Electrical Equipment and Machines: Finite Element Analysis

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Lecture 21

Computation of B and H Field and Method of Weighted Residuals

In the previous lectures, we saw 2-dimensional FE formulation in terms of magnetic vector potential. We saw the procedure to calculate A_z because a 2-dimensional formulation is developed in terms of A_z. But our main purpose is to get B and H fields.

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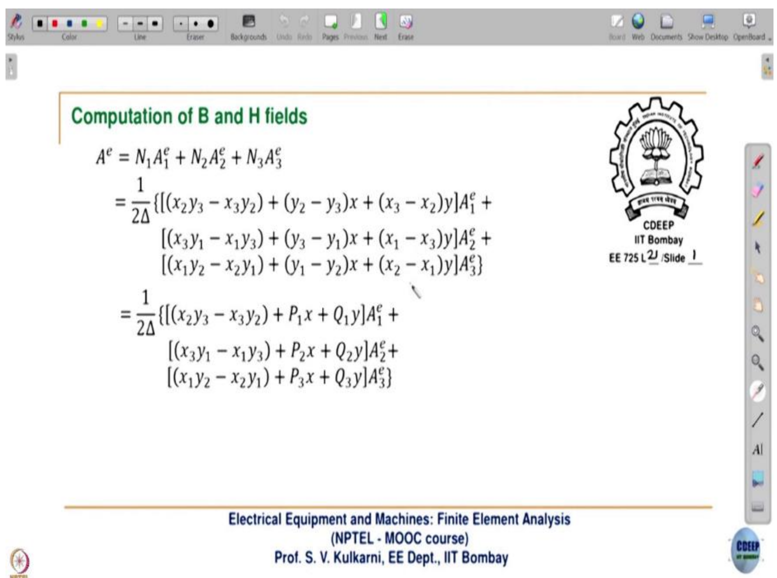

In this lecture, we will discuss a way to calaculate B and H using computed A_z values. We know already that A in an element is defined as $N_1A_1 + N_2A_2 + N_3A_3$ and if you substitute the expressions of N_1, N_2 and N_3, you will get the following equation.

$$
\begin{aligned}
A^e &= N_1A_1^e + N_2A_2^e + N_3A_3^e \\
&= \frac{1}{2\Delta}\{[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]A_1^e + \\
&\qquad [(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y]A_2^e + \\
&\qquad [(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y]A_3^e\}
\end{aligned}
$$

Then as we did in the previous lecture, by replacing $y_2 - y_3$, $y_3 - y_1$, $y_1 - y_2$ with P_1, P_2, and P_3 and $x_3 - x_2$, $x_1 - x_3$, $x_2 - x_1$ with Q_1, Q_2, and Q_3, the above equation can be written as

$$A^e = \frac{1}{2\Delta}\{[(x_2y_3 - x_3y_2) + P_1x + Q_1y]A_1^e + [(x_3y_1 - x_1y_3) + P_2x + Q_2y]A_2^e + [(x_1y_2 - x_2y_1) + P_3x + Q_3y]A_3^e\}$$

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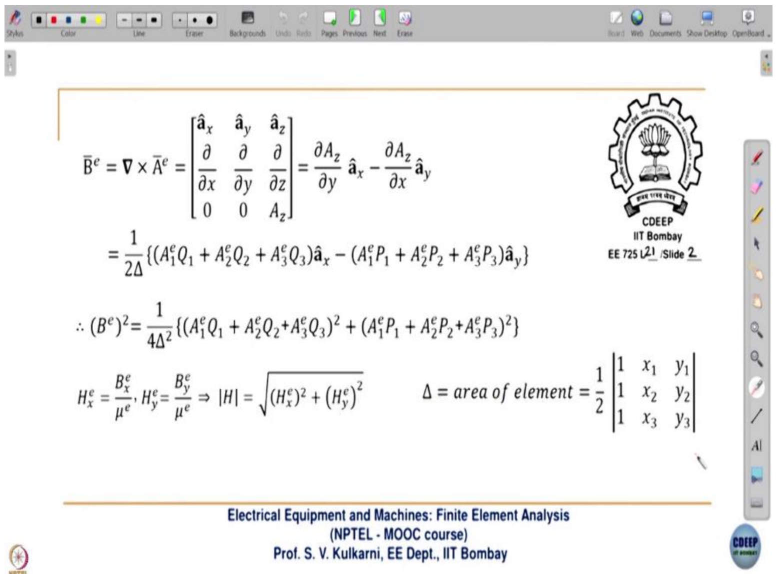

We know that $\mathbf{B}^e = \nabla \times \mathbf{A}^e$ is represented with the following determinant

$$\overline{B}^e = \boldsymbol{\nabla} \times \overline{A}^e = \begin{bmatrix} \hat{\mathbf{a}}_x & \hat{\mathbf{a}}_y & \hat{\mathbf{a}}_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & 0 & A_z \end{bmatrix} = \frac{\partial A_z}{\partial y}\hat{\mathbf{a}}_x - \frac{\partial A_z}{\partial x}\hat{\mathbf{a}}_y$$

Here, we have only A_z component because *x* and *y* components of A are 0. So $\nabla \times \mathbf{A}^e$ will reduce to $\frac{\partial A_z}{\partial y}\hat{\mathbf{a}}_x - \frac{\partial A_z}{\partial x}\hat{\mathbf{a}}_y$. So, B has $\hat{\mathbf{a}}_x$ and $\hat{\mathbf{a}}_y$ components, which is the case because the current is in z direction, so A is in z direction. This we have seen in the basics of electromagnetics.

