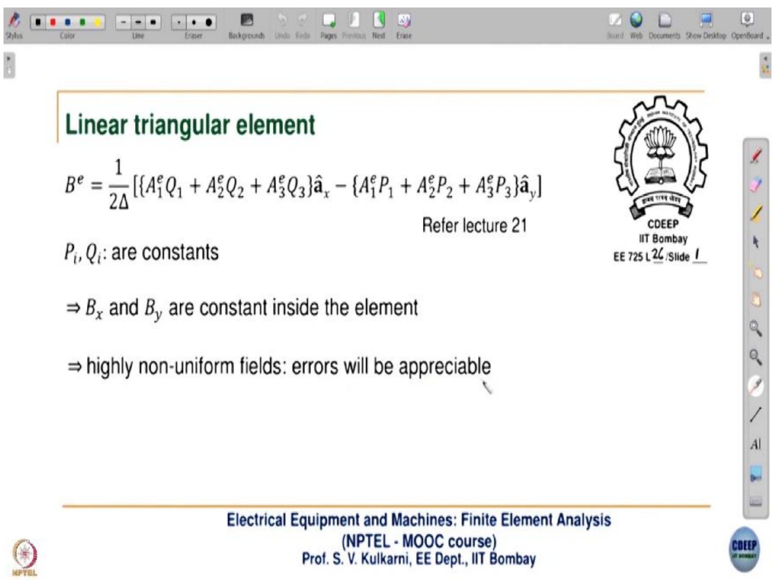
Electrical Equipment and Machines: Finite Element Analysis
Professor Shrikrishna V Kulkarni
Indian Institute of Technology Bombay
Lecture No 26
Quadratic Finite Elements

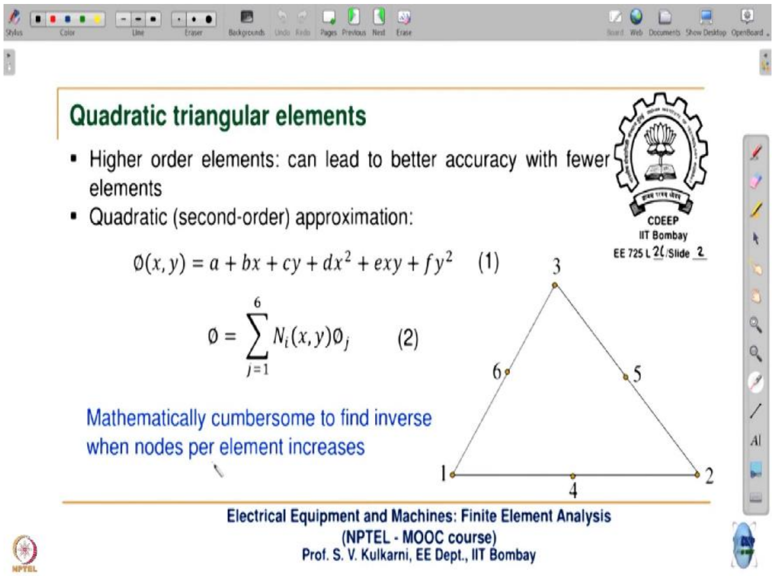
Welcome to the 26th lecture. In this lecture, we will see quadratic or second order elements. As mentioned earlier, linear elements may not be good if the field is varying drastically in some regions of the problem domain. If you don't want to use a very fine mesh with linear elements, then you can use a coarser mesh or with higher order elements for the same accuracy of the solution.

(Refer Slide Time: 00:52)



For a linear triangular element, we have seen the expressions of B, B_x, and B_y which are given in the above slide. Also in one of the previous lectures, we have seen that $B_x = \frac{\partial A_z}{\partial y}$ and $B_y = \frac{\partial A_z}{\partial x}$. We also saw that these expressions are constants because A_1, A_2, A_3 are nodal potential values that would have got after the FEM solution. Q_1, Q_2, Q_3 and P_1, P_2, P_3 are also constants because they depend on the coordinates of the vertices of the element under consideration. So B_x and B_y are constant over an element which may not be true in case of highly non-uniform fields and the errors will be appreciable. So we have to go for second order or quadratic formulation.

(Refer Slide Time: 01:50)



An example of quadratic triangular elements and the corresponding approximation are given in the above slide. In the figure given in the above slide, you can see that there are six nodes. The three nodes (1, 2, and 3) are usual nodes that we have seen in the previous lecture. Nodes 4, 5, and 6 are at the centres of the three edges of the triangle. Since there are six nodes the approximate potential function should have six unknowns a, b, …, f and the corresponding approximation is given in the following equation.

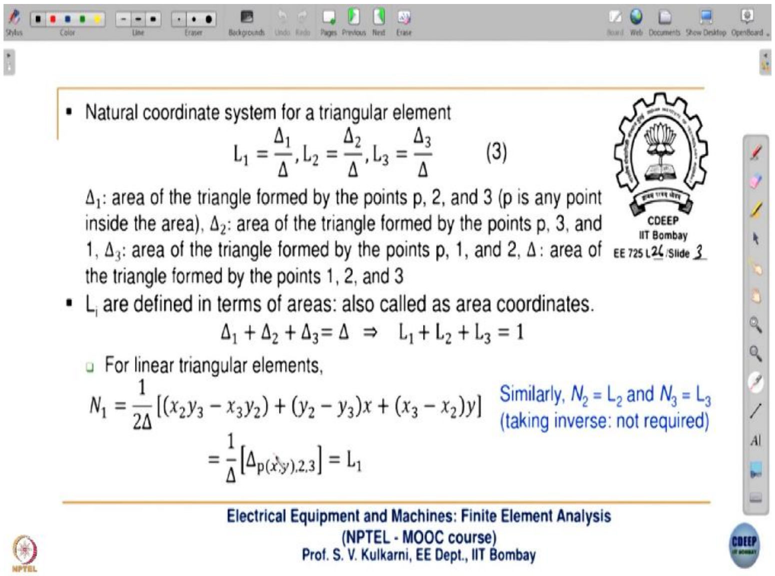
$$\emptyset(x,y) = a + bx + cy + dx^2 + exy + fy^2$$

