Electrical Equipment and Machines: Finite Element Analysis
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Lecture No 23
Calculation of Leakage Inductance of a Transformer

Welcome to the 23rd lecture, we will continue our discussion on applications of the finite element method. First, we will see leakage inductance calculations of a transformer and then we will also see the calculation of magnetizing inductance of a rotating machine.

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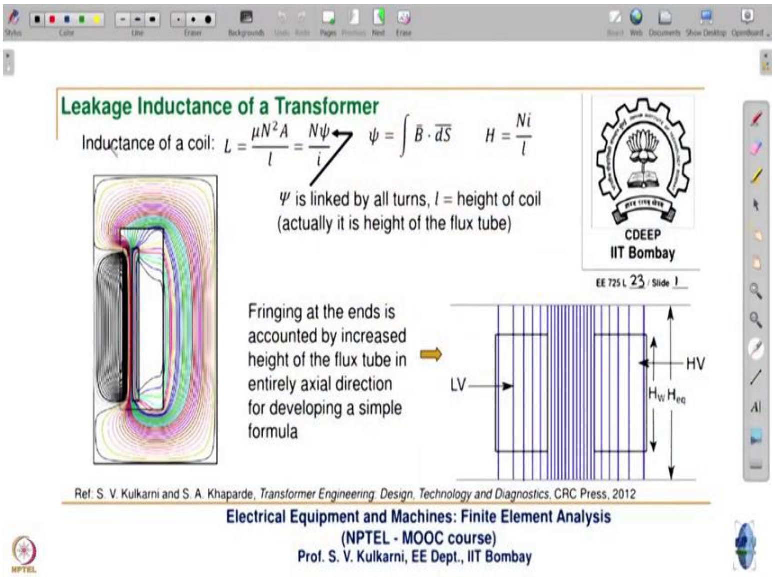
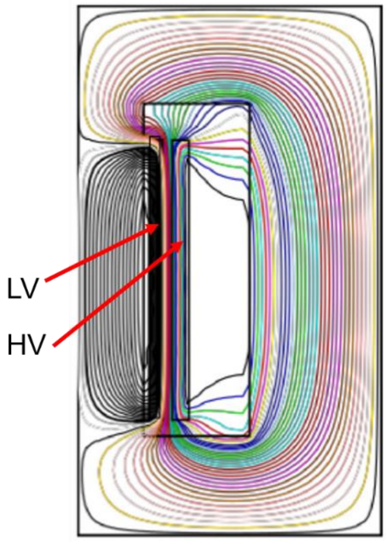


Now, we talk about the calculation of leakage inductance of a transformer. In finite element simulation, for a three phase transformer we take only one window with LV winding which is masked by flux contours and the high voltage winding as shown in the following figure. We actually force the ampere turns to be equal and opposite in the FEM simulation.



So, $N_1 i_1 = N_2 i_2$ and we make one of the ampere turns as minus so that net ampere turns enclosed by any contour in the core portion will be 0. So, that is why in the above figure we will see there is not a single flux contour which encloses both the windings. In the figure, you can see that so many contours are enclosing the LV ampere turns, and the rest are enclosing HV ampere turns, but there is no single flux contour which is enclosing both windings.

But practically this is not the case because there are magnetizing ampere turns and that is why you will get some magnetizing flux (or no load flux) in the core which moves around the core and encloses both windings. This flux is responsible for induced voltage in both windings. So that is why in FEM it is important to know what we want to model. Accordingly define the parameters which may not be the true case practically. For example, you will never have, transformer working with full currents being passed through both windings with zero magnetizing ampere turns. This condition would not be there because a practical core requires some ampere turns to magnetize. Since magnetising ampere turns are negligible and we are interested only in leakage field and the corresponding leakage inductance, so we are neglecting magnetizing ampere turns.

Effectively in the transformer equivalent circuit, we are assuming the magnetizing inductance as infinity, because the magnetizing current is 0. I hope you understood the leakage field plot shown in the above figure. Now, we will go to the following expression of inductance that we have seen in basics of electromagnetics.

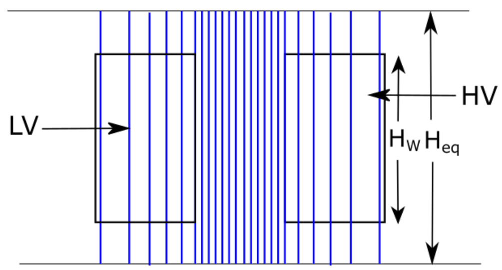
$$L = \frac{\mu N^2 A}{l} = \frac{N\psi}{i}$$

The above expression of L can be represented as $\frac{N^2}{\mathcal{R}}$ ($\mathcal{R}$ is the reluctance) because $\frac{l}{\mu A}$ is the reluctance. Of course that time we had used S instead of A and they represent the area.

