Electrical Equipment and Machines: Finite Element Analysis
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Lecture No. 22
Galerkin Method

In the previous lecture, we studied the weighted residual approach and the collocation method which is an example of that approach. The second popular approach is Galerkin Method and it is one of the weighted residual approaches.

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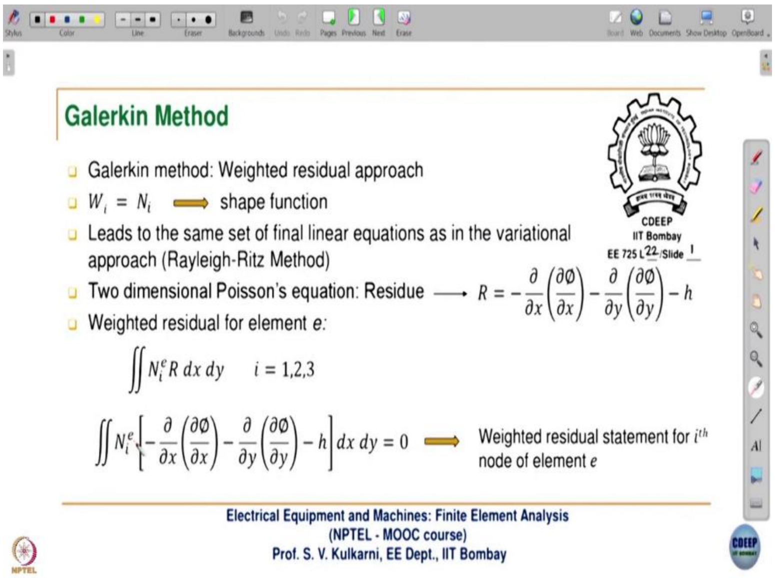

In this method, the shape function is chosen as the weighting function. This method is explained using the finite element procedure. In the case of the point collocation method, we used the whole domain approximation. But for the Galerkin method, we are using the finite element discretization procedure. So, we have divided the whole domain into a number of elements with a number of nodes.

For each node, there will be a shape function N_i for a corresponding element. For each node in a corresponding element, we will write a weighted residual statement . Now we will see how to do that. The advantage of choosing weight function as shape function is that it leads to the same linear system of equations as in the variational formulation and that we will prove now.

That is why this method became very popular and the two methods, Galerkin method and variational approach (Rayleigh-Ritz Method), can be proved to be equivalent. Now we will see how the equivalence can be proved. Let us consider a two dimensional Poisson's equation and its residue for an approximate solution which is given below.

$$R = -\frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) - h$$

Remember, the moment we say residue, that means ϕ is $\tilde{\phi}$. That means in the governing PDE you have substituted some approximate solution, and that is why you will get residue at every point in the domain. For simplicity, we have dropped ~ notation on ϕ. Now we will write the following weighted residual statement or expression for each element.

$$\iint WRdS = 0$$

Now in this approach the weighting function is substituted by corresponding shape function of that node for the element under consideration as given in the following equation.

$$\iint N_i^e R \, dx \, dy \qquad i = 1,2,3$$

This statement will be there for each node of the considered element. So, there will be 3 weighted residual statements for each element. The above equation defines that we are minimizing residue in a weighted integral sense wherein this weighting function is shape function. Weighted residual statement for an i^{th} node of element e is given in the following equation.

$$\iint N_i^e \left[-\frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) - h\right] dx \, dy = 0$$

There will be 3 statements for every node in each triangular element. If we are using a quadrilateral element with 4 nodes, there will be 4 weighted residual statements. But the corresponding shape function would be different for the rectangular and quadrilateral elements.

