Electrical Equipment and Machines: Finite Element Analysis
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Lecture No. 18
2D FEM: Procedure

In the previous lecture, we saw properties of shape functions and how to define potential at any point in a triangular element using the three shape functions of an element. Now, we will go further.

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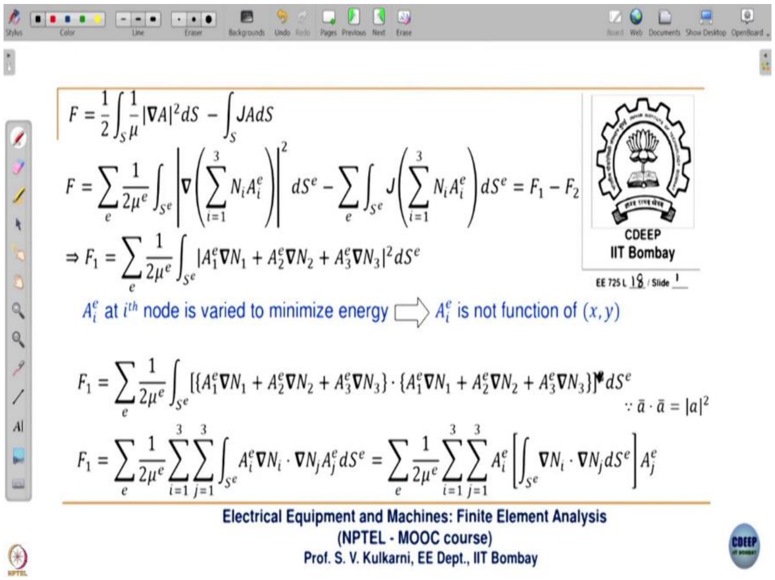

$$F = \frac{1}{2}\int_S \frac{1}{\mu}|\nabla A|^2 dS - \int_S JAdS$$

$$F = \sum_e \frac{1}{2\mu^e}\int_{S^e}\left|\nabla\left(\sum_{i=1}^{3} N_i A_i^e\right)\right|^2 dS^e - \sum_e \int_{S^e} J\left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e = F_1 - F_2$$

$$\Rightarrow F_1 = \sum_e \frac{1}{2\mu^e}\int_{S^e}|A_1^e\nabla N_1 + A_2^e\nabla N_2 + A_3^e\nabla N_3|^2 dS^e$$

A_i^e at i^{th} node is varied to minimize energy $\Rightarrow$ A_i^e is not function of (x, y)

$$F_1 = \sum_e \frac{1}{2\mu^e}\int_{S^e}[\{A_1^e\nabla N_1 + A_2^e\nabla N_2 + A_3^e\nabla N_3\}\cdot\{A_1^e\nabla N_1 + A_2^e\nabla N_2 + A_3^e\nabla N_3\}] dS^e$$

$$\because \bar{a}\cdot\bar{a} = |a|^2$$

$$F_1 = \sum_e \frac{1}{2\mu^e}\sum_{i=1}^{3}\sum_{j=1}^{3}\int_{S^e} A_i^e\nabla N_i\cdot\nabla N_j A_j^e dS^e = \sum_e \frac{1}{2\mu^e}\sum_{i=1}^{3}\sum_{j=1}^{3} A_i^e\left[\int_{S^e}\nabla N_i\cdot\nabla N_j dS^e\right]A_j^e$$

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The functional for a 2-dimensional magnetostatic problem is given below.

$$F = \frac{1}{2}\int_S \frac{1}{\mu}|\nabla A|^2 dS - \int_S JAdS$$

Here, A is the magnetic vector potential and the second integral in the above equation is from the source term (J). In this equation, μ is taken with the first integral and J is on the right-hand side of the governing PDE as shown below.

$$\frac{1}{\mu}\nabla^2 A = -J$$

That is why J gets multiplied by A in the functional expression.

Now, A in the functional expression is substituted as $\sum_{i=1}^{3} N_i A_i^e$ because A at any point within the element is $N_1A_1 + N_2A_2 + N_3A_3$. This substitution results in the following equation.

$$F = \sum_e \frac{1}{2\mu^e} \int_{S^e} \left| \nabla \left(\sum_{i=1}^{3} N_i A_i^e \right) \right|^2 dS^e - \sum_e \int_{S^e} J \left(\sum_{i=1}^{3} N_i A_i^e \right) dS^e = F_1 - F_2$$

Then, we are representing F as $F_1 - F_2$. In the above equation, ∇ operator will operate only on $N_1, N_2,$ and N_3 because they are the functions of *x* and *y* and A_is are not functions of *x* and *y* because A_i at an ith node is varied to minimize the energy in the variational procedure.