If you are confused with the symbol of magnetic vector potential, then better you can represent area with S. Now, this A is the area of the flux tube and l is the height of the corresponding flux tube. Generally, we consider the flux ψ in the above expression is linked by all the N turns. But there could be a case that we will see later, turns in the winding may not link with the same amount of flux.

That is why the area does not represent the actual physical area. The area is calculated by integration of flux linkages. Now if you want to calculate the leakage inductance, it is always better to understand analytical calculation which may be approximate because for FEM analysis, we should be clear about the background principles, so that we do not make any mistake in the simulation.

For example, the flux plot in the window can be approximated as shown in the above figure with LV and HV windings as indicated. In the figure you can see that there are end effects because the flux is fringing at the ends of the winding. With such fringing, it is difficult to find inductance by using a simple analytical formula. So we have to make some approximation, here we are approximating that the flux is entirely axial as shown in the following figure.



Effectively, we are increasing the height of the flux tube, because at the ends beyond the winding height there is flux, so the height of the flux tube (H_{eq}) is effectively more than the physical height of the coil. So the flux region beyond the winding height is accounted by some extra height of the flux tube, which is more than the height of the winding (H_W).

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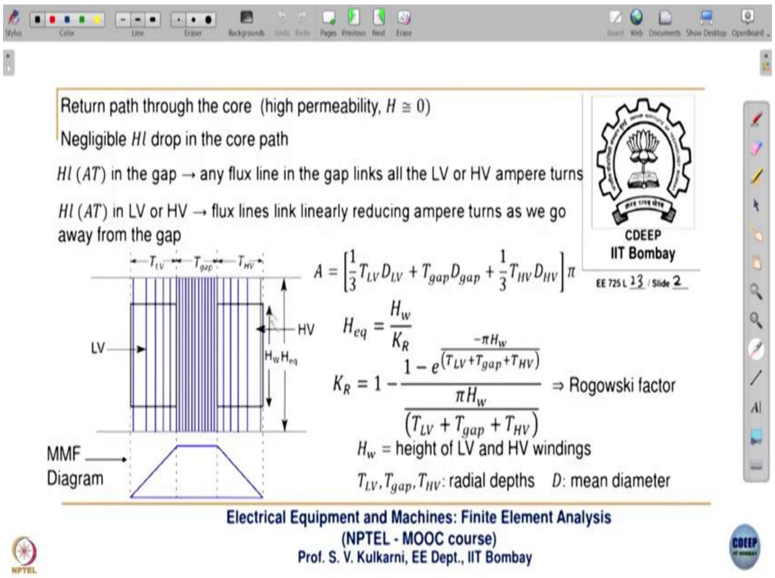


Going further in the plot shown in the above slide, we can see that in the gap between the LV and HV windings, the flux is uniformly spaced. But as you go away from the gap i.e, into the windings, the flux is initially closely spaced and then it is sparsely spaced. Because as you go away from the gap, the ampere turns enclosed by any flux contour would be reducing. Also, we know that we are neglecting magnetizing ampere turns, that means we are effectively making the permeability of core as infinite.

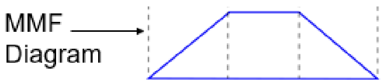
If permeability is infinite, $\frac{B}{\mu}$ is tending to 0, so H will tend to 0 in the core part. So that means if you evaluate $\oint H \cdot dl = Ni$ which is the ampere circuital law, since there is hardly any H in the core, if you take any closed contour the *Hl* drop will be only in the window region which is made of air which is non-magnetic and LV and HV windings that are made up of copper are also nonmagnetic.

So in the winding regions also $\mu_r \approx 1$. If you take any contour and calculate $\oint H \cdot dl$ along that contour, that value will be contributed by the flux lines or field values in the window portion. That is why any of the flux contours which are enclosing either full LV turns or full HV turns are enclosing the same turns.

The number of turns enclosed by any of the contours in the airgap region shown in the above slide is same, so H value will be the same in the entire gap. Now, as you go inside the winding, either through the LV winding or HV winding progressively, any of the flux lines will go on linking the lower ampere turns.

Remember the current in one of the windings is into the paper and in the other winding current is out of the paper because this is a 2D simulation.

If ampere turns are less, Ni is less and so H will be less and the corresponding B will be less. So that is the reason you get the magneto motive force (MMF) diagram as shown in the following figure.



Hl is constant in the air gap region and as you go into both the windings then H would reduce and it will go to 0 at the end points of the two windings. Because at the two points the ampere turn enclosed by this flux line will be 0.