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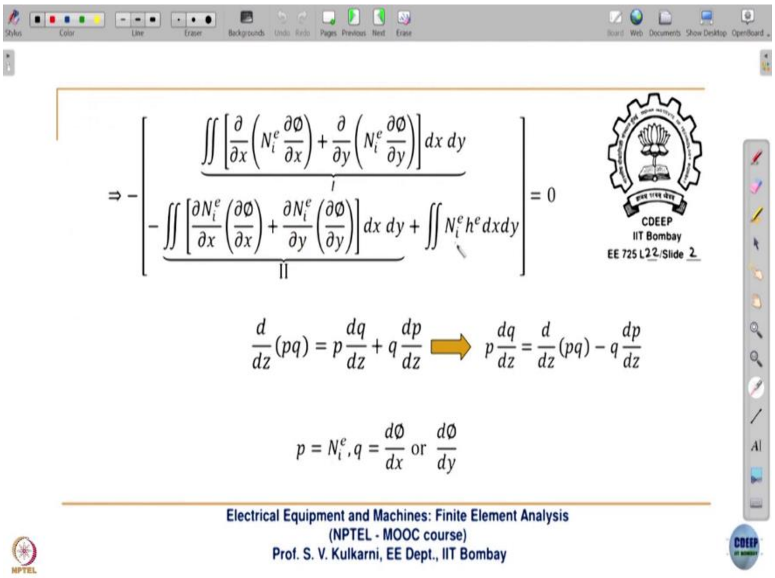

Now, as we did in the previous lecture we will represent the term $-\frac{\partial}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right)$ as $\left(\frac{\partial}{\partial x}\left(N_i^e \frac{\partial\emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_i^e \frac{\partial\emptyset}{\partial y}\right)\right) - \left(\frac{\partial N_i^e}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right) + \frac{\partial N_i^e}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right)\right)$ by using the chain rule of differentiation $\left(\frac{d}{dz}(pq) = p\frac{dq}{dz} + q\frac{dp}{dz}\right)$ and the weighted residual statement reduces to the following equation.

$$-\left[\begin{array}{c}\underbrace{\iint\left[\frac{\partial}{\partial x}\left(N_i^e \frac{\partial\emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_i^e \frac{\partial\emptyset}{\partial y}\right)\right] dx\,dy}_{I} \\ -\underbrace{\iint\left[\frac{\partial N_i^e}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right) + \frac{\partial N_i^e}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right)\right] dx\,dy}_{\mathrm{II}} + \iint N_i^e h^e\,dxdy\end{array}\right] = 0$$

So, now in the above expression we have 3 terms or integrals.

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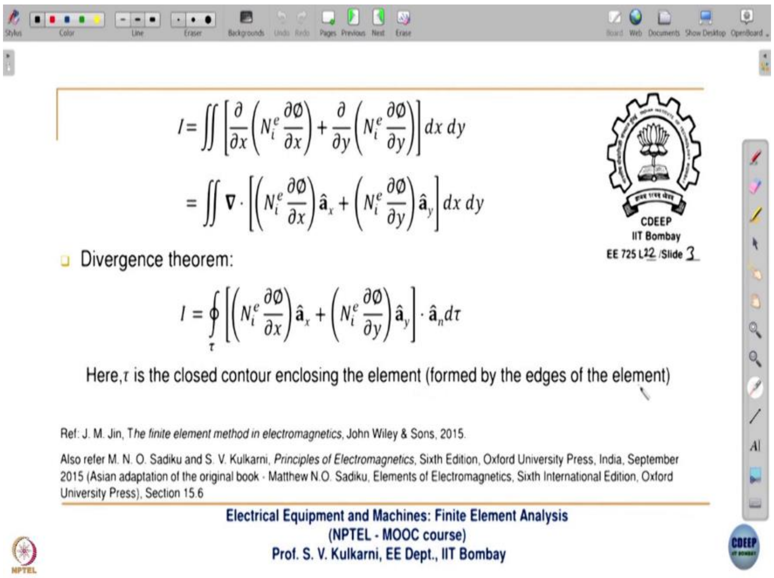

Now, let us consider the first integral which is given in the above equation. As we did earlier, we first apply the divergence theorem to the integrand and the integral reduces as given below.

$$I = \iint \left[\frac{\partial}{\partial x}\left(N_i^e \frac{\partial \emptyset}{\partial x}\right) + \frac{\partial}{\partial y}\left(N_i^e \frac{\partial \emptyset}{\partial y}\right)\right] dx\, dy = \iint \boldsymbol{\nabla} \cdot \left[\left(N_i^e \frac{\partial \emptyset}{\partial x}\right)\hat{\mathbf{a}}_x + \left(N_i^e \frac{\partial \emptyset}{\partial y}\right)\hat{\mathbf{a}}_y\right] dx\, dy$$

