an error then what you have to do is, you have to reduce this segment length further and make it even smaller and again what you will have to assume is E along the segment is same as E at the center of the segment or some other approximation, like considering E along the segment is same as average of E at both ends.

So, these are all the ways of numerical approximation, how best you can further and further make it fine or discretize to improve the solution and decide the accuracy, more about this little later, because, this course is on numerical methods and we will discuss this aspect in more detail when we see FEM theory.

So, by calculating this dot product, we will simply get a scalar number, which corresponds to the voltage drop across that i^{th} segment. So, this total integral along this arbitrary path will just then reduce to sum of voltage drops over all these segments. So, in a very simple way we have understood what is a numerical method. This is also a numerical technique to find line integral by subdividing an arbitrary path, still, this is not one dimensional. It has both dimensions, but what we are doing is this arbitrary shape is divided into segments and then we are doing this numerical procedure.

Now, an interesting thing in rotating machines which can be used to understand the dot product is calculation of flux linkages; a coil of a rotating machine is shown below.



So, this coil is supplied by a circuit (ckt). This circuit can be some source which is supplying the current. Current or current density (J) is flowing through this coil as shown in the figure.

Now, we need to understand, what is the concept of area as a vector. So, now this coil has this area which is a vector, and its magnitude is the area of the coil. And the direction is given by the unit normal to the plane containing the coil. So, that is why the way this area vector is shown is $dS\hat{\mathbf{a}}_n$, where $\hat{\mathbf{a}}_n$ is the unit normal to the plane of the coil and dS is an elemental area.

So, now actually if you calculate $\iint \mathbf{B} \cdot d\mathbf{S}$, the $\iint \mathbf{B} \cdot d\mathbf{S}$ all along the loop is 0, because B is parallel to the coil plane. The direction of B is normal to the direction of coil which is $\hat{\mathbf{a}}_n$ and that is why in this position of coil there is no flux linkage.

But even if flux linkage is 0, what about $\frac{d\phi}{dt}$? where ϕ is flux linkages and it is maximum. So, although $\phi = 0$, the $\frac{d\phi}{dt}$ is maximum. So, in fact for this position of coil, the voltage induced in some motor or generator application involving such a coil, we get maximum induced voltage as the position of coil varies with time.

And we will see some applications of this little later. The question that is asked by one of the participants is about this dS. dS is a small unit area of this loop. So, the area of this whole loop is all the area enclosed by this coil. So, the total flux linked by this coil will be $\iint \mathbf{B} \cdot d\mathbf{S}$.

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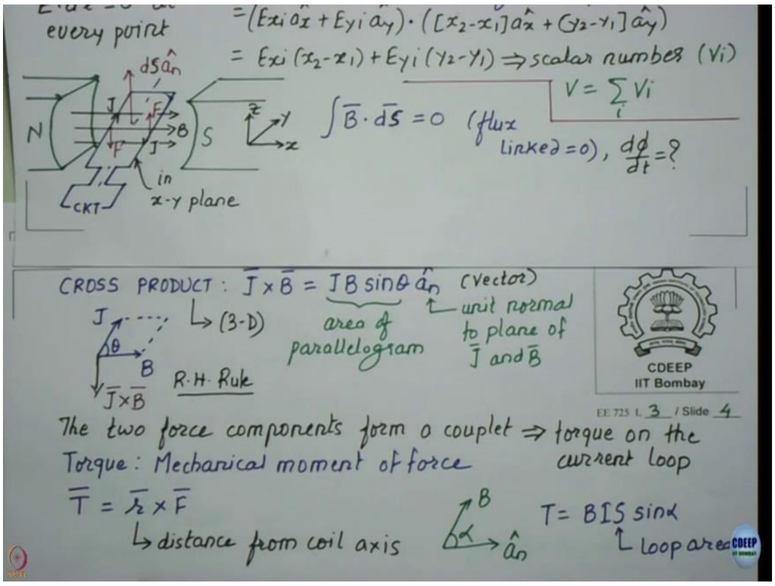

Now, coming to the cross product, again we will make use of this same geometry. Now the simplest example to take again is the Lorentz force or force density, which is given by $\mathbf{J} \times \mathbf{B}$ and we know the cross product is given by $JB\sin\theta\,\hat{\mathbf{a}}_n$. Now, cross product leads to a vector and that is why you can call it as a 3D product.