In the above equation, you can notice that B_x component is $\frac{\partial A_z}{\partial y}$ and B_y component is $\frac{\partial A_z}{\partial x}$. If you substitute the expression of A_z which we saw in the previous slide, then the magnetic flux density reduces to

$$\overline{B}^e = \frac{1}{2\Delta}\{(A_1^eQ_1 + A_2^eQ_2 + A_3^eQ_3)\hat{\mathbf{a}}_x - (A_1^eP_1 + A_2^eP_2 + A_3^eP_3)\hat{\mathbf{a}}_y\}$$

So the variation with respect to x is $A_1^e P_1 + A_2^e P_2 + A_3^e P_3$ and variation with respect to y is $A_1^e Q_1 + A_2^e Q_2 + A_3^e Q_3$

The terms $x_2 y_3 - x_3 y_2$, $x_3 y_1 - x_1 y_3$, and $x_1 y_2 - x_2 y_1$ will not count for B because they are constant. So derivatives of these terms will be 0. Then the magnitude of B^2 is given by the following equation.

$$(B^e)^2 = \frac{1}{4\Delta^2}\{(A_1^e Q_1 + A_2^e Q_2 + A_3^e Q_3)^2 + (A_1^e P_1 + A_2^e P_2 + A_3^e P_3)^2\}$$

Then you can calculate H_x^e and H_y^e by using the following equation.

$$H_x^e = \frac{B_x^e}{\mu^e}, H_y^e = \frac{B_y^e}{\mu^e} \Rightarrow |H| = \sqrt{(H_x^e)^2 + (H_y^e)^2}$$

As we discussed earlier, we can assume permeability over an element as constant so the expression of H will be straight forward.

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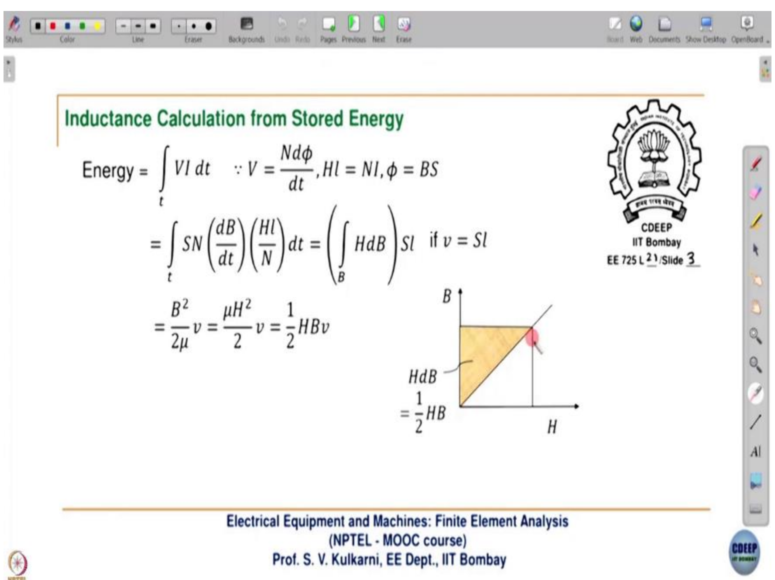


In basics of electromagnetics, we discussed why energy density is $\int_B HdB$. For the sake of completeness, again we are discussing in this slide. Here, we want to calculate the inductance of the isolated bar which we have simulated in the previous lecture using 2D FE code. So we will calculate the inductance and verify it with an analytical formula.

We know that energy is calculated by using $\int_t VI\,dt$. Now you replace I in this integral with $\frac{Hl}{N}$ and V with $N\frac{d\phi}{dt}$ and $\phi = BS$ where S is the area. If you substitute all these three terms in the above integral, the expression of energy is simplified as given below.

$$\int_t VI\,dt = \int_t SN\left(\frac{dB}{dt}\right)\left(\frac{Hl}{N}\right)dt = \left(\int_B HdB\right)Sl \text{ if } v = Sl$$

Here, Sl is the incremental volume and then if you evaluate $\int_B HdB$ by substituting $H = \frac{B}{\mu}$, the expression of energy reduces to $\frac{B^2}{2\mu}v$ as given below.

$$\left(\int_B HdB\right)v = \left(\int_B \frac{B}{\mu}dB\right)v = \frac{B^2}{2\mu}v = \frac{\mu H^2}{2}v = \frac{1}{2}HBv$$

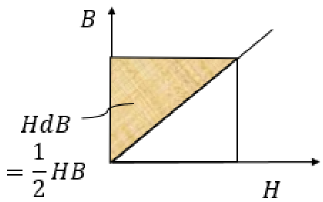
Consider a magnetic material divided into small elements and the volume of each element is v. In each of these elements, you calculate B by using the following expression which is in terms of A.

$$(B^e)^2 = \frac{1}{4\Delta^2}\{(A_1^e Q_1 + A_2^e Q_2 + A_3^e Q_3)^2 + (A_1^e P_1 + A_2^e P_2 + A_3^e P_3)^2\}$$

Remember that we calculate B using A and it is constant because we considered the first order approximation of A as $a + bx + cy$ which varies linearly with *x* and *y*.