The integrand of the first integral is nothing but the divergence of the vector in the second integral. Then by invoking divergence theorem, the surface integral reduces to contour integral as given below.

$$I = \oint_{\tau} \left[\left(N_i^e \frac{\partial \emptyset}{\partial x}\right)\hat{\mathbf{a}}_x + \left(N_i^e \frac{\partial \emptyset}{\partial y}\right)\hat{\mathbf{a}}_y\right] \cdot \hat{\mathbf{a}}_n d\tau$$

The τ in the above equation is the closed contour enclosing the element and it is formed by the edges of the element. Simplifications of this integral we will see little later. First, we will concentrate on the second integral of the weighted residual statement.

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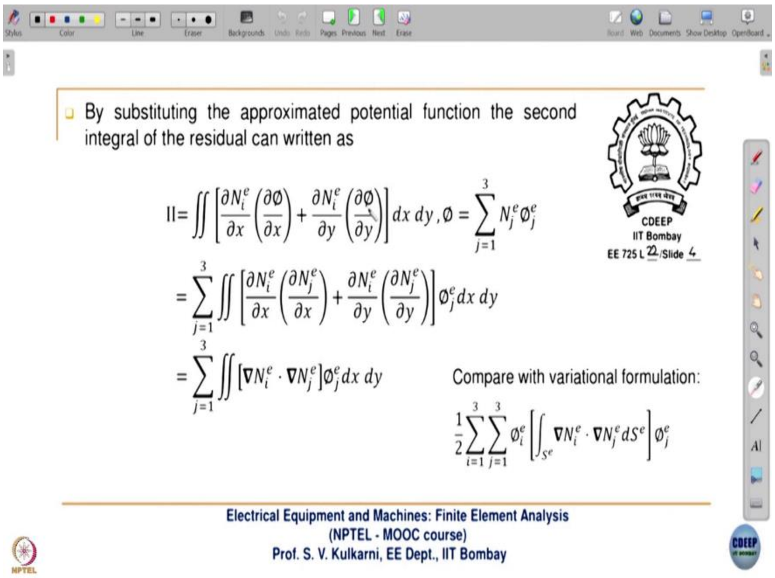

Now we substitute ϕ in the integral as $\sum_{j=1}^{3} N_j^e \emptyset_j^e$ So, when you substitute the summation in place of ϕ, the second integral of the weighted residual statement reduces to the following expression.

$$\text{II}=\iint\left[\frac{\partial N_i^e}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right)+\frac{\partial N_i^e}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right)\right]dx\,dy=\sum_{j=1}^{3}\iint\left[\frac{\partial N_i^e}{\partial x}\left(\frac{\partial N_j^e}{\partial x}\right)+\frac{\partial N_i^e}{\partial y}\left(\frac{\partial N_j^e}{\partial y}\right)\right]\emptyset_j^e dx\,dy$$

As discussed earlier, this $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ operate only on N_i and N_j because ϕ_j is not a function of x and y. In variational approach also we minimise the energy by varying the potential at a point.

So, that is why ϕ_j is taken out of the differential and the integral can be written as given below.

$$\text{II}=\sum_{j=1}^{3}\iint[\nabla N_i^e\cdot\nabla N_j^e]\emptyset_j^e dx\,dy$$

Now the above integral can be compared with the corresponding variational formulation expression at the element level which is given below.

$$\frac{1}{2}\sum_{i=1}^{3}\sum_{j=1}^{3}\emptyset_i^e\left[\int_{S^e}\nabla N_i^e\cdot\nabla N_j^e dS^e\right]\emptyset_j^e$$

We had got this integral when we found an element level energy. If you refer the previous lectures, you will easily recollect that.

In the above expression, you have two potential variables ϕ_i and ϕ_j. But integral that corresponds to the weighted residual method has only one potential variable. Because in the variational formulation, you will combine energies of all elements and then differentiate the total energy expression with respect to ϕ_i and equate it to 0, so one of the ϕ_i or ϕ_j will be cancelled when we differentiate the energy.