Then, you can use the same procedure that we followed for linear triangular element to determine the FE formulation. There the expression of ϕ is expressed in terms of ϕ_1, ϕ_2, and ϕ_3 by eliminating a, b, … f which involves inversion of a matrix. By eliminating the constants a, b, …f we get the following expression.

$$\emptyset = \sum_{j=1}^{6} N_i(x,y)\emptyset_j$$

You can do the same thing for quadratic elements also. But we will have to invert a 6 × 6 matrix to eliminate the constants a to f and with this the computational burden is becoming higher. Here we will use a simpler method called area coordinates approach and this does not require any matrix inversion.

(Refer Slide Time: 03:33)

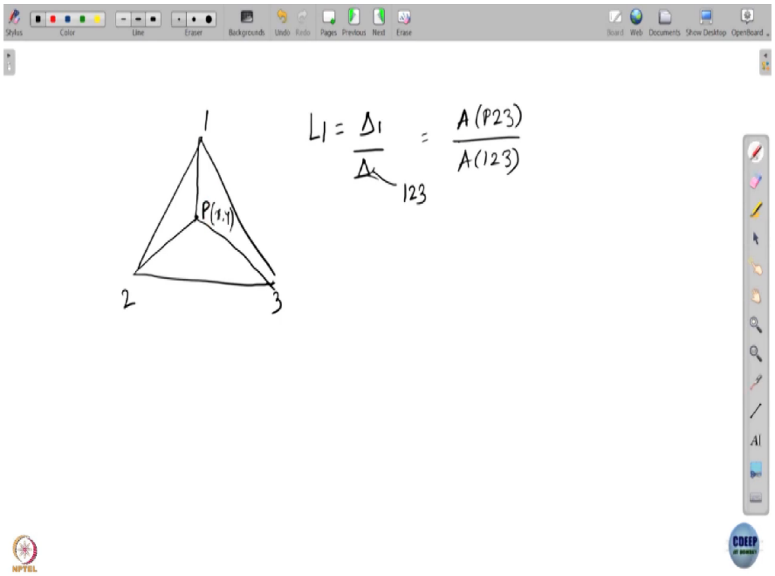


This approach uses area coordinates or natural coordinates and they are defined as given below.

$$L_1 = \frac{\Delta_1}{\Delta}, L_2 = \frac{\Delta_2}{\Delta}, L_3 = \frac{\Delta_3}{\Delta}$$

Now, what is Δ_1?

(Refer Slide Time: 4:01)



Let us consider a triangular element shown in the figure given in the above slide with nodes 1, 2 and 3. Consider a point p inside the element. L_1 is defined as $\frac{\Delta_1}{\Delta}$, here, Δ is the area of the whole triangle formed by nodes 1, 2, and 3 and Δ_1 is the area of triangle formed by nodes P, 2, and 3. So L_1 is defined as the area of the triangle formed by nodes P, 2, and 3 divided by the area triangle formed by nodes 1, 2, and 3. Δ_2 is the area of triangle formed by nodes P, 3, and 1. Again if you see here the sequence of nodes is P 3 1 and it is not written as P 1 3 because the area will become negative. Generally, when you calculate the area of a triangular element we have to take the three nodes in anticlockwise fashion. Similarly, Δ_3 is the area of triangle formed by nodes P, 1, and 2.

Also, remember that $\Delta_1 + \Delta_2 + \Delta_3 = \Delta$. This is obvious because L_1 is a function which is divided by the total area. So the addition of all these L_1, L_2, L_3 will be given as

$$\mathrm{L}_1 + \mathrm{L}_2 + \mathrm{L}_3 = \frac{\Delta_1 + \Delta_2 + \Delta_3}{\Delta} = 1$$

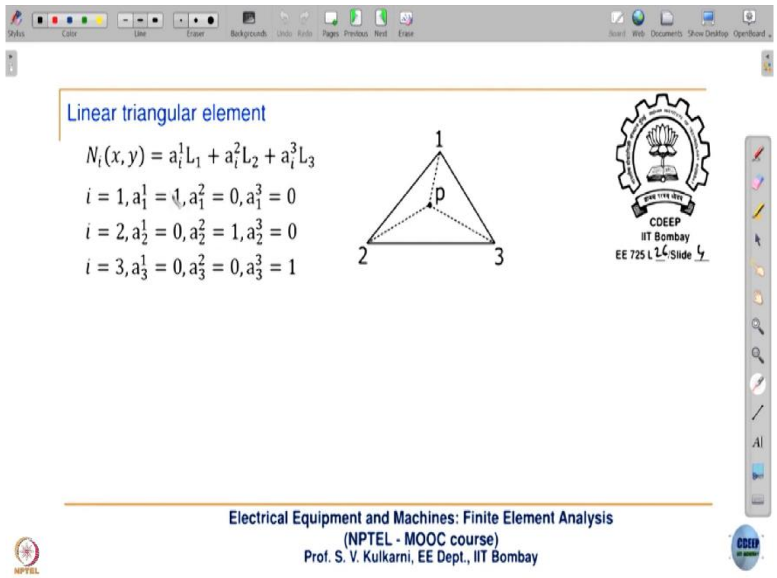
In fact for linear triangular elements, we have seen the following expression of N_1many times.

$$N_1 = \frac{1}{2\Delta}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$$
$$= \frac{1}{\Delta}\left[\Delta_{\mathrm{p}(x,y),2,3}\right] = \mathrm{L}_1$$

Similarly, we saw the expressions N_2 and N_3 for other nodes.