The derivative of A with respect to *x* and *y* are constant and therefore B in each element will be constant. So over each element $\frac{B^2}{2\mu}v$ is also constant. If it is a 2D formulation, then $v = Sl = S \times 1$ which means 1 m depth in z direction. So the energy associated with each element is $\frac{B^2}{2\mu}S$. Then to calculate the energy for the entire core you have to add energies of all the elements calculated using $\frac{B^2}{2\mu}S$.

Also, when we were studying basics of electromagnetics we understood that the energy is represented by the shaded area and co energy is represented by the remaining area of the following figure.



The physical interpretation of energy and co-energy and usefulness of co energy will be discussed later, when we see calculation of forces. So till that time, we will be deferring the discussion on co-energy.

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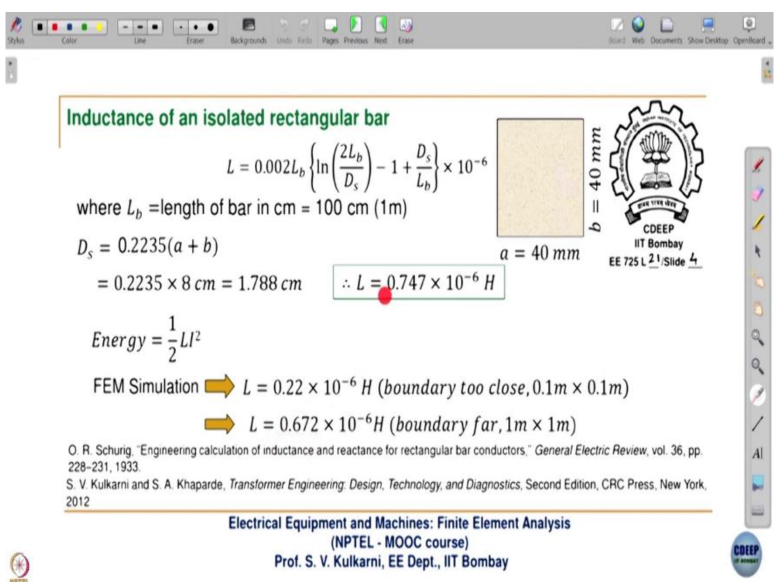
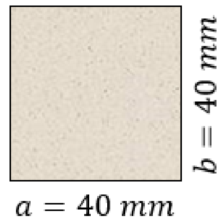


Going further, using the code discussed in the previous lecture, we have simulated and got the field solution for the rectangular bar which was enclosed in another rectangle. The geometry and dimensions of the bar are given in the following figure.



Then using the FEM solution, we can calculate the inductance of the bar conductor. First we will see the following analytical formula to calculate the inductance of an isolated rectangular conductor.

$$L = 0.002L_b\left\{\ln\left(\frac{2L_b}{D_s}\right) - 1 + \frac{D_s}{L_b}\right\} \times 10^{-6}$$

Here L_b is the length of the bar (in cm) along the direction of current. Here it is z direction. In 2D analysis, we are assuming 1 meter length in z direction. So we are calculating the inductance per meter depth.

D_s in the above equation is represented using the empirical formula $D_s = 0.2235(a + b)$ where a and b are the dimensions of the rectangular conductor. So if you substitute all the dimensions in centimeters in the above expression of inductance, we will get the value of L as 0.747×10^{-6} H. Now we can verify the finite element solution with the value calculated using the analytical expression.

We already have got the FE solution and we can calculate the B values in each of the elements using the formulation discussed in the previous slides. We can calculate the energy of each element using $\frac{B^2}{2\mu}S$ and that will be the elemental energy. We add energies of all elements and then equate it to $\frac{1}{2}LI^2$, where I is the current flowing through the conductor.

In the previous lecture, we have specified current in our FEM simulation in terms of J and we have to take that current and the expression $\frac{1}{2}LI^2$ is equated to the energy obtained from the FEM simulation. Each elemental energy is $\frac{B^2}{2\mu}v$. When you do that then the FEM simulation gives 0.22×10^{-6} H when we take the boundary dimensions as 0.1×0.1 which are closer to the rectangular conductor. In the previous lecture, we had mentioned that the boundary which is too close will give an inaccurate result and it is evident here that the inductance calculated using the FE simulation is quite far from the one calculated using the analytical formula.

The FE solution for the configuration that we saw in the previous lecture is quite approximate because the boundary is very close to the conductor and the analytical formula is for an isolated bar. In that lecture, we had taken the boundary as too close to understand FEM coding for a simple geometry with few number of elements.