ϕ_i will be eliminated when you differentiate that total energy by ϕ_i. Another difference is that there is no half in the integral that correspond to weighted residual method. Because in the case of integral that corresponds to the variational method when you differentiate with respect to the diagonal terms, 2 from ϕ_i^2 will cancel the half.

In the case of off diagonal terms, they will come twice because of symmetry. So, again that 2 will cancel the half. That is why the above two integrals are equivalent after energy minimization. This integral is going to give you the global coefficient matrix. Finally, the linear system of equation is $CA = B$.

The above two integrals will give you C matrix. Here, C is the global coefficient matrix whose dimensions are $n \times n$. So, now we have already seen the equivalence between the Galerkin method and the variational method which is known by Rayleigh Ritz.

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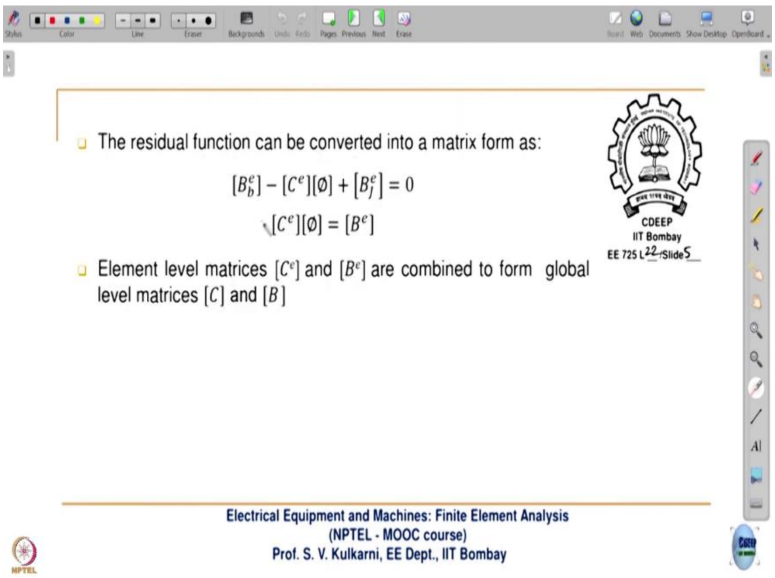

Now, the residual statement can be converted into a matrix form $[B_b^e] - [C^e][\emptyset] + [B_f^e] = 0$. The C^e will directly come from the above integrals. Remember, the element coefficient matrix derived using variational formulation will have two summations as given in the above equation. One of the two summations will have i goes from 1 to 3 and the other will have j goes from 1 to 3. Then this equation will result in a 3×3 matrix. But integral II derived using weighted residual method will have only 3 terms because it has only one summation.

Other 6 terms will be determined when you write the corresponding residual statement for the other two nodes. So, $[C^e]$ will be again a 3×3 element level matrix. We will again see the same concept. Suppose there is an element e with nodes 1, 2, 3, and the integral II given above will be for node 1 and this term will give 3 terms. When you execute the same procedure for second and third nodes of that element, you will get 6 terms of element coefficient matrix. So, you all again get 9 terms and it will lead to a 3×3 matrix at the element level using this weighted residual procedure. Now, there are two more matrices in the above slide. Matrix $[B_f^e]$ will be derived from the source term which is given in the following integral.

$$\iint N_i^e h^e dxdy$$

The matrix $[B_b^e]$ stands for boundary conditions and the entries of this matrix are given by the following integral.

$$I = \oint_\tau \left[\left(N_i^e \frac{\partial \emptyset}{\partial x}\right)\hat{\mathbf{a}}_x + \left(N_i^e \frac{\partial \emptyset}{\partial y}\right)\hat{\mathbf{a}}_y\right] \cdot \hat{\mathbf{a}}_n d\tau$$

Now going back to the following weighted residual statement for node i which we started with the following equation.

$$\iint N_i^e \left[-\frac{\partial}{\partial x}\left(\frac{\partial \emptyset}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \emptyset}{\partial y}\right) - h\right] dx\, dy = 0$$

Then we got the following 3 terms or integrals.