Now, we have to calculate the total inductance because of all the leakage fields, which is our objective, using the following expression.

$$A = \left[\frac{1}{3} T_{LV} D_{LV} + T_{gap} D_{gap} + \frac{1}{3} T_{HV} D_{HV}\right] \pi$$

Since the flux is uniform in the air gap region the corresponding contribution by that gap is $T_{gap} \pi D_{gap}$, where, D_{gap} is the mean diameter and πD_{gap}will give the circumferential length and when it is multiplied with T_{gap}, width of the air gap will give the corresponding cross-sectional area of that gap.

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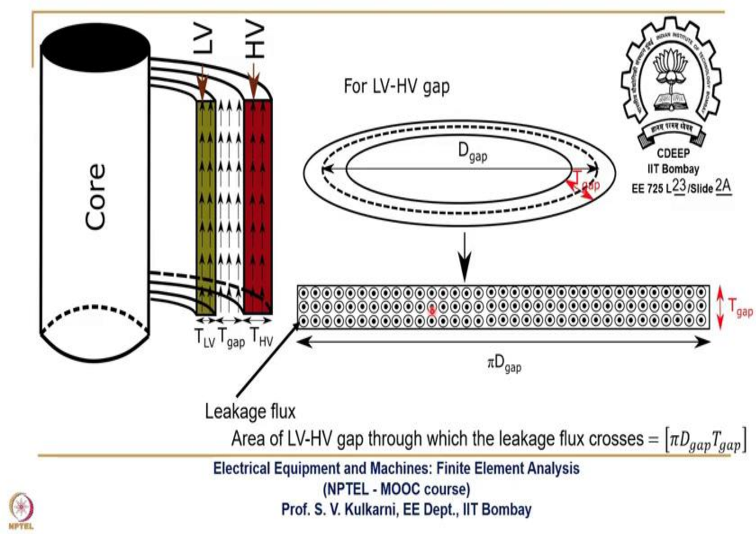
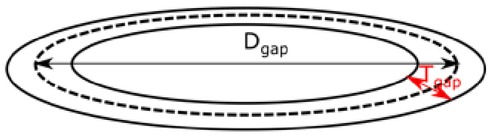
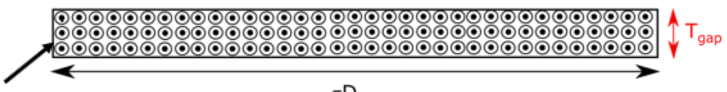


The cross sectional area through which the flux is crossing will be given by $\pi T_{gap} D_{gap}$. In the above slide, we will understand the area of the air gap through which the leakage flux is passing. Now, let us consider the LV-HV gap. From the top, the LV HV gap will be seen as shown below.



T_{gap} is the thickness or width of the gap and πD_{gap} is the circumferential distance as indicated in the above figure. Now, if you develop the area in the above figure along a horizontal line, then the area would be as shown in the following figure and at the considered instant of time flux is directed in the upward direction.



In the developed area shown in the above figure the flux is represented by dots. Then the area of the LV HV gap through which the leakage flux crosses would be $\pi D_{gap} T_{gap}$ where πD_{gap} is the length and T_{gap} is the width of rectangular area. So the cross sectional area that corresponds to this LV HV gap through which the leakage flux is crossing is shown in the above figure. $\pi D_{gap} T_{gap}$ is the corresponding cross sectional area through which the leakage flux is passing. But in the above expression of inductance the T_{LV} and T_{HV} are the corresponding radial depths of LV and HV windings and D_{LV} and D_{HV} are the corresponding mean diameters of LV and HV windings. Also in the above expression, one third appears, why?

Because flux is not uniform in the winding regions and if you integrate the flux linkages over the LV or HV areas and after some simplifications, you will get this 1/3 term and this is obvious because the flux lines or density of flux lines is reducing. So it is quite logical that the factor will be less than 1 and its value comes close to 1/3. The derivation is given in the third chapter of the following book.

Ref: S. V. Kulkarni and S. A. Khaparde, *Transformer Engineering: Design, Technology and Diagnostics*, CRC Press, 2012

So, you should understand that the area term that appears in the inductance formula will have the 1/3 factor because of non uniform spaced flux lines in LV and HV windings and for the gap there is no 1/3 term because the flux is uniform in the gap and any of the flux lines in the gap is linking the full LV or HV ampere turns.