So, A_i is not a function of *x* and *y*. That is why A_1, A_2 and A_3 are taken out of the integral in the above equation. Now, the expression of F_1 can be expanded as given below.

$$F_1 = \sum_e \frac{1}{2\mu^e} \int_{S^e} |A_1^e \nabla N_1 + A_2^e \nabla N_2 + A_3^e \nabla N_3|^2 dS^e$$

Now if we go further, the whole square term in the above equation can be simplified as given below using the identity $a \cdot a = |a|^2$.

$$F_1 = \sum_e \frac{1}{2\mu^e} \int_{S^e} [\{A_1^e \nabla N_1 + A_2^e \nabla N_2 + A_3^e \nabla N_3\} \cdot \{A_1^e \nabla N_1 + A_2^e \nabla N_2 + A_3^e \nabla N_3\}]\, dS^e$$

So that is why the square of the whole integrand in the previous equation can be written as the above expression using a dot product.

When you expand the dot product in the above equation, you will get 9 terms. These 9 terms can be written using 2 summations as given in the following equation.

$$F_1 = \sum_e \frac{1}{2\mu^e} \sum_{i=1}^{3} \sum_{j=1}^{3} \int_{S^e} A_i^e \nabla N_i \cdot \nabla N_j A_j^e dS^e = \sum_e \frac{1}{2\mu^e} \sum_{i=1}^{3} \sum_{j=1}^{3} A_i^e \left[\int_{S^e} \nabla N_i \cdot \nabla N_j dS^e \right] A_j^e$$

We are taking A_i and A_j outside the integral as given in the above equation because they are independent of *x* and *y*.

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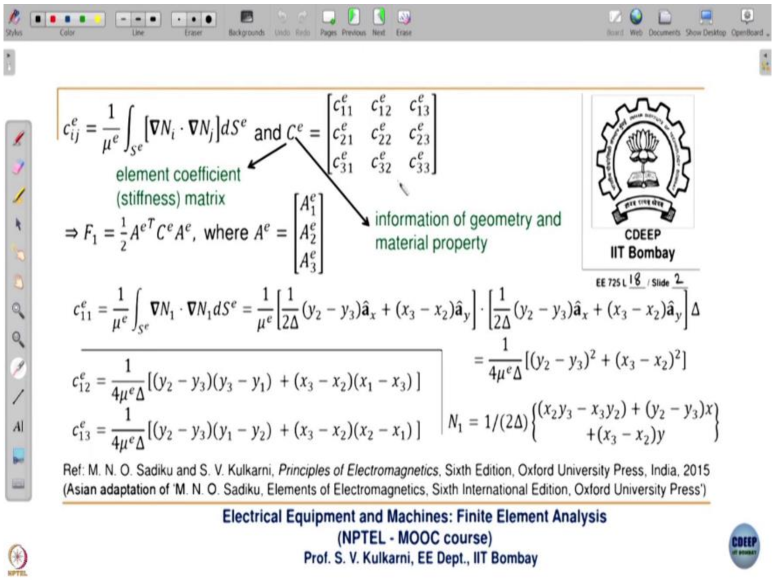

$$c_{ij}^e = \frac{1}{\mu^e}\int_{S^e}[\nabla N_i \cdot \nabla N_j]dS^e \text{ and } C^e = \begin{bmatrix} c_{11}^e & c_{12}^e & c_{13}^e \\ c_{21}^e & c_{22}^e & c_{23}^e \\ c_{31}^e & c_{32}^e & c_{33}^e \end{bmatrix}$$

element coefficient (stiffness) matrix

information of geometry and material property

$$\Rightarrow F_1 = \frac{1}{2}A^{e^T}C^eA^e, \text{ where } A^e = \begin{bmatrix} A_1^e \\ A_2^e \\ A_3^e \end{bmatrix}$$

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$$c_{11}^e = \frac{1}{\mu^e}\int_{S^e}\nabla N_1 \cdot \nabla N_1 dS^e = \frac{1}{\mu^e}\left[\frac{1}{2\Delta}(y_2 - y_3)\hat{\mathbf{a}}_x + (x_3 - x_2)\hat{\mathbf{a}}_y\right] \cdot \left[\frac{1}{2\Delta}(y_2 - y_3)\hat{\mathbf{a}}_x + (x_3 - x_2)\hat{\mathbf{a}}_y\right]\Delta$$

$$= \frac{1}{4\mu^e\Delta}[(y_2 - y_3)^2 + (x_3 - x_2)^2]$$

$$c_{12}^e = \frac{1}{4\mu^e\Delta}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)]$$

$$c_{13}^e = \frac{1}{4\mu^e\Delta}[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)]$$

$$N_1 = 1/(2\Delta)\begin{Bmatrix} (x_2y_3 - x_3y_2) + (y_2 - y_3)x \\ +(x_3 - x_2)y \end{Bmatrix}$$