$$-\left[\begin{array}{c}\underbrace{\iint\left[\frac{\partial}{\partial x}\left(N_i^e\frac{\partial\emptyset}{\partial x}\right)+\frac{\partial}{\partial y}\left(N_i^e\frac{\partial\emptyset}{\partial y}\right)\right]dx\,dy}_{I}\\ -\underbrace{\iint\left[\frac{\partial N_i^e}{\partial x}\left(\frac{\partial\emptyset}{\partial x}\right)+\frac{\partial N_i^e}{\partial y}\left(\frac{\partial\emptyset}{\partial y}\right)\right]dx\,dy}_{\mathrm{II}}+\iint N_i^e h^e dxdy\end{array}\right]=0$$

The third integral in the above equation will lead to $[B_J^e]$, the source matrix at the element level and we have seen this term already. The second integral will lead to the element coefficient matrix $[C^e]$. Now the first integral indicated by I is converted into the following equation.

$$I=\oint_{\tau}\left[\left(N_i^e\frac{\partial\emptyset}{\partial x}\right)\hat{\mathbf{a}}_x+\left(N_i^e\frac{\partial\emptyset}{\partial y}\right)\hat{\mathbf{a}}_y\right]\cdot\hat{\mathbf{a}}_n d\tau$$

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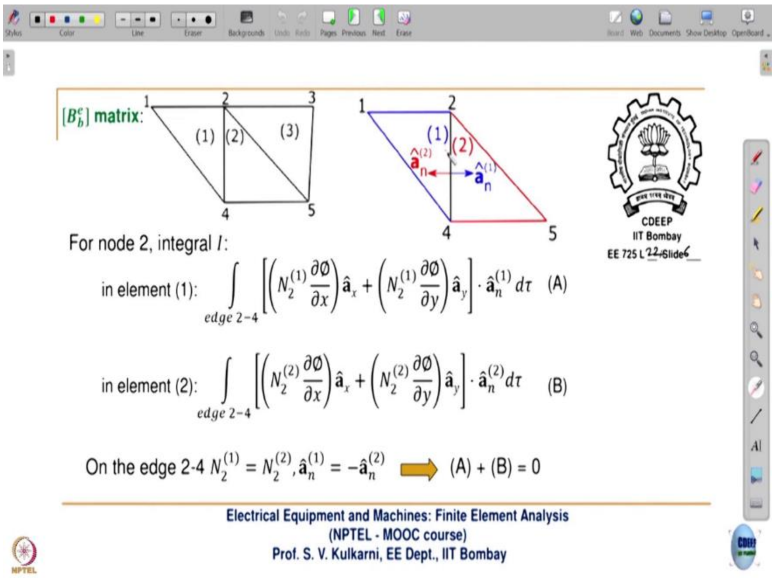
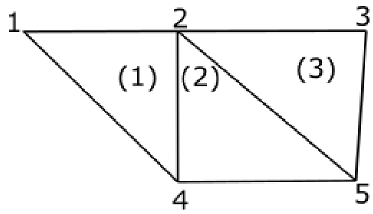


Now, we will see what happens to the integral I and we will discuss the entries of $[B_b^e]$ which is an element level matrix. Now, consider a geometry with 5 nodes which is shown in the following figure.



In this geometry, we have 5 global nodes and 3 elements. Now we write weighted residual statement for node 2. The integral I for this node is given below.

$$\int_{edge\ 2-4} \left[\left(N_2^{(1)}\frac{\partial \emptyset}{\partial x}\right)\hat{\mathbf{a}}_x + \left(N_2^{(1)}\frac{\partial \emptyset}{\partial y}\right)\hat{\mathbf{a}}_y\right]\cdot \hat{\mathbf{a}}_n^{(1)}\, d\tau$$

In element 1, we are considering the edge 2-4 and we will see how this integral will reduce. We derived the above integral from the contour integral which is deduced from the surface integral.

We will see what are the contributions of this integral I along all the segments of an element. The closed contour of an element is formed by the edges of the element. For element 1, the contour is formed by nodes 1, 4, 2. So now, we are considering edge 2-4. Edge 2-4 is common to elements 1 and 2. In element 1, the corresponding integral I for node 2 will be given by the above expression.