Going further, now we need to worry about the height or length of flux tube and we already discussed that we should only consider the axial field and increase the height of that flux tube which should be more than the physical height of winding to consider the fringing fields. This is considered by using the following empirical formula.

$$H_{eq} = \frac{H_w}{K_R} \qquad K_R = 1 - \frac{1 - e^{\frac{-\pi H_w}{(T_{LV}+T_{gap}+T_{HV})}}}{\frac{\pi H_w}{(T_{LV} + T_{gap} + T_{HV})}}$$

Here, K_R is the Rogowski factor and H_w is the physical height of the winding which increases to H_{eq} which is given by H_w/K_R and the value of $K_R$will be less than 1. That is why $H_{eq} > H_w$. Again K_R is a function of radial depths and heights of the windings. This expression is used to find the effective height of the flux tube by neglecting the fringing effects and considering only the axial field.

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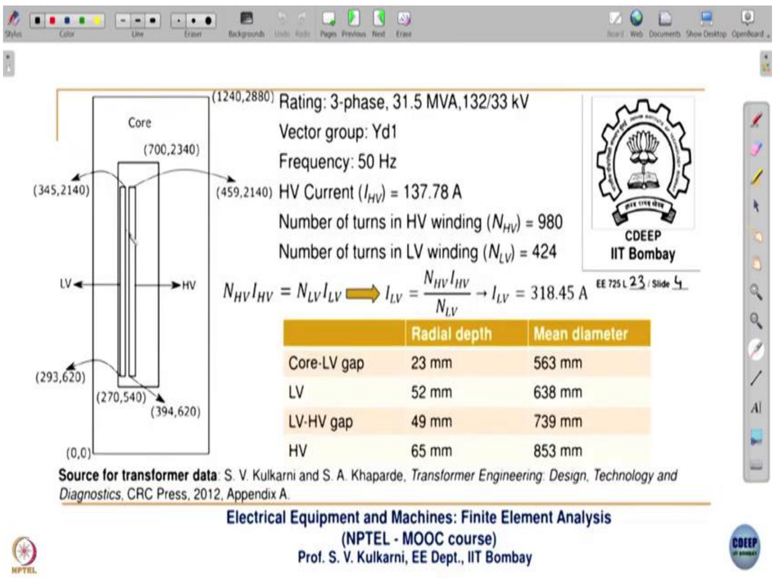

Now before we see the FEM formulation let us not go into details of all these dimensions and co-ordinates which are shown in the above slide. In the geometry, there are two windings (LV and HV winding) and corresponding radial depths and mean diameters are given in the above slide. Remember we have modelled only one phase and there will be other phases also. But we do not have to model them because we can get the per phase value of inductance by considering the window model of the windings shown in the above slide.

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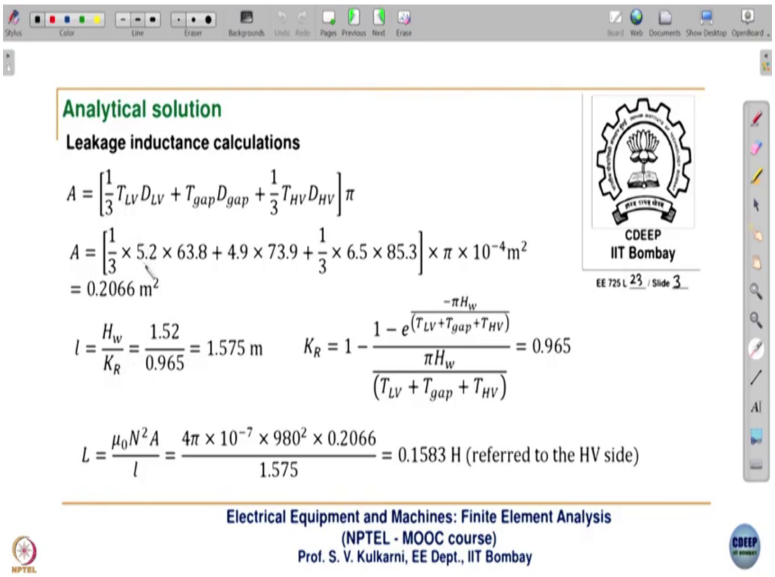

Now, we substitute all the values of radial depths and mean diameters in the following expression

$$A = \left[\frac{1}{3}T_{LV}D_{LV} + T_{gap}D_{gap} + \frac{1}{3}T_{HV}D_{HV}\right]\pi$$

All the values given in the above slide are in centimetre here, that is why it is multiplied by 10^{-4} in the following equation because the product of radial depth and mean diameter is in cm^2 so we converted into m^2 by multiplying with 10^{-4}.

$$A = \left[\frac{1}{3} \times 5.2 \times 63.8 + 4.9 \times 73.9 + \frac{1}{3} \times 6.5 \times 85.3\right] \times \pi \times 10^{-4}\text{m}^2$$
$$= 0.2066\ \text{m}^2$$

Using the above expression, you can calculate the area of cross section for the flux tube. l is the length of the flux tube. The equivalent length of the complete axial flux tube is $\frac{1.52}{0.965} =$ 1.575. Now this 1.52 and 1.575 are in metres and 0.965 is the Rogowski factor and the height of the equivalent flux tube is 1.575 which is more than the physical height of the winding (1.52

metres). Then you substitute the values of A and l in the following expression of L$\left(=\frac{\mu N^2 A}{l}\right)$. The number of turns N in the above expression is the HV turns.