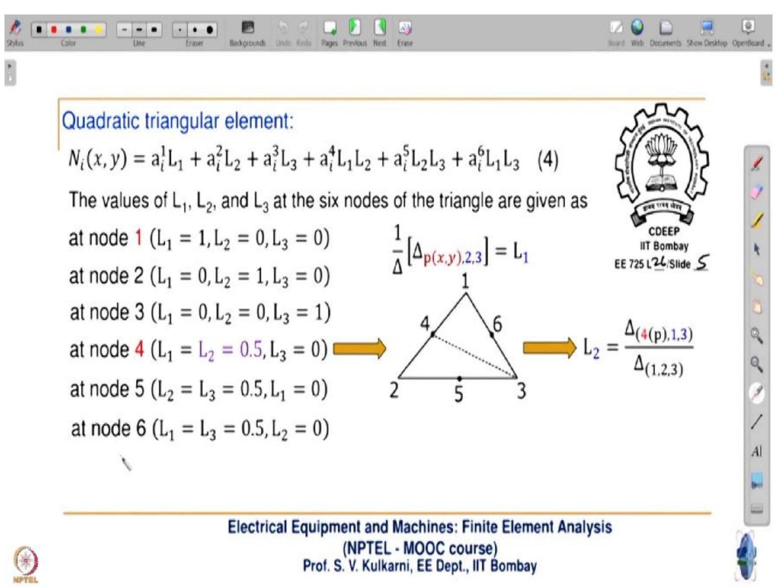
If you take this 1/2 inside the bracket that will give the area of the triangle formed by P, 2, and 3. So that is why for a linear triangular element $L_1 = N_1$. That means when we saw the procedure for a linear triangular element we need not have inverted the 3×3 matrix to eliminate a, b, and c and we could have directly use this property and derived the shape functions. We did not use it there because we have not seen the theory of natural coordinates and there we wanted you to understand the general procedure of FEM. So that is why we have taken the inversion there. But using this approach we can directly get the expression for N_1 by just calculating the ratio of areas of two triangles.

(Refer Slide Time: 07:42)



Similarly, we can write the expressions of N_2 and N_3. That is why for a linear triangular element we can write a general expression of shape function in terms of area coordinates as given in the above slide. For $i = 1$ that is node 1, $a_1^1 = 1$ and other constants are zero because $N_1 = L_1$. Similarly, for $i = 2$, $N_2 = L_2$ and that is why $a_2^2 = 1$ and other constants are zero. Similarly, N_3 can be determined by using the generalized expression which is given in the above equation. So N_i can be represented as a general function of L_1, L_2, and L_3. The corresponding coefficient values for each shape function are also given in the above equation.

(Refer Slide Time: 08:47)



Now for a quadratic element we can extend the generalized expression that we have seen earlier as given below.

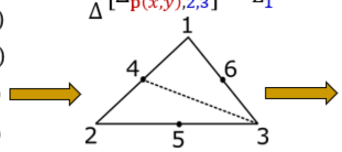
$$N_i(x,y) = \mathrm{a}_i^1\mathrm{L}_1 + \mathrm{a}_i^2\mathrm{L}_2 + \mathrm{a}_i^3\mathrm{L}_3 + \mathrm{a}_i^4\mathrm{L}_1\mathrm{L}_2 + \mathrm{a}_i^5\mathrm{L}_2\mathrm{L}_3 + \mathrm{a}_i^6\mathrm{L}_1\mathrm{L}_3$$

For a linear element, N_i was a function of only the three terms L_1, L_2, and L_3. Now it is a quadratic element with six nodes and so we need to have six coefficients so that is why the generalized expression will have extra terms as given in the above equation. It is natural to have L_1L_2, L_2L_3, L_3L_1 to get quadratic terms in the shape function. So N_i is not only a function of L_1, L_2, and L_3 as linear function but also their products. Now, for each node a_i^1, a_i^2, …, a_i^6 are unknowns. Remember that for shape function at every node there are six unknowns.

We have already seen that at node 1 the value of L_1 is 1, $L_2 = 0$ and $L_3 = 0$. Similarly, for the other two nodes. At node 4, L_1 and L_2 are equal to 0.5 and L_3 is 0 because L_2 at node 4 will be Δ_{413}/Δ and Δ_{413}= 0.5Δ and that is why $L_2 = 0.5$. Similarly, you can calculate the rest of the coefficients. In the following figure, we have calculated the values of L_1, L_2, and L_3 at all the nodes.