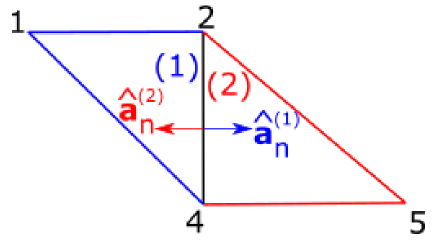
$N_2^{(1)}$ is the shape function of node 2 in element 1. Remember shape function for a node will have different expressions for the adjacent elements. So, $N_2^{(1)}$ in element 1 will be different form $N_2^{(2)}$ of element 2. But the value of $N_2^{(1)}$ at node 2 will be the same as $N_2^{(2)}$ at node 2. In general, at other points, the values of shape functions will be different except on edge 2-4. Now, for the segment 2-4, the values of $N_2^{(1)}$and $N_2^{(2)}$ are same. We can prove this by substituting any value of (x, y) and equation of this segment in the shape functions $N_2^{(1)}$and $N_2^{(2)}$.

We know that any line can be represented as $y = mx + c$. We know the end coordinates of the segment 2-4 and you can express this segment with $y = mx + c$. If we substitute the equation in $N_2^{(1)}$and $N_2^{(2)}$, we can prove this.

The expression of shape function for node 2 in an element is $\frac{1}{\Delta}[(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]$ and this expression varies for different elements because the coordinates (x_1, y_1) and (x_3, y_3) are different.

But in the shape function expression of node 2 for the two elements, if you substitute the equation of segment 2-4, $y = mx + c$, that is, if you substitute y in terms of x, you will find that N_2 expressions for both the elements will reduce to the same on the segment 2-4.

If that being the case, then the bracketed term in the above equation becomes equal on segment 2-4 for both the elements. What happens to this $\hat{\mathbf{a}}_n$? $\hat{\mathbf{a}}_n$ (outward normal) for this edge that corresponds to the two elements are exactly opposite as shown in the following figure.



In the above figure, $\hat{\mathbf{a}}_n^{(1)}$in blue colour is outward normal for element 1 and $\hat{\mathbf{a}}_n^{(2)}$ in red colour is outward normal for element 2. So, the outward normals for the segment 2-4 for the two elements are exactly opposite. Effectively, the contribution of the above integral for all the inner segments will cancel.

So the inside segments which are common to any two elements, the contribution of integral I is 0. The contribution to the integral I will be only from the edges of elements on the outermost boundary. That means when integral I is evaluated for the entire geometry and when you combine all the element level contributions, then only the contribution of the outermost boundary segments will remain.

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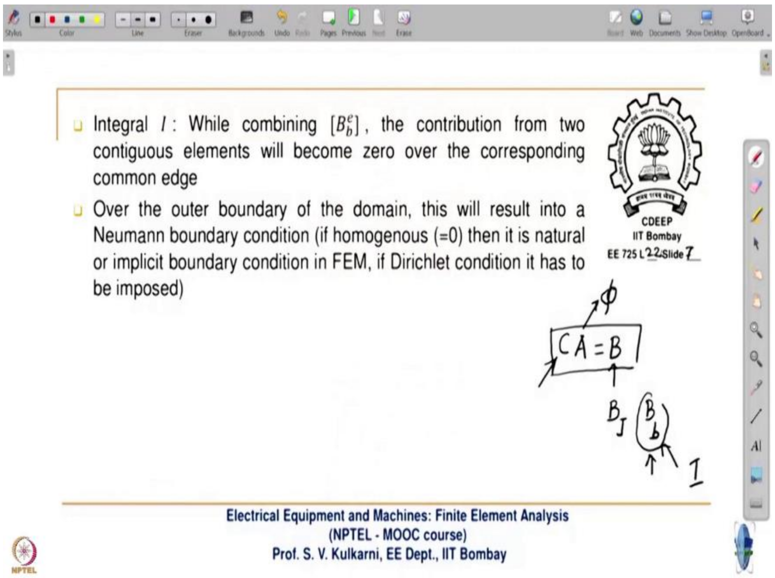

Now, it is summarised in the statements given in the above slide. Integral I: while combining all individual element level matrices, the sum of contributions from the two con tiguous elements will become 0 over the corresponding common edge. Over the outer boundary of the domain, this integral will result in a Neumann boundary condition.