Later on, we developed a code. Using that code, if we take the boundary far from the conductor say, $1\ \text{m} \times 1\ \text{m}$, which is a much bigger boundary then you will get that value of L as 0.672×10^{-6}H which is closer to the one calculated using the analytical formula. Remember this value of inductance will be more correct than the one calculated using the analytical formula. Because there are some empirical factors in the analytical formula, which depend upon the dimensions, so the accuracy will vary.

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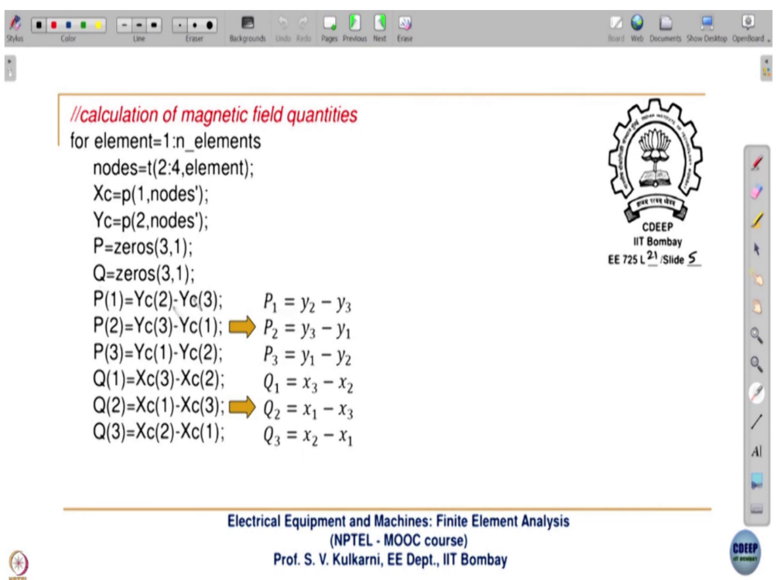

Now, how do we calculate the same thing using a code. We have already got the nodal potential (A) values using the code in the previous lecture. Then we run the code, the for loop, given in the above slide for each element. Global nodes of each element are taken by using the command nodes=t(2:4,element) and you are familiar about this.

By this command, second, third and fourth entries of each column from the t matrix which are the global node numbers of each element will come into the 'nodes' matrix for that element. Then the two commands Xc=p(1,nodes'); Yc=p(2,nodes') will save the x and y coordinates of those global nodes in the Xc and Yc matrices whose dimensions are 3×1.

Now we will assign P_1, P_2, and P_3 and Q_1, Q_2, and Q_3 and then we calculate the values of these variables using the following code that we have already seen.

```
P(1)=Yc(2)-Yc(3);
P(2)=Yc(3)-Yc(1);
P(3)=Yc(1)-Yc(2);
Q(1)=Xc(3)-Xc(2);
Q(2)=Xc(1)-Xc(3);
Q(3)=Xc(2)-Xc(1);
```

$$P_1 = y_2 - y_3$$
$$P_2 = y_3 - y_1$$
$$P_3 = y_1 - y_2$$
$$Q_1 = x_3 - x_2$$
$$Q_2 = x_1 - x_3$$
$$Q_3 = x_2 - x_1$$

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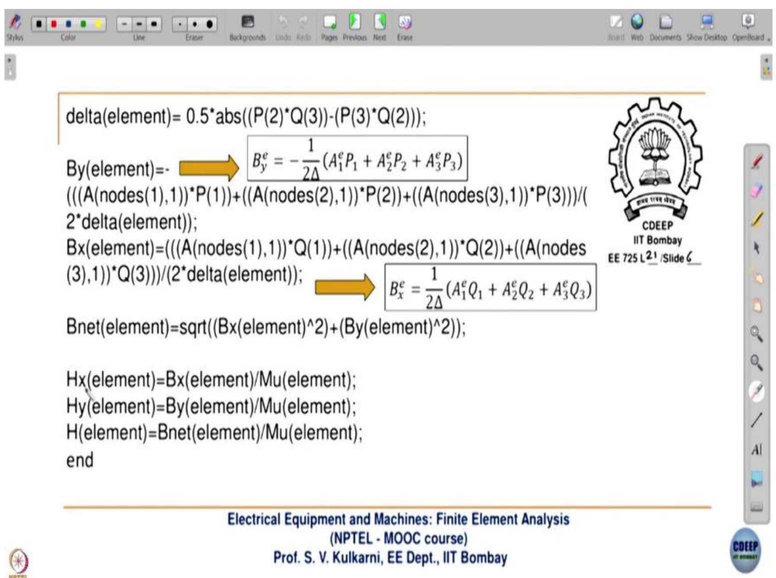

Then we saw the following expressions of B_x and B_y.

$$\overline{\mathrm{B}}^e = \frac{1}{2\Delta}\{(A_1^e Q_1 + A_2^e Q_2 + A_3^e Q_3)\hat{\mathbf{a}}_x - (A_1^e P_1 + A_2^e P_2 + A_3^e P_3)\hat{\mathbf{a}}_y\}$$