As shown in the above slide, you have J, B and θ, again, θ is the smaller angle between the two vectors. And for the direction of $\mathbf{J} \times \mathbf{B}$ you have to always use the right-hand rule, so from the first vector J, you turn the fingers of right hand towards the second vector B and the direction of the thumb will give you the direction of the cross product. So, $\mathbf{J} \times \mathbf{B}$ you will get force in the downward direction.

And $JB\sin\theta$ is the area of the parallelogram formed by the vectors $\mathbf{J}$ and $\mathbf{B}$ as shown in the above slide. So, now, if we find out the force directions, on the two coil sides, you have J and B, so in this case, if you apply right hand rule, the force will be in the downward direction for the coil side on the right hand side, whereas, here the B direction is same, now the current or J direction is reversed for the other coil side, so the direction of force becomes upward.

So, now these two force components, they form a couplet around this axis of the coil and that couplet will produce torque and this is one of the main principles in rotating machines. That torque is given by $\mathbf{r} \times \mathbf{F}$, where r is the distance of the coil side from the coil axis. Note that the other two coil sides will not contribute to the torque because the direction of current is along the B.

So, now starting with this equation $\mathbf{T} = \mathbf{r} \times \mathbf{F}$ with some mathematical manipulation which is given in all textbooks on electromagnetics, you will find that $T = BIS\sin\alpha$, where S is the loop area. Don't get confused with the S in the diagram here. I have marked N and S as the north and south poles. In this course because we are short of symbols and units you will find same letter is being used for more than once, but I will try to differentiate when such thing is there. Then you can get the magnitude of the torque as $BIS\sin\alpha$ and this again is quite widely used in machines.

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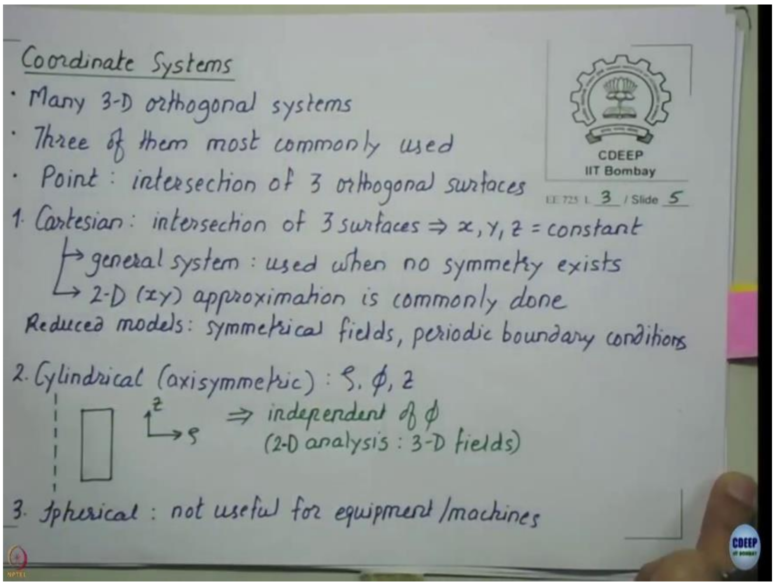
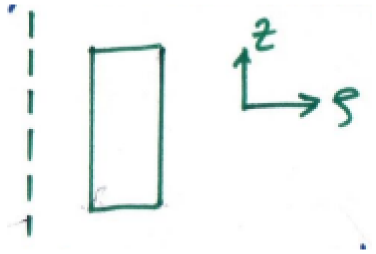


Next topic for discussion is coordinate systems. There are many coordinate systems but most commonly used are Cartesian, Cylindrical and Spherical systems. Now, the spherical system is of almost no use for this course, because you do not have something which is a very small entity and from there, something is coming out. For example, in antennas, the spherical system is widely used because an antenna can be a Hertz dipole and can be considered as a very small structure it radiates field. In those kinds of applications, the spherical system is used, but here we generally use only Cartesian and Cylindrical systems.