Why Neumann? Because the integral has $\frac{\partial \phi}{\partial x}$ and $\frac{\partial \phi}{\partial y}$ which are Neumann conditions. Suppose if we take the parallel plate capacitor problem that we have seen earlier, this $\hat{\mathbf{a}}_n$ is either in x direction or y direction. For example, for vertical boundaries, the unit normal will be in the x direction. Then one of the two terms in the following equation will be 0.

$$\int_{edge\ 2-4} \left[\left(N_2^{(1)} \frac{\partial \emptyset}{\partial x}\right) \hat{\mathbf{a}}_x + \left(N_2^{(1)} \frac{\partial \emptyset}{\partial y}\right) \hat{\mathbf{a}}_y\right] \cdot \hat{\mathbf{a}}_n^{(1)}\, d\tau$$

For example, if we consider a vertical boundary, then $\hat{\mathbf{a}}_n$ will be $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_x \cdot \hat{\mathbf{a}}_y$ will be 0. So, there will be only one contribution because of $\frac{\partial \phi}{\partial x}$ term. If there is a Neumann boundary condition, i.e, $\frac{\partial \phi}{\partial x}$ will get imposed. If it is homogeneous Neumann condition, then $\frac{\partial \phi}{\partial x} = 0$. So the value of $\frac{\partial \phi}{\partial x}$ depends on the boundary condition that is imposed on the boundary.

So, if it is a homogeneous Neumann condition as in case of the parallel plate capacitor problem with fringing effect neglected, then there will not be any contribution from this integral.

Because for homogeneous Neumann condition, all the derivatives in the above equation will become 0.

Suppose on the outermost boundary, if you have Dirichlet boundary condition, that means, on the top and bottom plates of that capacitor, then that will get imposed in the final set of equations as we have done earlier. Then the boundary condition will get imposed at that stage. In case of non-homogenous Neumann boundary condition, the two differential terms in the above equation will be non-zero and then $[B_b^e]$ will be there. In case of homogeneous Neumann condition, this $[B_b^e]$ will go down to 0. Then homogeneous Neumann condition is called as a natural or implicit boundary condition in FEM.

Why implicit? If nothing is defined on a boundary, then boundary condition will be automatically taken as homogenous Neumann. Because the matrix $[B_b^e]$ will not be there in the final set of equations. That is why it is called as implicit. So, we do not have to do anything, if homogeneous Neumann condition have to be imposed. The moment you do not consider $[B_b^e]$ matrix, automatically the terms in the above integral are made to 0.

The Dirichlet condition has to be imposed when you get the final matrix equation $CA = B$. Here, A is ϕ in our case. Because we are taking ϕ as the potential variable.The B matrix has two contributions, one from B_J (current density) if it is a magnetostatic problem and then there is a contribution from the boundary conditions $[B_b]$.

$[B_b]$ matrix will come from the segments which are on the outermost boundary when you evaluate that integral I. $[C]$ is the global coefficient matrix which is same in both variational and weighted residual methods . So, finally, $CA = B$ will be the same in both methods.

With this, we have understood both variational and weighted residual methods. A weighted residual approach is more of mathematical because here we have not talked of energy and we only talked of minimizing the residue or the error in the weighted integral sense. So, it is purely a mathematical technique which involves minimizing the error or the residue at each point.

So, we will stop at this point and then we will see the applications and the corresponding changes in the FE formulations of different PDEs. From the next lecture, we will see new formulations like diffusion equation, transient, etc. We will discuss only the governing equation and the corresponding changes in the FE formulation and we will not get into coding because the coding part is more or less now completed. We have explained 2-3 codes in detail.

Using those codes and the explanations about the modifications in the FE formulation you can develop a code for any two-dimensional problem. As I said earlier, devoloping a code for a 3-D formulation is difficult from the point of view of coding. Unless it is necessary, you should not go for 3-D coding to start with. First, you should do 2-D coding, verify the results and then only you can go for three dimensional coding. Thank you.

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