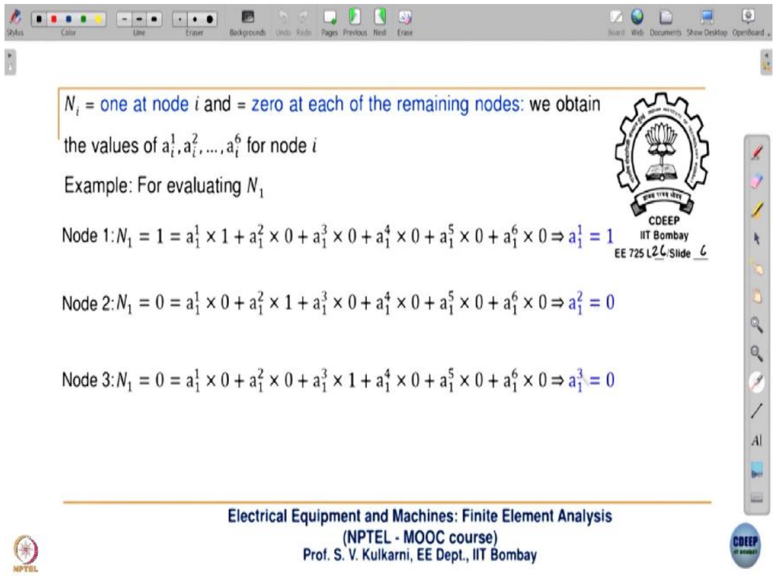
at node 1 $(\mathrm{L}_1 = 1, \mathrm{L}_2 = 0, \mathrm{L}_3 = 0)$
at node 2 $(\mathrm{L}_1 = 0, \mathrm{L}_2 = 1, \mathrm{L}_3 = 0)$
at node 3 $(\mathrm{L}_1 = 0, \mathrm{L}_2 = 0, \mathrm{L}_3 = 1)$
at node 4 $(\mathrm{L}_1 = \mathrm{L}_2 = 0.5, \mathrm{L}_3 = 0)$
at node 5 $(\mathrm{L}_2 = \mathrm{L}_3 = 0.5, \mathrm{L}_1 = 0)$
at node 6 $(\mathrm{L}_1 = \mathrm{L}_3 = 0.5, \mathrm{L}_2 = 0)$

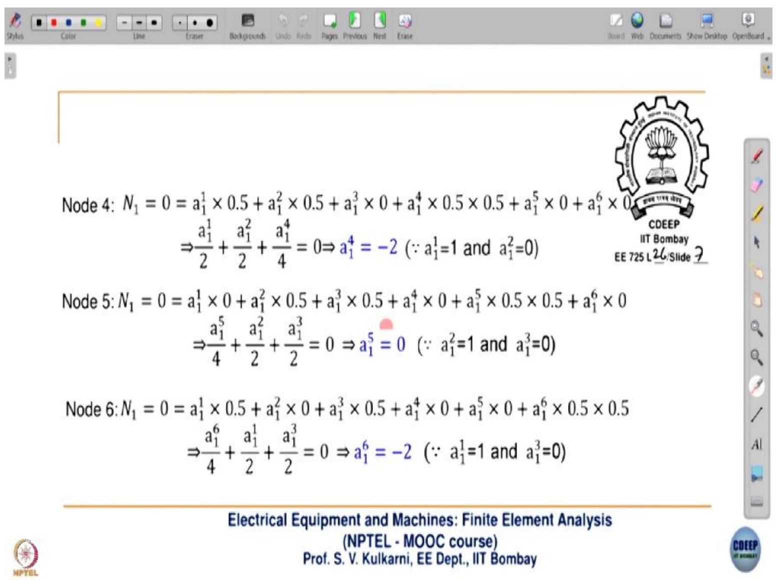
$$\frac{1}{\Delta}\left[\Delta_{p(x,y),2,3}\right] = \mathrm{L}_1$$



$$\frac{\Delta_{(4(p),1,3)}}{\Delta_{(1,2,3)}}$$

(Refer Slide Time: 10:56)





Now, we will use the values determined in the previous slide to calculate the six coefficients for every node. We need six equation or six conditions because we have to determine six coefficients for every node. $N_1 = 1$ at node number 1 and it is 0 at all other five nodes. So, we substitute the values of all the coordinates where the values of L_1, L_2, and L_3 are known in the expression of N_1 as given below to calculate the unknown constants.

$$\text{Node 1}: N_1 = 1 = a_1^1 \times 1 + a_1^2 \times 0 + a_1^3 \times 0 + a_1^4 \times 0 + a_1^5 \times 0 + a_1^6 \times 0 \Rightarrow a_1^1 = 1$$

$$\text{Node 2}: N_1 = 0 = a_1^1 \times 0 + a_1^2 \times 1 + a_1^3 \times 0 + a_1^4 \times 0 + a_1^5 \times 0 + a_1^6 \times 0 \Rightarrow a_1^2 = 0$$

$$\text{Node 3}: N_1 = 0 = a_1^1 \times 0 + a_1^2 \times 0 + a_1^3 \times 1 + a_1^4 \times 0 + a_1^5 \times 0 + a_1^6 \times 0 \Rightarrow a_1^3 = 0$$

$$\text{Node 4}: N_1 = 0 = a_1^1 \times 0.5 + a_1^2 \times 0.5 + a_1^3 \times 0 + a_1^4 \times 0.5 \times 0.5 + a_1^5 \times 0 + a_1^6 \times 0$$

$$\Rightarrow \frac{a_1^1}{2} + \frac{a_1^2}{2} + \frac{a_1^4}{4} = 0 \Rightarrow a_1^4 = -2 \quad (\because a_1^1 = 1 \text{ and } a_1^2 = 0)$$

$$\text{Node 5}: N_1 = 0 = a_1^1 \times 0 + a_1^2 \times 0.5 + a_1^3 \times 0.5 + a_1^4 \times 0 + a_1^5 \times 0.5 \times 0.5 + a_1^6 \times 0$$

$$\Rightarrow \frac{a_1^5}{4} + \frac{a_1^2}{2} + \frac{a_1^3}{2} = 0 \Rightarrow a_1^5 = 0 \quad (\because a_1^2 = 1 \text{ and } a_1^3 = 0)$$

$$\text{Node 6}: N_1 = 0 = a_1^1 \times 0.5 + a_1^2 \times 0 + a_1^3 \times 0.5 + a_1^4 \times 0 + a_1^5 \times 0 + a_1^6 \times 0.5 \times 0.5$$

$$\Rightarrow \frac{a_1^6}{4} + \frac{a_1^1}{2} + \frac{a_1^3}{2} = 0 \Rightarrow a_1^6 = -2 \quad (\because a_1^1 = 1 \text{ and } a_1^3 = 0)$$