So, in any of these systems, a point gets defined by intersection of three surfaces. In case of Cartesian, a point is an intersection of x = constant surface, y = constant surface and z = constant surface. And Cartesian system is a very general system, whenever no geometrical symmetries exist and then you do not have a choice and you have to use Cartesian because if you use Cylindrical system, calculations will become more complicated.

Cylindrical system is useful when there is a cylindrical symmetry. For example, consider   a coil which is circular and around this axis as shown below and if you enclose this into a cylinder with top and bottom surfaces, then it becomes a perfect cylindrical geometry and effectively you can do 2D analysis because there are two dimensions involved: z and ρ.



The field distribution is independent of ϕ in this case, the ϕ is into the paper. As it is independent of ϕ effectively when you do the 2D analysis, the fields that you will get are really three dimensional fields, because all along the ϕ direction which is normal to this plane of the paper, the direction and the magnitude of the field is not changing. So, that is the advantage of using Cylindrical system. But although you have Cylindrical system, in most of the times you have no choice but to use Cartesian system.

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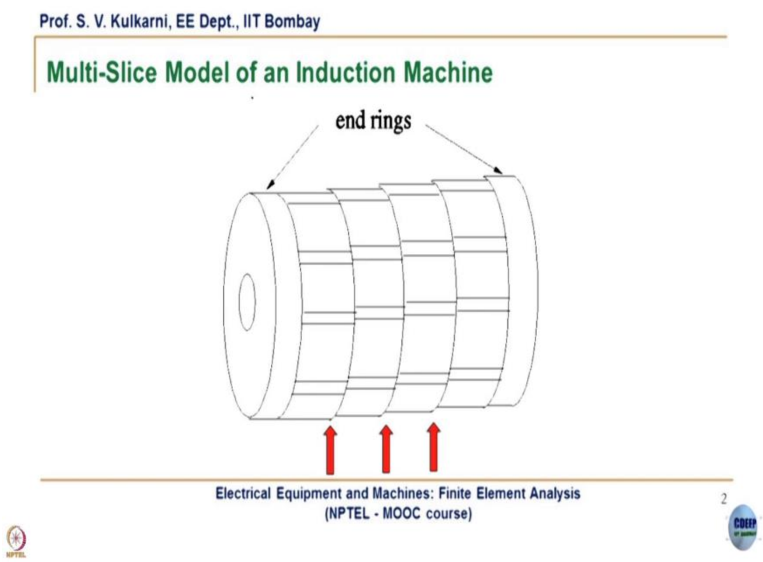