Δ is the area of the element and we have seen its expression in the previous lecture. Then you calculate B_x and B_y using the following commands.

```
By(element)= -
(((A(nodes(1),1))*P(1))+((A(nodes(2),1))*P(2))+((A(nodes(3),1))*P(3)))/(2*delta(element));
Bx(element)=(((A(nodes(1),1))*Q(1))+((A(nodes(2),1))*Q(2))+((A(nodes(3),1))*Q(3)))/(2*del
ta(element));
```

Then $B_{net} = \sqrt{B_x^2 + B_y^2}$ and H_x and H_y and H_{net} are calculated by using the following commands.

```
Bnet(element)=sqrt((Bx(element)^2)+(By(element)^2));
Hx(element)=Bx(element)/Mu(element);
Hy(element)=By(element)/Mu(element);
H(element)=Bnet(element)/Mu(element);
```

So by this, we have got B and H values.

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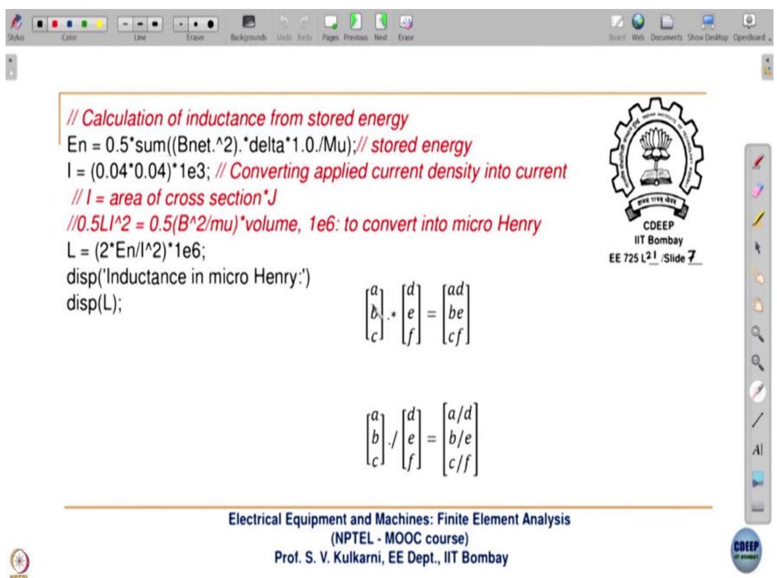

Now we calculate the energy using the expression $\frac{B^2}{2\mu} delta \times 1$. $\frac{B^2}{2\mu}$ will be the energy density and multiplying it with the area of the element will give energy per meter depth. $delta \times 1$ will be the volume. So, by multiplying $\frac{B^2}{2\mu}$ with $delta \times 1$ will give the energy of the element. The sum function in the following command adds energies of all elements in the domain and the total energy is equated to $\frac{1}{2}LI^2$ to calculate L.

```
// Calculation of inductance from stored energy
En = 0.5*sum((Bnet.^2).*delta*1.0./Mu);// stored energy
```

I in the expression is calculated by multiplying the current density that is imposed in the FE simulation with cross sectional area of the conductor (0.4 m × 0.4 m) and it is calculated by using the following command.

```
I = (0.04*0.04)*1e3; // Converting applied current density into current
```

The above command gives you current and then L is calculated by using the following command.

```
L = (2*En/I^2)*1e6;
```

This commands gives the value of L in μH because we have multiplied the expression with 10^6.

In the command to calculate the energy, you can see that there is a .* command. If you have two column vectors and if you want corresponding entries of the two column vectors to be multiplied then you have to use .* between the two vectors as shown in the following equations.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} .* \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} ad \\ be \\ cf \end{bmatrix} \quad \begin{bmatrix} a \\ b \\ c \end{bmatrix} ./ \begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} a/d \\ b/e \\ c/f \end{bmatrix}$$

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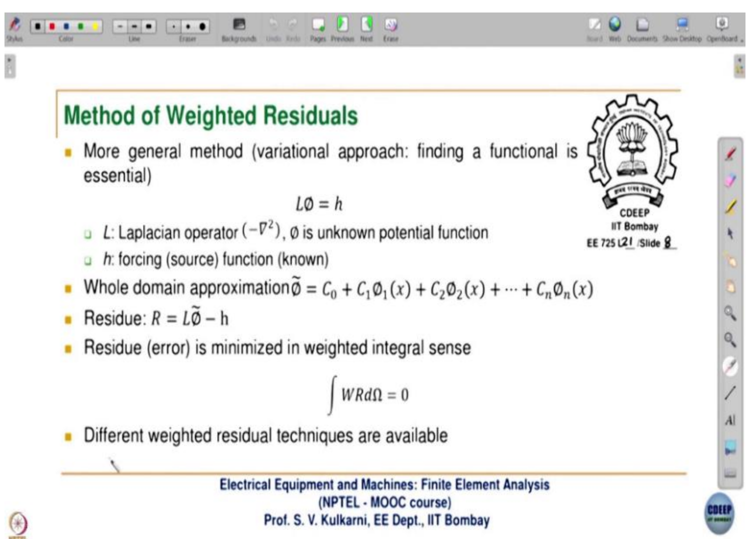

Till now we have seen variational formulation and the corresponding FE procedures. In one of the first lectures, I mentioned that there are two distinct approaches; one is the variational approach and other is the weighted residual method. The variational approach is based on physical principles because we are minimizing energy to determine the solution whereas weighted residual approach is more mathematical and it is based on error minimization.