If you take LV turns as N, then you will get the value of inductance refered to the LV side. Here we are taking HV turns so the calculated inductance is referred to the HV side. Remember the area (A) is the effective area of the whole flux tube.

Now we calculate the inductance by using finite element method. For the sake of completeness all the parameters of this transformer are: 3 phase 31.5 MVA 132/ 33 kV transformer, vector group is Yd1 which says that HV is star winding, LV is delta winding, and vector group is 1, frequency of excitation is 50 Hz, and HV current is 137.78 A which can be calculated based on the transformer rating.

The number of turns in HV winding is 980 and that in LV winding is 424 and the radial depth and mean diameters are given in the following table

	Radial depth	Mean diameter
Core-LV gap	23 mm	563 mm
LV	52 mm	638 mm
LV-HV gap	49 mm	739 mm
HV	65 mm	853 mm

We have used the above parameters in the analytical formula. In the analytical formulation, we are not considering the LV-core gap and HV -core gap because there is no flux in these regions. So that physical area does not contribute to the area in the expression of inductance, even in this case we have made all the radial fringing flux as vertical, so effectively there is no flux in this area and it is an approximation in the analytical formula. We do not have to do this approximation in FEM and it will be considered on its own. Then we have already defined all the geometrical parameters and we have $N_{LV}I_{LV} = N_{HV}I_{HV}$, so I_{LV} can be calculated from I_{HV} by using the relation of ampere turn balance. So that we are exactly matching the ampere turns. Even if there was some difference, for example, if you make 318.2 instead of 318.45 then the entire flux will be in the core and there will be a lot of magnetizing flux. That means the small difference in ampere turns will result in magnetizing ampere turns and field is high since the core permeability is defined as very high. So even a very small difference in ampere turns will set up a reasonably high flux density in the core, that is why we need to exactly match the ampere turns in the FEM simulation to calcualate leakage inductance.

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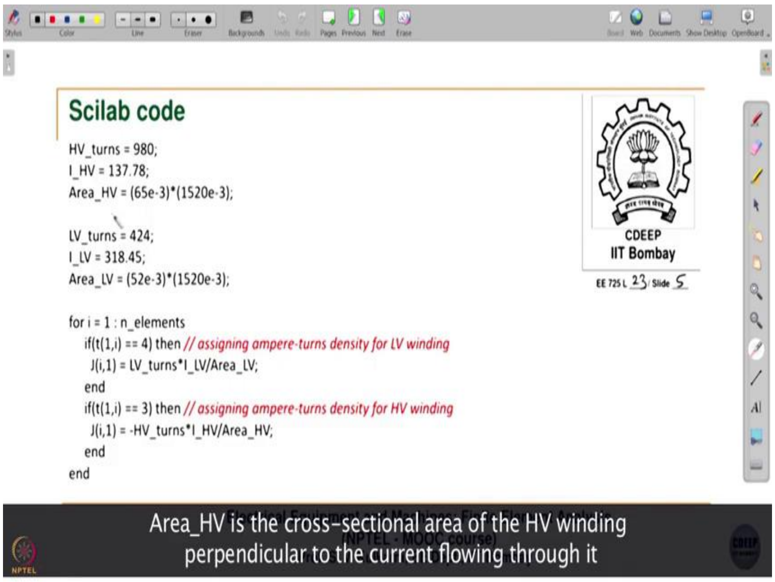

Now we will see how do we calculate the inductance using a Scilab code. In this slide we are only explaining extra parts of the code that is required for this problem and the rest of the code remains same. As compared to the magnetostatic code that we saw in the previous lecture, here we have to define ampere turn density in the windings and it is the main new thing. For example here, HV turns and current is 980 and 137.78 A respectively, similarly LV turns is 424. LV current is calculated on the previous slide, area of cross-section is calculated by multiplying 52 mm (radial depth) and the height LV and HV windings. The corresponding code to calculate HV and LV ampere turn densities is given below.

```
HV_turns = 980;
I_HV = 137.78;
Area_HV = (65e-3)*(1520e-3);

LV_turns = 424;
I_LV = 318.45;
Area_LV = (52e-3)*(1520e-3);
```