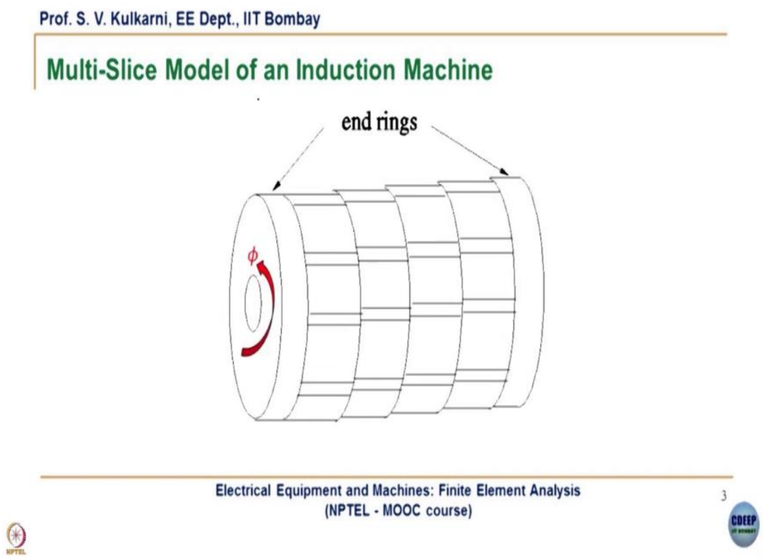

But even when you use Cartesian system, you can do some approximations like what is shown in the above slide. For example, in the figure, the rotor structure of a Squirrel Cage Induction motor with skewed rotor bars is shown. Now of course, what is shown here is a multi-slice model since the induction motor is with a skew of some finite angle, so it becomes a 3-dimensional analysis, but what you generally do here is you divide this model into a number of slices.

And then for each of those slices, you assume that there is no skew and then properly join these models through mathematical equations. And then get the total solution. So here, this is the way you can actually reduce the complexity of 3D modeling by using multiple 2D models.

Now, remember one may get confused whether this geometry has a Cylindrical symmetry. No, it is not, because there are slots and as you go along the ϕ direction, these slots are not actually through the entire circumference So, this does not have Cylindrical symmetry along the ϕ direction, that should be borne in mind.

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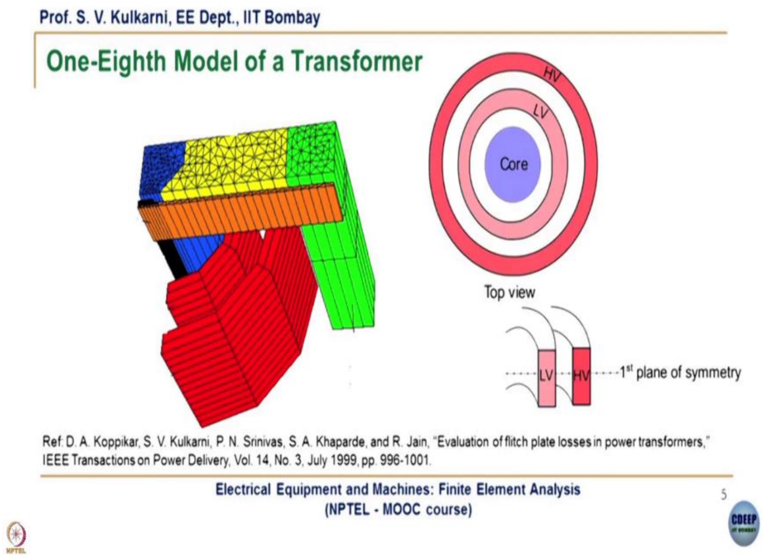

Now, this is another example to show how do we reduce the complexity. In the above slide, a section of a large power transformer is shown, here what is done is, although as you can see again here, there is no Cylindrical symmetry at all, because, around this axis of the coil, the core is not symmetrical, and the core is more or less a planar structure so there is no Cylindrical symmetry.

But here we have exploited three field symmetries. The field is symmetrical about the half of the height of the transformer. So, we assume symmetry although the clearances at the top and bottom are different in practice, but that again is the approximation made. So, if you do that approximation then there is a symmetry of field along the middle plane which cuts the transformer height into half, so that is the first symmetry.