Now we will quickly see details about the method of weighted residuals and we will also see the equivalence between the two approaches (variational and weighted residual approaches) and we will prove that both approaches lead to the same final system of matrix equations. Sometimes the weighted residual approach is more preferred because it is a generalized method whereas in the variational approach you need to find a functional for a given PDE.

For standard PDEs, you know functional expressions. But if you have a non-standard PDE then you have to first find the corresponding functional to determine the final matrix equations and solution. Instead of that in weighted residual approach, there is no need to to find a functional. Now let us start with a partial differential equation which is given by the equation $L\phi = h$ where L in this case is a Laplacian operator $-\nabla^2$ for the case of a Poisson's equation.

Here, ϕ is the unknown potential function and h is the forcing or source function which is known. Now again we start with a whole domain approximation and we approximate the unknown function as $\widetilde{\emptyset} = C_0 + C_1\emptyset_1(x) + C_2\emptyset_2(x) + \cdots + C_n\emptyset_n(x)$.

By substituting the approximate function $(\widetilde{\emptyset})$, we get residue as $R = L\widetilde{\emptyset} - h$. This residue or error is minimized in the weighted integral sense which is given by the following equation.

$$\int WR d\Omega = 0$$

Now $d\Omega$ in the above equation stands for the area and we could also call this as dS. In the earlier lectures, we have been using dS. Three different residual approaches are available.

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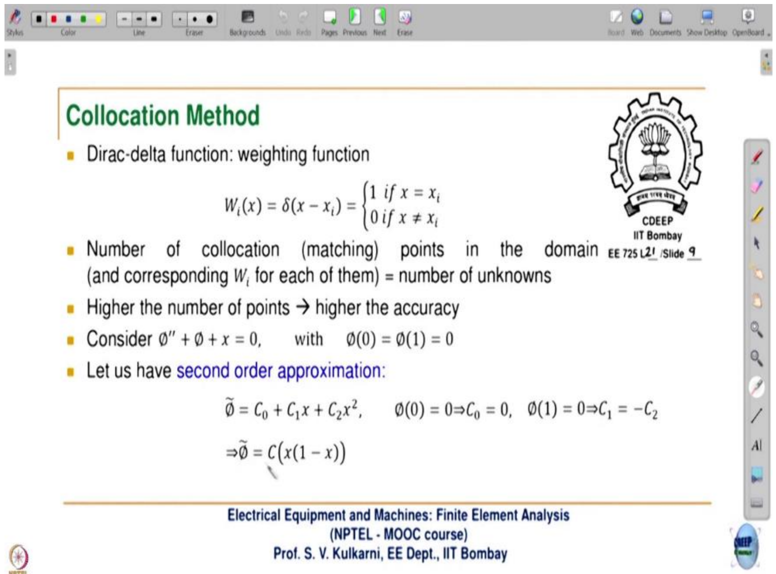

The first method is collocation method, in which we use Dirac-delta function as the weighting function (W). It is defined as given in the following equation.

$$W_i(x) = \delta(x - x_i) = \begin{cases} 1 \;\; if \;\; x = x_i \\ 0 \; if \;\; x \neq x_i \end{cases}$$

In signal and systems, this is very popularly used. Basically this function is used to focus on a particular point in the domain and to apply an equation at that point only and at rest of the points you make the function as 0. The number of collocation or matching points in the domain should be equal to the number of unknowns. Because our objective is to find the number of equations which should be equal to the number of unknowns. Higher number of points or collocation points then higher will be the accuracy.

Now consider the following differential equation that we have been seeing for quite sometime.

$$\emptyset'' + \emptyset + x = 0, \;\; \text{with} \qquad \emptyset(0) = \emptyset(1) = 0$$

Let us consider a second order approximation for the unknown potential as $\tilde{\emptyset} = C_0 + C_1 x + C_2 x^2$ and if you apply the boundary conditions $\emptyset(0) = \emptyset(1) = 0$, then $C_0 = 0$, and $C_1 = -C_2$, so $\tilde{\emptyset} = C(x(1-x))$.