For example, N_1 and L_1 at node 1 is 1 and the values of L_2 and L_3 are 0. So all the terms except the first term will go down to 0 as given in the following equation.

$$\text{Node 1}: N_1 = 1 = a_1^1 \times 1 + a_1^2 \times 0 + a_1^3 \times 0 + a_1^4 \times 0 + a_1^5 \times 0 + a_1^6 \times 0 \Rightarrow a_1^1 = 1$$

So that is why you will the value of $a_1^1 = 1$. Similarly, using the values at nodes 2 and 3, we will get $a_1^2 = a_1^3 = 0$.

Then let us take the value of $N_1 = 0$ at node 4 by definition and property of shape function. Now at node 4, we have to substitute all these values of L_1 to L_3 as given below. At node 4, $L_1 = L_2 = 0.5$ and $L_3 = 0$.

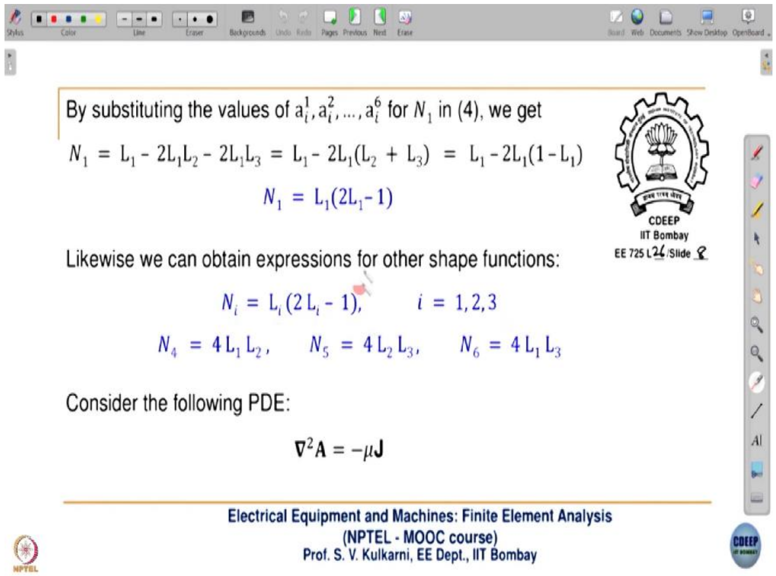
$$\text{Node 4}: N_1 = 0 = a_1^1 \times 0.5 + a_1^2 \times 0.5 + a_1^3 \times 0 + a_1^4 \times 0.5 \times 0.5 + a_1^5 \times 0 + a_1^6 \times 0$$

After simplification, you will get the following expression and the values of a_1^1, a_1^2, and a_1^3 are already calculated. Using these values, a_1^4 is calculated as given below.

$$\Rightarrow \frac{a_1^1}{2} + \frac{a_1^2}{2} + \frac{a_1^4}{4} = 0 \Rightarrow a_1^4 = -2 \quad (\because a_1^1 = 1 \text{ and } a_1^2 = 0)$$

Likewise, you can calculate the remaining two coefficients also.

(Refer Slide Time: 13:49)



Substituting the values of a_i^1 to a_i^6 in the generalized expression of N_1, we will get the shape function of N_1 as given below.

$$N_1 = L_1 - 2L_1L_2 - 2L_1L_3 = L_1 - 2L_1(L_2 + L_3) = L_1 - 2L_1(1 - L_1)$$

$$N_1 = L_1(2L_1 - 1)$$

See the difference, for linear element N_1 was just equal to L_1, but for quadratic element $N_1 = L_1(2L_1 - 1)$.

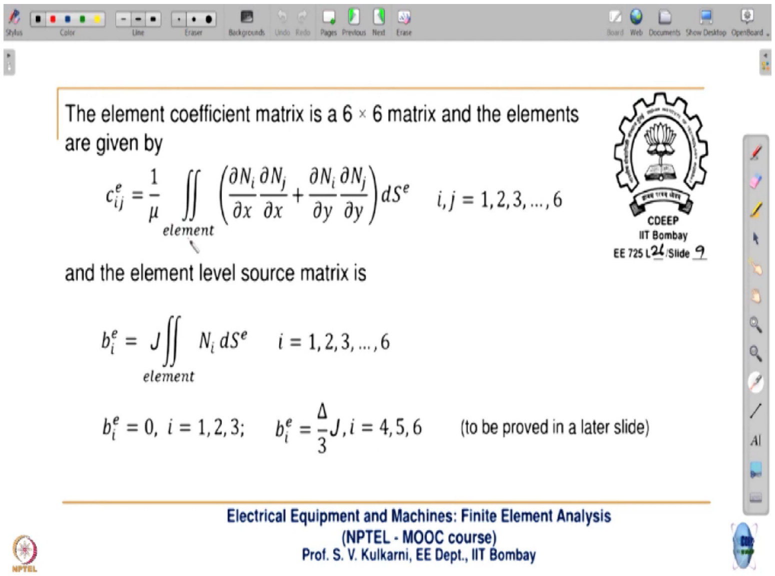
Likewise, we can obtain the expressions for other shape functions as given below.

$$N_i = L_i(2L_i - 1), \qquad i = 1, 2, 3$$

$$N_4 = 4L_1L_2, \qquad N_5 = 4L_2L_3, \qquad N_6 = 4L_1L_3$$

These expression can be verified by following the procedure that was explained earlier.