We have defined the ampere turn density. Now we have to set up an element wise for loop to go across all the elements and check which of those elements lie in this sub region number 4 which corresponds to LV. This we would have assigned while generating mesh using Gmsh software. At that time, we would have defined LV as sub-region number 4, so automatically all the elements in the LV will get the sub-region number 4. For all those elements which are in the LV region we will define ampere turn density by dividing ampere turns of LV with LV area. Always remember that when number of turns is more than one turn then you have to consider ampere turns divided by area as the current density and it is not just current divided by the area. So ampere turns divided by area is J. Similarly we define ampere turn density for sub-region number 3 which corresponds to HV. The following part of the code is used to define the ampere turn density.

```
for i = 1 : n_elements
    if(t(1,i) == 4) then // assigning ampere-turns density for LV winding
      J(i,1) = LV_turns*I_LV/Area_LV;
    end
    if(t(1,i) == 3) then // assigning ampere-turns density for HV winding
      J(i,1) = -HV_turns*I_HV/Area_HV;
    end
end
```

In the above code you can observe that there is a minus sign for the current density of HV, that means ampere turns of HV are taken as negative of ampere turns of LV which effectively means that the current directions of LV and HV are in the opposite directions. If current of LV is going in, the current of HV will come out. So the net ampere turns will be 0 and that is why there will not be any magnetizing ampere turns in the core, which is essential to calculate the leakage inductance of a transformer. If we are analysing the core, then the magnetizing ampere turns are important. Analysis under no load condition is an example.

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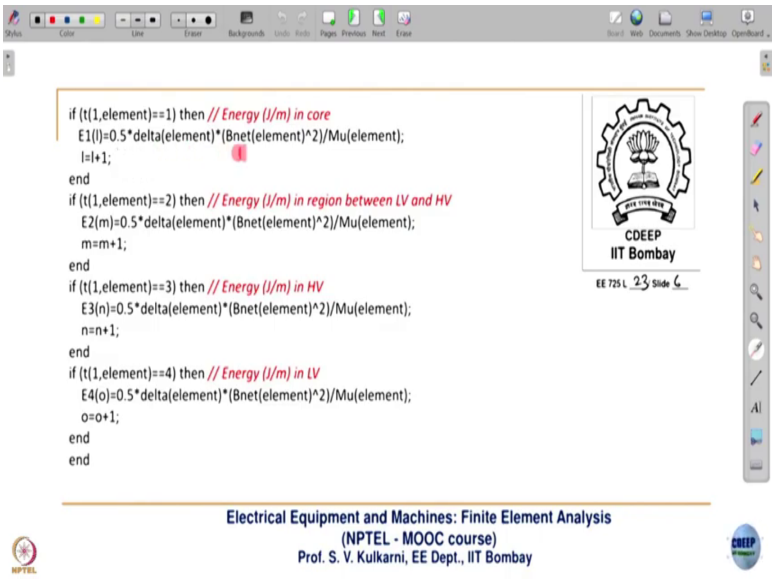

After getting the solution (nodal potential and flux density values) by following the normal procedure that we have seen, we can calculate the energy stored in each part of the problem domain. In the code given in the above slide, the energy is calculated in the core first. In the code, B^2 is associated with each element and $\frac{B^2}{2\mu}$ is the energy density which is nothing but energy per unit volume. Now we have to multiply this energy density by volume. In this code, we are multiplying by area and later multiply by mean πD_m (D_m is the mean diameter) to calculate energy from the energy density. Similarly, energy stored in LV and HV windings can be calculated using the same procedure for the corresponding elements in sub-regions 1, 2, 3 and 4.