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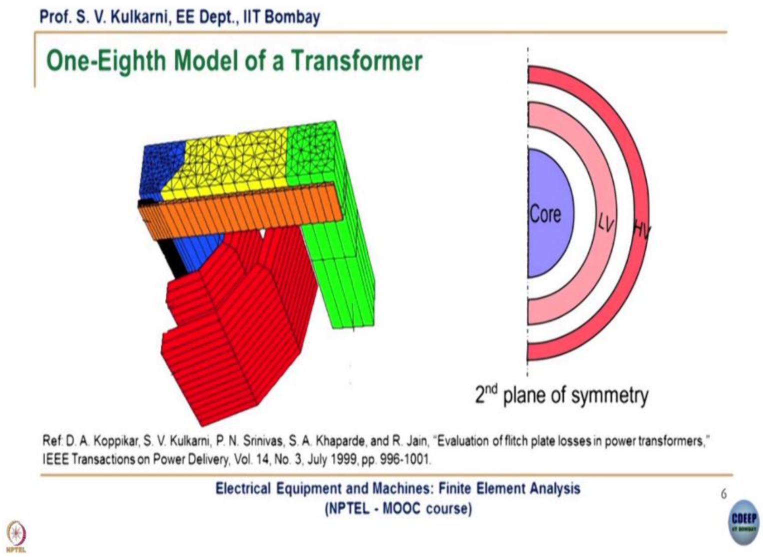

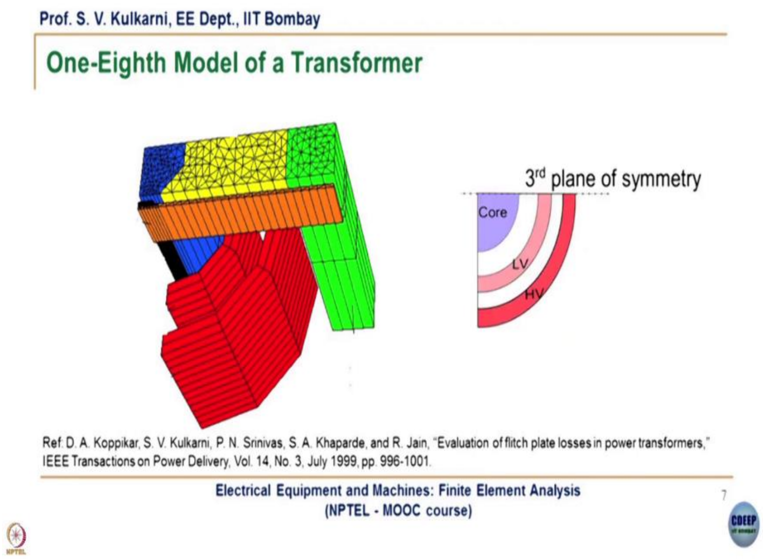

Second symmetry is on the vertical plane of the coils and the third symmetry is on the other side of the coils. So, the coil is modeled only one fourth along the ϕ direction. So, it is a quarter model along the ϕ direction because we are exploiting the field symmetry at the two points separated by 90 degrees along the ϕ direction.

And then there is one more symmetry we have exploited in the vertical direction, so one fourth in the ϕ direction and one half in the vertical direction. So, that is why it becomes one eighth of the full model. And that gives a big relief in terms of computational efforts, when you go from the full model to the one eighth model.

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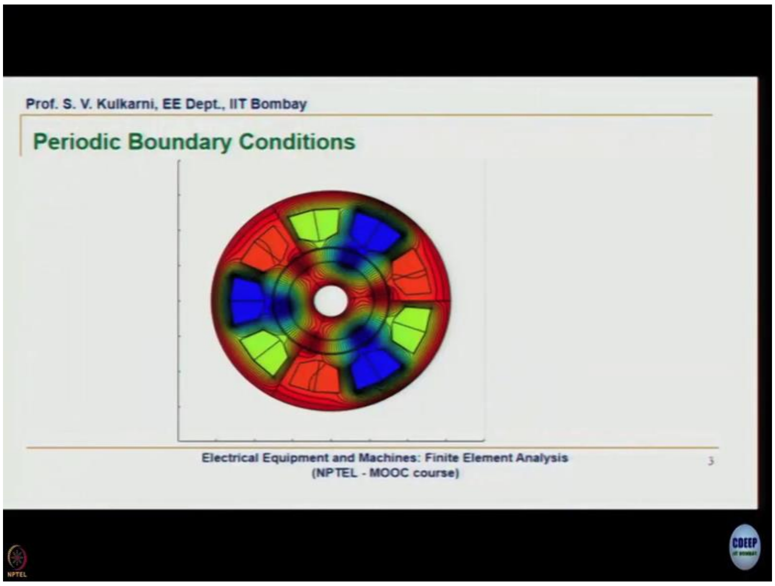