(Refer Slide Time: 15:06)



Suppose we want to solve this Poisson's equation in magnetostatics which is defined as $\nabla^2 A = -\mu J$. As we discussed earlier, μ should not be associated with J because mu can vary with space in the given geometry. So we bring μ on the left hand side and we associate it with ∇^2. When we integrate, depending upon which element we are considering the corresponding μ will be taken for that element. The following equation defines c_{ij} expression.

$$c_{ij}^e = \frac{1}{\mu} \iint_{element} \left(\frac{\partial N_i}{\partial x} \frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y} \frac{\partial N_j}{\partial y} \right) dS^e \qquad i,j = 1,2,3,\dots,6$$

The dimensions of the element coefficient matrix will be 6×6. The entries of b_i^e of the right hand side matrix which represents the source are defined by the following equation.

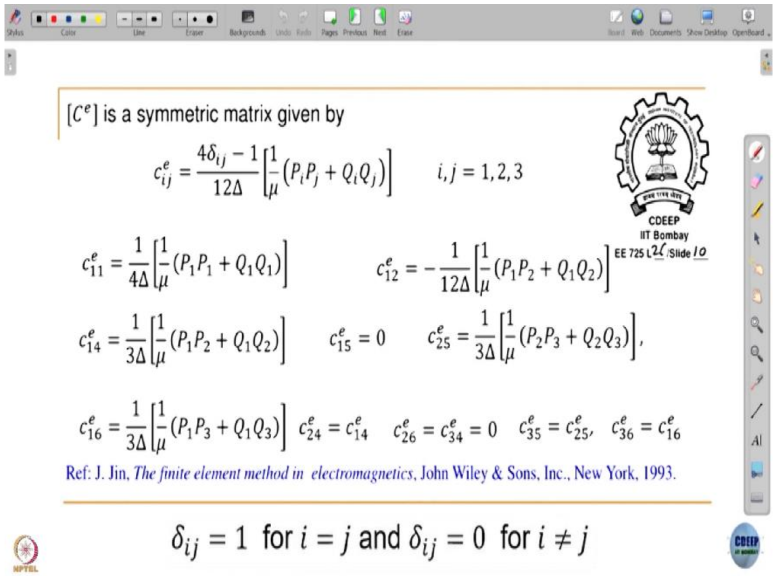
$$b_i^e = J \iint_{element} N_i \, dS^e \qquad i = 1,2,3,\dots,6$$

Earlier when it was a linear element we apportioned J equally to the three nodes as $J\Delta/3$. For quadratic elements, that J gets apportioned equally to the middle nodes of each node but not to nodes 1, 2, and 3. So the entries of b_i^e are defined as given below.

$$b_i^e = 0, \; i = 1,2,3; \qquad b_i^e = \frac{\Delta}{3} J, i = 4,5,6$$

We will see the derivations of these terms later.

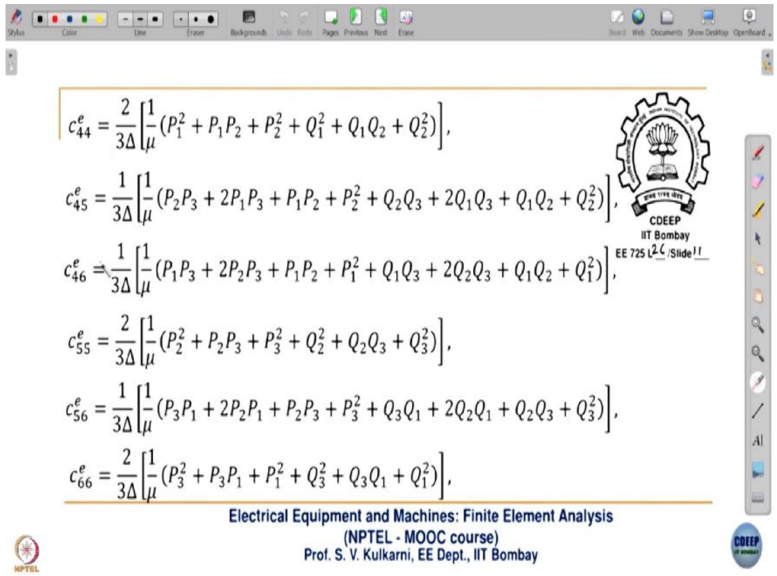
(Refer Slide Time: 16:50)



$[C^e]$ is a symmetric matrix. The element level matrix entries c_{ij}^e are given in the above slide and the next slide. From the above slide, we can see that c_{11} for this case is identical to the linear triangular element. But off diagonal entries like c_{12} are different. Again c_{15}, c_{26}, and c_{34} are 0 because there is no connection between the corresponding nodes.