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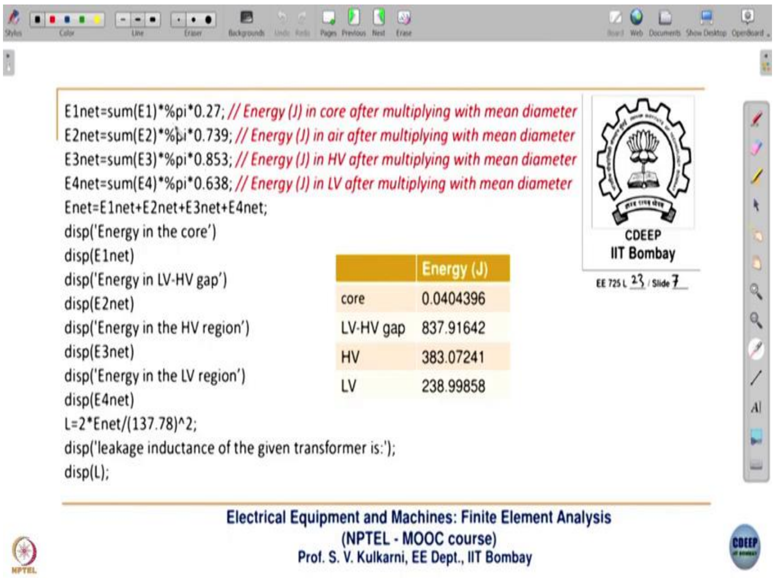
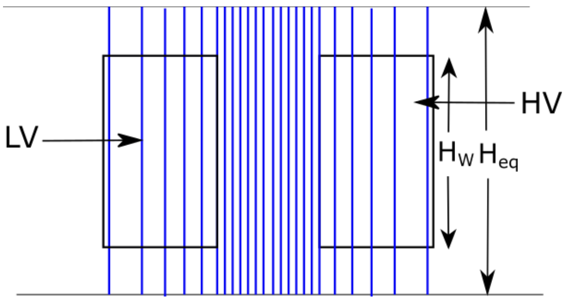


The corresponding energies per unit length for each element are calculated in the previous slide. The total energy associated with the core is calculated by multiplying total energy per length with πD_m which is the circumferential length. So, we multiply the area with circumferential length to get the volume. By multiplying the volume with the corresponding energy density $\left(\frac{B^2}{2\mu}\right)$ would give you the energy.

Similarly, the energies in the other three regions, air gap which means the gap between the LV and HV windings which is the main contributor of the energies and then the third and fourth contributors are HV and LV windings. Then we are adding all the energies and we are displaying them in the console using the code given in the above slide. The energies in the different regions are given in the following table.

	Energy (J)
core	0.040
LV-HV gap	837.92
HV	383.07
LV	239.99

From this, we can observe that the LV-HV gap has the maximum contribution in the total energy. The reason is obvious, you can see the following flux plot, all these flux lines which are uniformly spaced and the flux density will be maximum. Also, the radial depth and mean diameter of the HV winding are more compared to LV winding, so the area of HV will be more compared LV, that is why the energy of HV is more than LV. Then the total energy is calculated by summing up energies of all the regions and then you equate it to $\frac{1}{2}LI_{LV}^2$ to calculate L. The calculated total leakage of inductance of the transformer is referred to HV side and since we used HV current to calculate inductance form energy.



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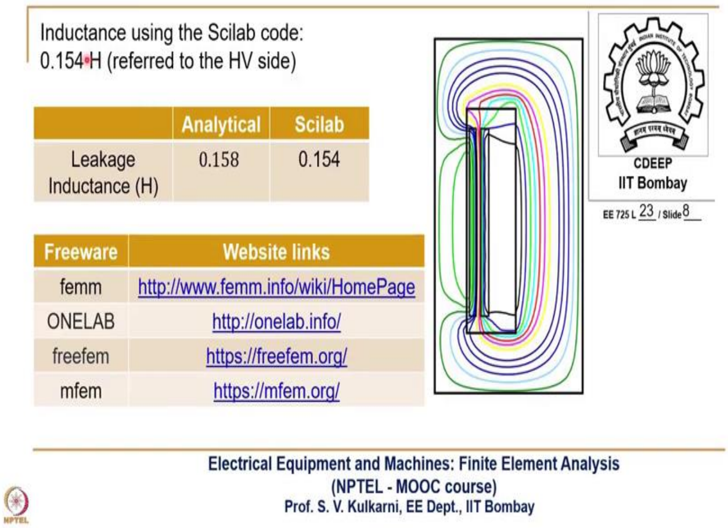

In this slide, you can see the comparison of the leakage inductance values calculated by using the analytical formula and the developed scilab code. In the above slide, you can see that they are quite close to each other and this verifies the both approaches of calculation (analytical and FEM). In analytical formulation, we did a number of approximations and the main approximation is considering the leakage field as axial.

Whereas in finite element method this approximation is not required and the flux can be axial as well as radial. However, in FEM also we approximated the field as two dimensional, but the actual field is three dimensional. In this problem, we have taken only one cross-section, so to that extent the value calculated by using 2D FEM will have an error as compared to more accurate 3D field based computation.

You can also use freeware FEM software as listed below and corresponding website links are given. For magnetostatic, electrostatic and even time-harmonic eddy current problems, some of these software can be used easily to compute performance parameters. Those who are interested they can go to the given websites, download the software and try such simple FEM simulations and then you can use them for more complicated problems.