Ref: M. N. O. Sadiku and S. V. Kulkarni, *Principles of Electromagnetics*, Sixth Edition, Oxford University Press, India, 2015
(Asian adaptation of 'M. N. O. Sadiku, Elements of Electromagnetics, Sixth International Edition, Oxford University Press')

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The above summation can be written in a 3 × 3 matrix as given below,

$$\Rightarrow F_1 = \frac{1}{2}A^{e^T}C^eA^e, \text{ where } A^e = \begin{bmatrix} A_1^e \\ A_2^e \\ A_3^e \end{bmatrix}$$

C^e is the elemental coefficient matrix whose entries are given by the following equation.

$$c_{ij}^e = \frac{1}{\mu^e}\int_{S^e}[\nabla N_i \cdot \nabla N_j]dS^e \text{ and } C^e = \begin{bmatrix} c_{11}^e & c_{12}^e & c_{13}^e \\ c_{21}^e & c_{22}^e & c_{23}^e \\ c_{31}^e & c_{32}^e & c_{33}^e \end{bmatrix}$$

Now, we have simplified the integral expression of F_1 as a matrix form. The bracketed term involving integral is called as c_{ij} which is expressed as the integral in the above equation.

Also, remember that we can assume μ^e as constant over the element area. So, that is why μ^e is coming outside the integral because it is constant over the element.

Now, we will see how do we calculate c_{ij}. Remember that the C^e matrix is called as the element coefficient matrix and it has information about the geometry and the material properties.

This is because, in $\nabla N_i \cdot \nabla N_j$, N_i and N_j are functions of *x* and *y* and it has information about all the coordinates of the 3 vertices. Also in general, they are functions of *x* and *y* (any arbitrary point within the element under consideration). So, it has information about the geometry and material properties which will determine the energy stored in that element. By combining all such element coefficient matrices we will eventually form the global coefficient matrix.

Now, c_{11} is calculated by substituting $i = j = 1$ and it is given in the following equation.

$$c_{11}^{e} = \frac{1}{\mu^{e}}\int_{S^{e}} \nabla N_1 \cdot \nabla N_1 dS^{e} = \frac{1}{\mu^{e}}\left[\frac{1}{2\Delta}(y_2 - y_3)\hat{\mathbf{a}}_x + (x_3 - x_2)\hat{\mathbf{a}}_y\right] \cdot \left[\frac{1}{2\Delta}(y_2 - y_3)\hat{\mathbf{a}}_x + (x_3 - x_2)\hat{\mathbf{a}}_y\right]\Delta$$

$$= \frac{1}{4\mu^{e}\Delta}[(y_2 - y_3)^2 + (x_3 - x_2)^2]$$

Because the expression of N_1 is given by the equation $N_1 = \frac{1}{2\Delta}[(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y]$. So, ∇N_1 will be given by $\frac{1}{2\Delta}(y_2 - y_3)\hat{\mathbf{a}}_x + (x_3 - x_2)\hat{\mathbf{a}}_y$ and this is anyway constant.

Now, $\nabla N_1 \cdot \nabla N_1$ will be simplified as given in the above equation. Since the integrand of the above integral is constant and integral dS will result in Δ. Similarly, you can calculate c_{12} and its expression is given below.

$$c_{12}^{e} = \frac{1}{4\mu^{e}\Delta}[(y_2 - y_3)(y_3 - y_1) + (x_3 - x_2)(x_1 - x_3)]$$

In the integral, there will be two terms (∇N_1 and ∇N_2) because i and j are not equal. Then corresponding expressions of N_2 and N_3 are given in the previous lecture. Similarly we can determine the expressions of C_{13} as given below. By following the same method, you can find expressions for all other remaining entries of C^{e}matrix.

$$c_{13}^{e} = \frac{1}{4\mu^{e}\Delta}[(y_2 - y_3)(y_1 - y_2) + (x_3 - x_2)(x_2 - x_1)]$$

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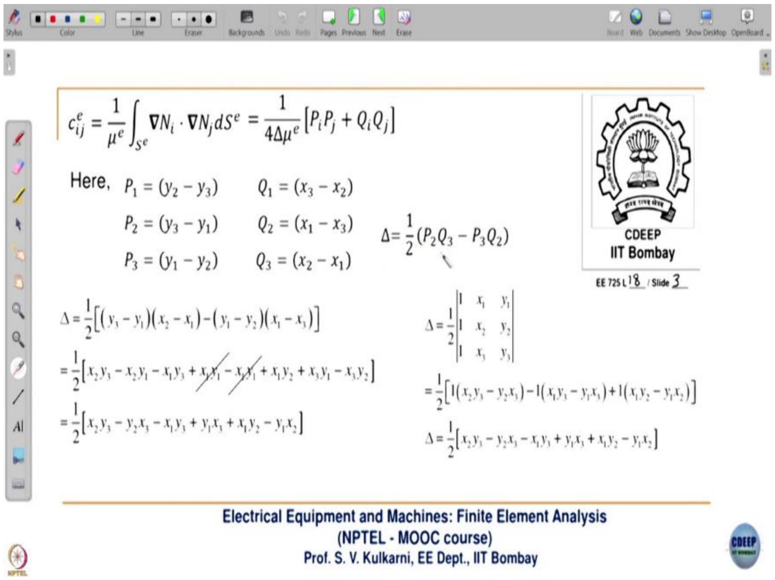