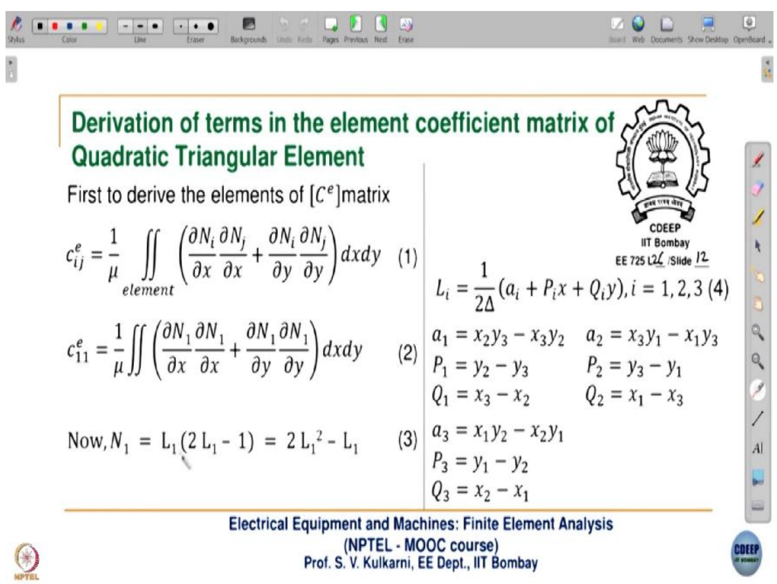
That means, there is no connection between nodes 1 and 5, nodes 2 and 6 and nodes 3 and 4. At the bottom of this slide, we have given one reference book titled 'The finite element method in electromagnetics' for high frequency electromagnetic mostly. If you want to verify the expressions of these entries you can refer this book. Some of these coefficients are already derived and we will see in the next lecture.

(Refer Slide Time: 18:15)



You do not have to be worried about the expressions given in the above slide because deriving them is a one-time effort if you develop a code to form element coefficient matrix that can be used for any problem as we have seen earlier. Now we will see the derivation of some of these coefficients that we have seen in the previous slide.

(Refer Slide Time: 18:36)



For example, we know the following expression for c_{ij}.

$$c_{ij}^{e} = \frac{1}{\mu} \iint_{element} \left(\frac{\partial N_i}{\partial x}\frac{\partial N_j}{\partial x} + \frac{\partial N_i}{\partial y}\frac{\partial N_j}{\partial y} \right) dxdy$$

c_{11} can be evaluated by replacing i and j with 1 and 1 so c_{11} will be simply given by the following expression.

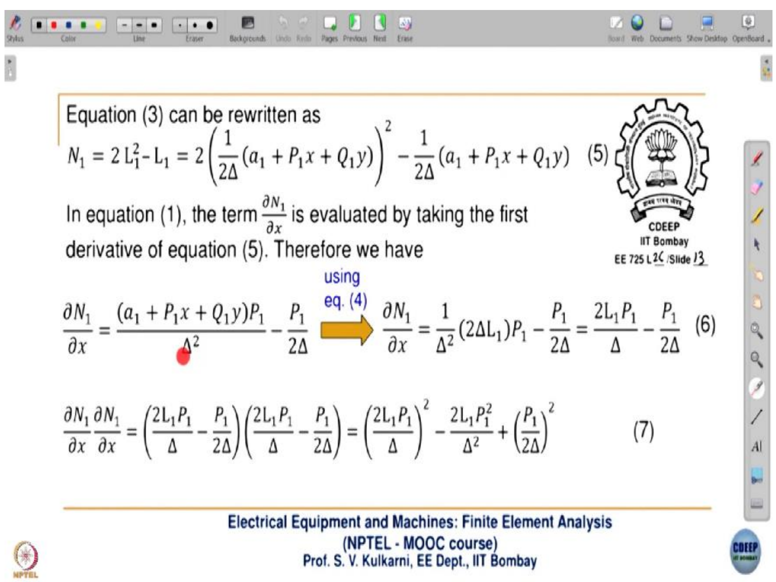
$$c_{11}^{e} = \frac{1}{\mu} \iint \left(\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} + \frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y} \right) dxdy$$

In the previous slides we have derived the expression of N_1 as $L_1(2L_1 - 1)$ and $L_1 = \Delta_{p23}/\Delta$. We had already seen the following expression of L_i.

$$L_i = \frac{1}{2\Delta}(a_i + P_i x + Q_i y), i = 1, 2, 3$$

$$a_1 = x_2 y_3 - x_3 y_2 \quad a_2 = x_3 y_1 - x_1 y_3$$
$$P_1 = y_2 - y_3 \quad P_2 = y_3 - y_1$$
$$Q_1 = x_3 - x_2 \quad Q_2 = x_1 - x_3$$
$$a_3 = x_1 y_2 - x_2 y_1$$
$$P_3 = y_1 - y_2$$
$$Q_3 = x_2 - x_1$$

(Refer Slide Time: 19:30)



So if you substitute the expression of L_1 in $N_1 = 2L_1^2 - L_1$, we get

$$N_1 = 2\,L_1^2 - L_1 = 2\left(\frac{1}{2\Delta}(a_1 + P_1 x + Q_1 y)\right)^2 - \frac{1}{2\Delta}(a_1 + P_1 x + Q_1 y)$$

The value of c_{11} is determined by evaluating the first derivative $\partial N_1/\partial x$ and it is given by the following expression.

$$\frac{\partial N_1}{\partial x} = \frac{(a_1 + P_1 x + Q_1 y)P_1}{\Delta^2} - \frac{P_1}{2\Delta}$$