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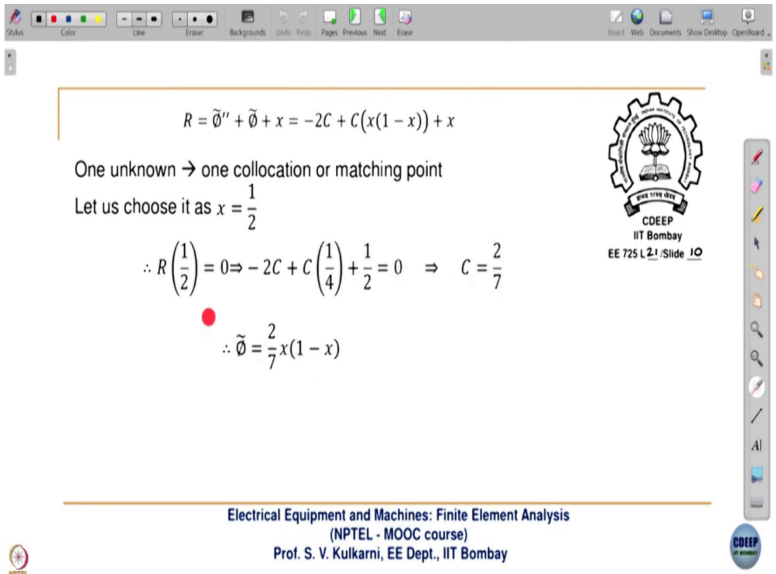

Now residue is given by $R = \tilde{\emptyset}'' + \tilde{\emptyset} + x$. If you substitute the expression of the approximate potential $\tilde{\emptyset} = C\big(x(1-x)\big)$ in the residual, then you will get the residual as $R = -2C + C\big(x(1-x)\big) + x$ which is the residue at each point because it depends upon the value of x. So the value of residue will vary in the one dimensional domain. Now in the approximate function you have only one unknown (C), so only one matching point is required.

The Dirac delta function $\big(\delta(x - x_i)\big)$ is operated only at that one matching point that you have chosen. Here, the selected matching point is 0.5 which is the midpoint of the whole domain, if we are choosing only one point.

So when you execute $\int WRd\Omega = 0$ with $W = \delta(x - 0.5)$ and if you substitute it in the above integral then you will get $R(0.5) = 0.5$. Now if you substitute $x = 0.5$ in the residual expression then it will lead to $C = 2/7$. So we will get the solution as $\tilde{\emptyset} = \frac{2}{7}x(1-x)$.

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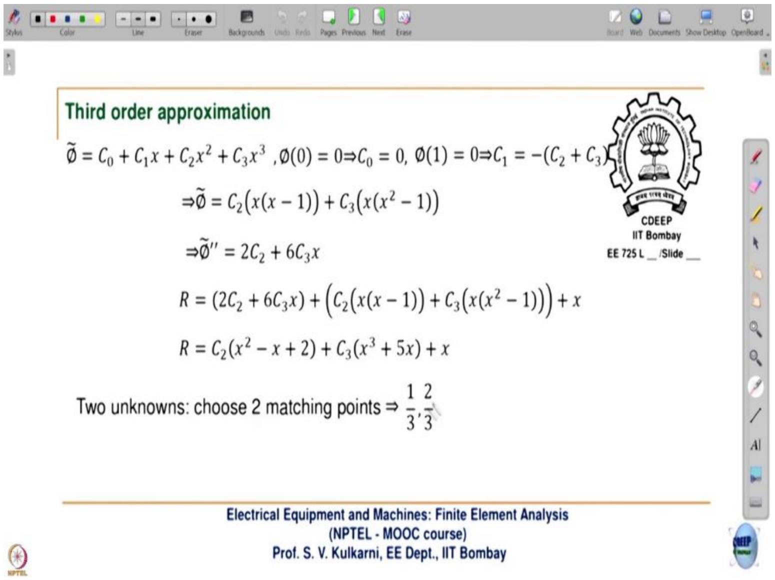

Now let us consider approximation $\tilde{\emptyset} = C_0 + C_1x + C_2x^2 + C_3x^3$ with 4 coefficients C_0 to C_3. Then after applying boundary conditions which we have seen earlier the approximate solution reduces to $\tilde{\emptyset} = C_2\big(x(x-1)\big) + C_3\big(x(x^2-1)\big)$. Then $\tilde{\emptyset}'' = 2C_2 + 6C_3x$. The residue for this approximation is given below.

$$R = C_2(x^2 - x + 2) + C_3(x^3 + 5x) + x$$

The above equation which is a function of x is the residue at every point . Now in this residual expression, there are two unknowns C_2 and C_3. So, we have to choose two matching points to get two equations. Now let us choose two matching points $x = \frac{1}{3}$ and $x = \frac{2}{3}$ which are equidistant.

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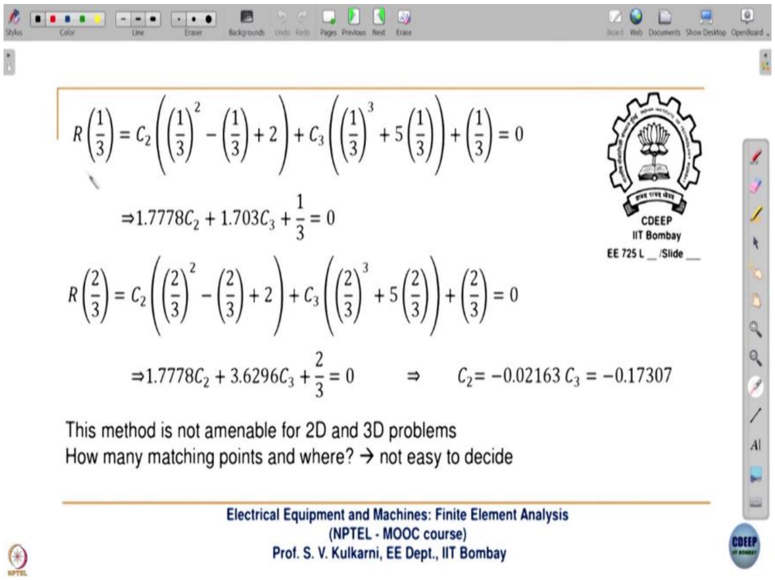