A rotating machine is shown in the above slide; now here there are no symmetries. Later on we will see there are what are called as Dirichlet and Neumann boundary conditions. In fact, in the previous example involving a transformer we exploited Neumann conditions at those three symmetry planes. I will explain you later what are these Dirichlet and Neumann conditions, but when those conditions are not possible to be exploited, then particularly in rotating machines you can use what are known as periodic boundary conditions.

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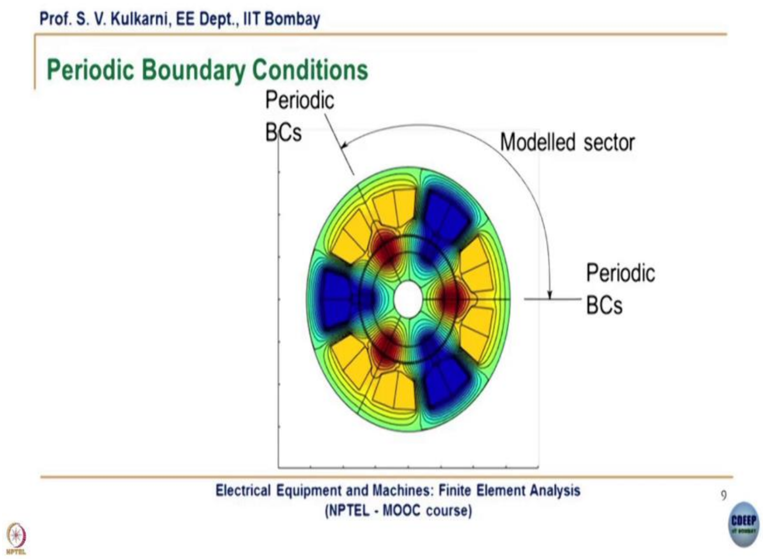

So, you do not have to model the entire 360 degrees geometry. You can model only a sector and then impose what are known as periodic boundary conditions and in the post processing stage, you will get the entire field distribution. How do we do that? There will be a separate lecture in this course on how to use periodic boundary conditions and improve the analysis, that we will see later.

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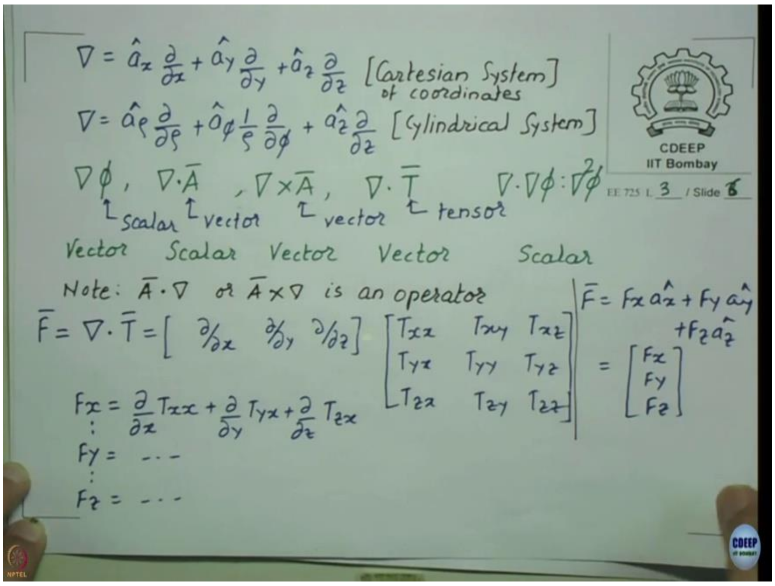
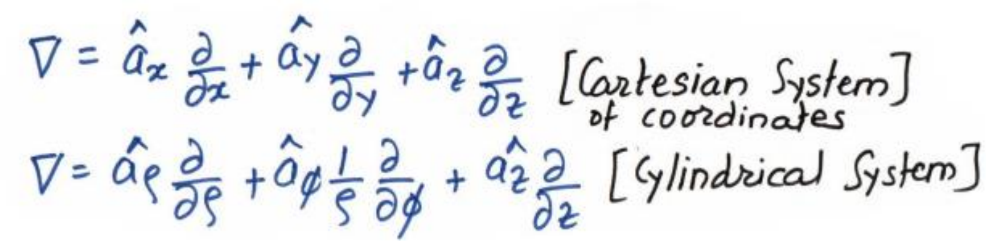