Freeware	Website links
femm	http://www.femm.info/wiki/HomePage
ONELAB	http://onelab.info/
freefem	https://freefem.org/
mfem	https://mfem.org/

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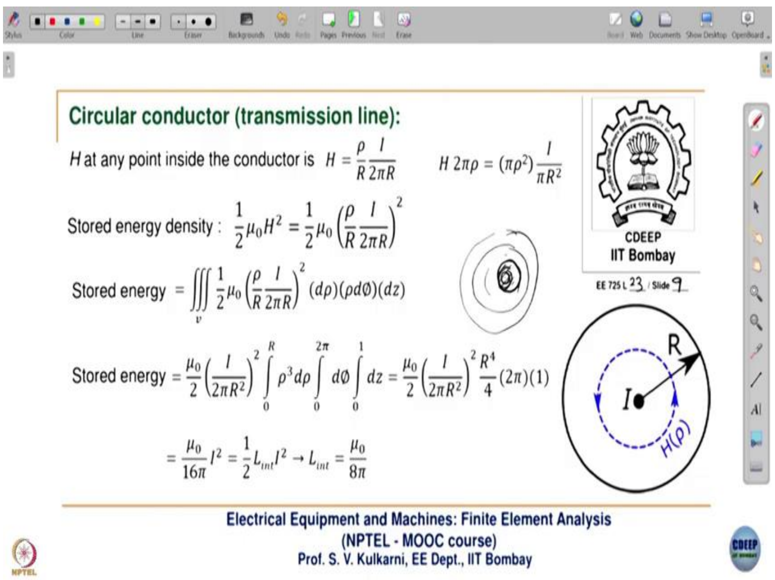

Now before going further into other problems like motor analysis, let us quickly see the analysis of a transmission line conductor. In typical power system books, they will calculate the inductance of a conductor.

In any conductor, there is internal and external flux. The internal flux will contribute to internal inductance (L_{int}) and the flux which is outside the conductor will give the external inductance (L_{ext}) of the conductor and the total inductance of the conductor will be $L_{int} + L_{ext}$. In this slide, we are calculating the internal inductance.

Again, we calculate the stored energy inside this conductor because of the internal flux. How do you do that?

H at any point (radius) inside the conductor is given by the following expression.

$$H = \frac{\rho}{R}\frac{I}{2\pi R}$$

Here, the radius is ρ in cylindrical coordinates and R is the conductor radius. The above expression is derived by evaluating the integral of the following expression.

$$H\ 2\pi\rho = (\pi\rho^2)\frac{I}{\pi R^2}$$

Here, the right hand side of the above expression will be the current enclosed. The expression of H can be derived by rearranging the terms in the above equation. The stored energy density will be $\frac{1}{2}\mu_0 H^2$, why μ_0?

The current-carrying conductor is made of copper, so $\mu_r \approx 1$. Then substitute the above expression of H, you will get

$$\frac{1}{2}\mu_0 H^2 = \frac{1}{2}\mu_0\left(\frac{\rho}{R}\frac{I}{2\pi R}\right)^2$$

The above expression of stored energy will be energy density i.e, energy per unit volume. The energy can be calculated by integrating the above expression over the volume of the conductor as given below. Here, $\rho d\rho d\phi dz$ is the incremental volume in cylindrical systems.

$$\text{Stored energy} = \iiint_v \frac{1}{2}\mu_0 \left(\frac{\rho}{R}\frac{I}{2\pi R}\right)^2 (d\rho)(\rho d\emptyset)(dz)$$

Remember, we are considering dz = 1 because we are doing per meter depth calculations, that means the determined values are for a 1 metre length of the transmission line in z direction. Because the 2D representation of conductor shown in the above slide is like a $\rho - \phi$ plane and you do not see the z direction. So that is why dz is taken as 1 or we integrate along z from 0 to 1. If you simplify the above integral, you will get stored energy as given by the following expression.

$$\text{Stored energy} = \frac{\mu_0}{16\pi} I^2$$

Then you equate this energy to $\frac{1}{2}LI^2$ to calculate as given below.

$$\text{Stored energy} = \frac{\mu_0}{16\pi} I^2 = \frac{1}{2} L_{int} I^2 \rightarrow L_{int} = \frac{\mu_0}{8\pi}$$

Here, you can see the value of internal inductance is constant and it is not a function of geometrical details. If you increase the radius of the conductor, the total stored energy will remain same $\left(\frac{\mu_0}{16\pi} I^2\right)$, because the current is constant and that is why the L_{int} is always constant. This ends lecture number 23. In lecture 24, we will see calculation of magnetizing inductance for a rotating machine. Thank you.

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