When you execute $\int WRd\Omega = 0$ for Dirac-delta function operating at $x = \frac{1}{3}$ and $x = \frac{2}{3}$, you will get the two equations with two unknowns given in the above slide. If you solve these two equations, you will get $C_2 = -0.02163$, $C_3 = -0.17307$. But this method is not amenable for 2D and 3D problems, because how many matching points that you should choose and their positions will be a matter of judgement. For 1D, it is straightforward that is why you logically chose equidistant points. But for a 2D or even more complicated problems, it will be very difficult.

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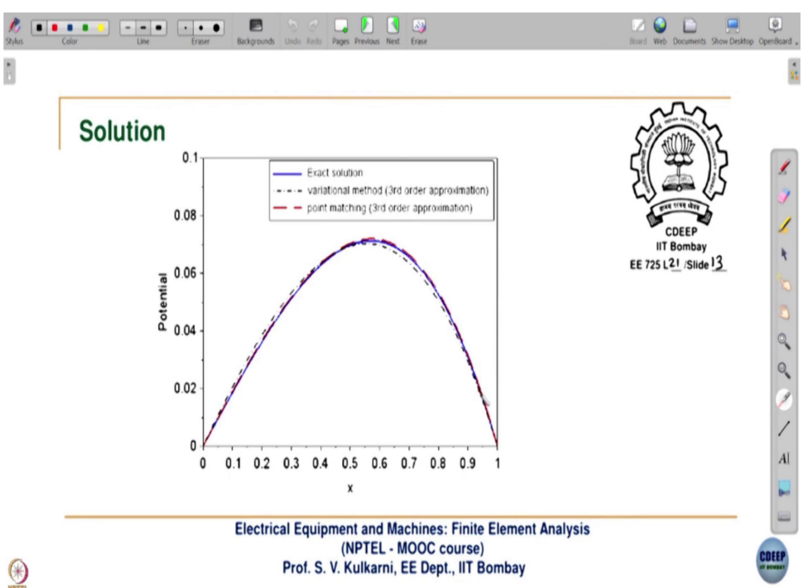

Before going further, we will see the corresponding errors and compare the errors in the solutions determined using the two approaches that we have seen till now. This slide shows the exact solution which we have been seeing and it is represented by the blue colour line. The black dashed line is the variational method with third order approximation and red line is the collocation method. From this comparison in the above slide, one can say that the results are quite close in both cases.

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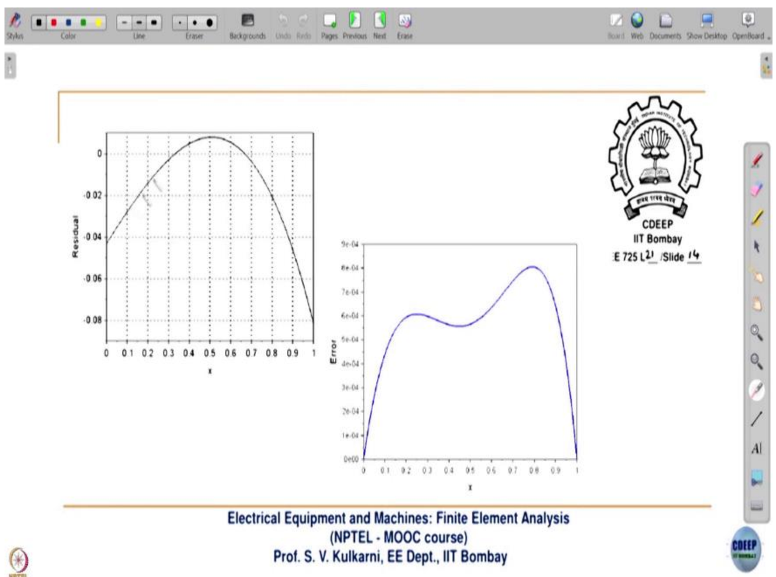

Now we will see the residual of the solution determined using the weighted residual method as shown in the figure on the left hand side. This solution is determined by using $x = 0.3333$ and $x = 0.6666$ as matching points. At these two points, the residue is coming equal to 0. But at other points, residue is not equal to 0 because we are not forcing the residue to be 0 at the other points. Residue at these two points being 0 does not mean that we will get the minimum error in potential at those two points.

The figure on the right hand side represents the variation of error in potential with x. Although you are getting lower values of errors in potential values but they are not equal to 0. So in this method what we are ensuring is the residues at the matching points are 0 but we are not ensuring that the error in potential is 0. We will stop at this point and continue our discussion in the next lecture.

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