Now, the next topic is the ∇ operator. So, in the Cartesian system and in Cylindrical system, the expressions of the ∇ operator are given below.



See here why this $\frac{1}{\rho}$ appears in the middle term of the ∇ expression in the cylindrical system? First of all, this ρ is the distance, do not confuse this ρ with ρ_v which is the volume charge density.

So, this $\frac{1}{\rho\partial\phi}$ will give you the distance because this $\partial\rho$ and ∂z are distances. So, $\rho\partial\phi$ is also (angular) distance along the ϕ direction. So, this way whenever you see any expression you match the units, then the understanding becomes better.

Now, when this ∇ operator operates on a scalar, you get a vector, when it operates on a vector with a dot product, you get a scalar, when it operates on a vector with a cross product, you get a vector, when it operates on a tensor, I will explain you, what the tensor means. Already I have explained you a little bit when we discussed permeability tensor. Again now, this T is denoting tensor not the torque, just be sure about that.

So, divergence of a tensor gives you a vector and divergence of gradient of ϕ gives you $\nabla^2\phi$ which is a scalar. Remember that $\mathbf{A}\cdot\nabla$ or $\mathbf{A}\times\nabla$ is an operator, where A is a vector. So, this whole thing ($\mathbf{A}\cdot\nabla$ or $\mathbf{A}\times\nabla$) will operate on another vector or scalar. For example, $\mathbf{A}\cdot\nabla\phi$, is a vector dot product. So, $\mathbf{A}\cdot\nabla$ is an operator and not a vector.

Now, coming to tensors. So now here any vector F, which has three components written in a column matrix form as given here $\begin{bmatrix} F_x \\ F_y \\ F_z \end{bmatrix}$, when you write it like that, you do not have to write $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$, that is implicit and assumed.

So, now when we say force, it is divergence of a tensor. ∇ operator can be written as $\nabla = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$. Now what is implicit here is $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$, but since we have written it in matrix form, those unit vectors are not mentioned.

Similarly, the tensor (T) which is given below has got T_{xx}, implicit for the columns of the matrix are: $\hat{\mathbf{a}}_x, \hat{\mathbf{a}}_y$ and $\hat{\mathbf{a}}_z$.

$$\begin{bmatrix} T_{xx} & T_{xy} & T_{xz} \\ T_{yx} & T_{yy} & T_{yz} \\ T_{zx} & T_{zy} & T_{zz} \end{bmatrix}$$

So now, when you actually take the product of the two matrices $\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$ and $\begin{bmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{bmatrix}$, the three terms $\begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{bmatrix}$ will get multiplied with three terms $\begin{bmatrix} T_{xx} \\ T_{yx} \\ T_{zx} \end{bmatrix}$ to give F_x.

So, now $F_x = \frac{\partial T_{xx}}{\partial x} + \frac{\partial T_{yx}}{\partial y} + \frac{\partial T_{zx}}{\partial z}$ is a component of the total force and similarly F_y and F_z can be calculated. So, this is how you get F force as divergence of a tensor. Again, this will be required in one of the lectures later, when we calculate forces on conductors or magnetic system by using this tensor concept.

So, this ends this lecture, we have discussed some aspects of vector calculus and electromagnetics. In next lecture, we will see further, thank you.

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