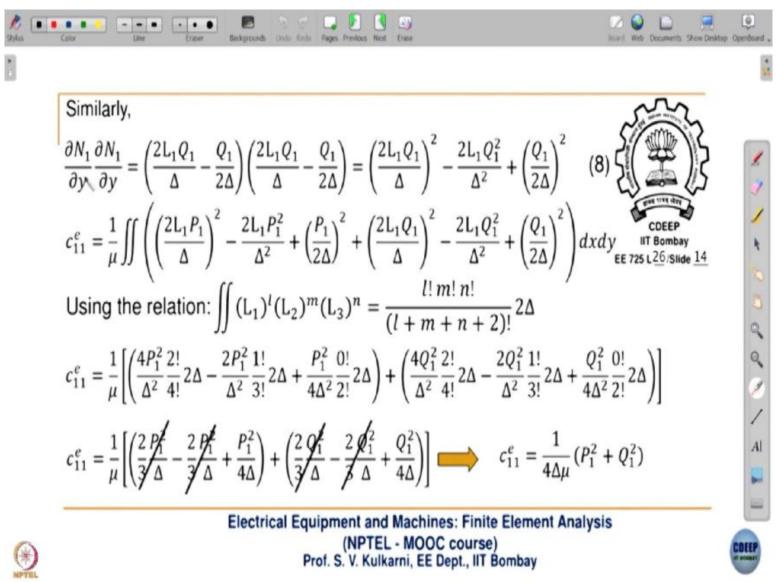
Then, you resubstitute $a_1 + P_1 x + Q_1 y$ with $2\Delta L_1$ based on the expression of L_1 and we finally get the following simplified expression.

$$\frac{\partial N_1}{\partial x} = \frac{1}{\Delta^2}(2\Delta \mathrm{L}_1)P_1 - \frac{P_1}{2\Delta} = \frac{2\mathrm{L}_1 P_1}{\Delta} - \frac{P_1}{2\Delta}$$

Then the expression of $\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}$ is given below

$$\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x} = \left(\frac{2\mathrm{L}_1 P_1}{\Delta} - \frac{P_1}{2\Delta}\right)\left(\frac{2\mathrm{L}_1 P_1}{\Delta} - \frac{P_1}{2\Delta}\right) = \left(\frac{2\mathrm{L}_1 P_1}{\Delta}\right)^2 - \frac{2\mathrm{L}_1 P_1^2}{\Delta^2} + \left(\frac{P_1}{2\Delta}\right)^2$$

(Refer Slide Time: 21:01)



Similarly, you can calculate $\frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y}$ as given below.

$$\frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y} = \left(\frac{2\mathrm{L}_1 Q_1}{\Delta} - \frac{Q_1}{2\Delta}\right)\left(\frac{2\mathrm{L}_1 Q_1}{\Delta} - \frac{Q_1}{2\Delta}\right) = \left(\frac{2\mathrm{L}_1 Q_1}{\Delta}\right)^2 - \frac{2\mathrm{L}_1 Q_1^2}{\Delta^2} + \left(\frac{Q_1}{2\Delta}\right)^2$$

In contrast to the expression of $\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}$, you will have Q_1 instead of P_1 and that is the only difference. Because the derivateive of N_1 with respect to x we will give P_1s and derivative with respect to y will give only Q_1s. So for $\frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y}$ you have expressions with only Q_1s . So c_{11} is equal to the integral of sum of the two terms $\frac{\partial N_1}{\partial x}\frac{\partial N_1}{\partial x}$ and $\frac{\partial N_1}{\partial y}\frac{\partial N_1}{\partial y}$ as given below.

$$c_{11}^{e} = \frac{1}{\mu}\iint\left(\left(\frac{2\mathrm{L}_1 P_1}{\Delta}\right)^2 - \frac{2\mathrm{L}_1 P_1^2}{\Delta^2} + \left(\frac{P_1}{2\Delta}\right)^2 + \left(\frac{2\mathrm{L}_1 Q_1}{\Delta}\right)^2 - \frac{2\mathrm{L}_1 Q_1^2}{\Delta^2} + \left(\frac{Q_1}{2\Delta}\right)^2\right) dxdy$$

Now the integrand in the above equation has six terms. The above integral is evaluated by using the following equation.

$$\iint (\mathrm{L}_1)^l (\mathrm{L}_2)^m (\mathrm{L}_3)^n \, dxdy = \frac{l!\, m!\, n!}{(l+m+n+2)!} 2\Delta$$

We have seen this formula earlier and now actually for each of the terms in the integral of c_{11} if you use the above formula, we will get the final expression of c_{11}. For example, in the first term $\frac{4P_1^2}{\Delta^2}$ you have L_1^2 that means $l = 2$ and $m = n = 0$ and if you substitute these values in the above formula you will get

$$\iint \frac{4L_1^2 P_1^2}{\Delta} dxdy = \frac{4P_1^2}{\Delta}\frac{2!}{4!} 2\Delta$$

Likewise, you can calculate the integrals of all the terms using the above formula and then you can simplify as given below.

$$c_{11}^{e} = \frac{1}{\mu}\left[\left(\frac{4P_1^2}{\Delta^2}\frac{2!}{4!}2\Delta - \frac{2P_1^2}{\Delta^2}\frac{1!}{3!}2\Delta + \frac{P_1^2}{4\Delta^2}\frac{0!}{2!}2\Delta\right) + \left(\frac{4Q_1^2}{\Delta^2}\frac{2!}{4!}2\Delta - \frac{2Q_1^2}{\Delta^2}\frac{1!}{3!}2\Delta + \frac{Q_1^2}{4\Delta^2}\frac{0!}{2!}2\Delta\right)\right]$$

$$c_{11}^{e} = \frac{1}{\mu}\left[\left(\frac{2P_1^2}{3\Delta} - \frac{2P_1^2}{3\Delta} + \frac{P_1^2}{4\Delta}\right) + \left(\frac{2Q_1^2}{3\Delta} - \frac{2Q_1^2}{3\Delta} + \frac{Q_1^2}{4\Delta}\right)\right] \Rightarrow c_{11}^{e} = \frac{1}{4\Delta\mu}(P_1^2 + Q_1^2)$$

We will stop here and continue in the next lecture.

(Refer Slide Time: 23:05)