$$c_{ij}^{e} = \frac{1}{\mu^{e}}\int_{S^{e}} \nabla N_i \cdot \nabla N_j dS^{e} = \frac{1}{4\Delta\mu^{e}}\left[P_iP_j + Q_iQ_j\right]$$

Here, $P_1 = (y_2 - y_3)$ $Q_1 = (x_3 - x_2)$

$P_2 = (y_3 - y_1)$ $Q_2 = (x_1 - x_3)$ $\Delta = \frac{1}{2}(P_2Q_3 - P_3Q_2)$

$P_3 = (y_1 - y_2)$ $Q_3 = (x_2 - x_1)$

$$\Delta = \frac{1}{2}\left[(y_3 - y_1)(x_2 - x_1) - (y_1 - y_2)(x_1 - x_3)\right]$$
$$= \frac{1}{2}\left[x_2y_3 - x_2y_1 - x_1y_3 + \cancel{x_1y_1} - \cancel{x_1y_1} + x_1y_2 + x_3y_1 - x_3y_2\right]$$
$$= \frac{1}{2}\left[x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2\right]$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
$$= \frac{1}{2}\left[1(x_2y_3 - y_2x_3) - 1(x_1y_3 - y_1x_3) + 1(x_1y_2 - y_1x_2)\right]$$
$$\Delta = \frac{1}{2}\left[x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2\right]$$

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In general, we can write the expression of c_{ij} as given below.

$$c_{ij}^{e} = \frac{1}{\mu^{e}}\int_{S^{e}} \nabla N_i \cdot \nabla N_j dS^{e} = \frac{1}{4\Delta\mu^{e}}\left[P_iP_j + Q_iQ_j\right]$$

The expressions of P_is and Q_is are given below.

Here, $P_1 = (y_2 - y_3)$ $Q_1 = (x_3 - x_2)$

$P_2 = (y_3 - y_1)$ $Q_2 = (x_1 - x_3)$

$P_3 = (y_1 - y_2)$ $Q_3 = (x_2 - x_1)$

Then the area of the triangle will be given by $\Delta = \frac{1}{2}(P_2Q_3 - P_3Q_2)$. This can be verified by using the derivations given below.

$$\Delta = \frac{1}{2}(P_2Q_3 - P_3Q_2)$$
$$\Delta = \frac{1}{2}\left[(y_3 - y_1)(x_2 - x_1) - (y_1 - y_2)(x_1 - x_3)\right]$$
$$= \frac{1}{2}\left[x_2y_3 - x_2y_1 - x_1y_3 + \cancel{x_1y_1} - \cancel{x_1y_1} + x_1y_2 + x_3y_1 - x_3y_2\right]$$
$$= \frac{1}{2}\left[x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2\right]$$

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$
$$= \frac{1}{2}\left[1(x_2y_3 - y_2x_3) - 1(x_1y_3 - y_1x_3) + 1(x_1y_2 - y_1x_2)\right]$$
$$\Delta = \frac{1}{2}\left[x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2\right]$$

We know that the area of a triangle can be calculated by using the determinant on the right hand side. If you expand this determinant, you will get the area as $\Delta = \frac{1}{2}[x_2y_3 - y_2x_3 - x_1y_3 + y_1x_3 + x_1y_2 - y_1x_2]$. If you substitute P_is and Q_is in the Δ expression given on the right hand side, we will again get the same expression as shown in the above derivation.

We will be using the expression of c_{ij} and area of the triangle in terms of P_is and Q_is often in this course. Now, we will see the matrix representation of F_2 which is related to the source current density J. The integral representation of F_2 which is applicable for the whole domain is given below.

$$F_2 = \int_S JAdS$$

Then we go to discretized domain where magnetic vector potential is approximated as

$$A^e = \sum_{i=1}^{3} N_i(x,y)A_i^e \Rightarrow A^e = N_1A_1^e + N_2A_2^e + N_3A_3^e$$

That means from whole domain functional we went to element level representation. So the second term which is representing the source is converted from the whole domain to the element level.

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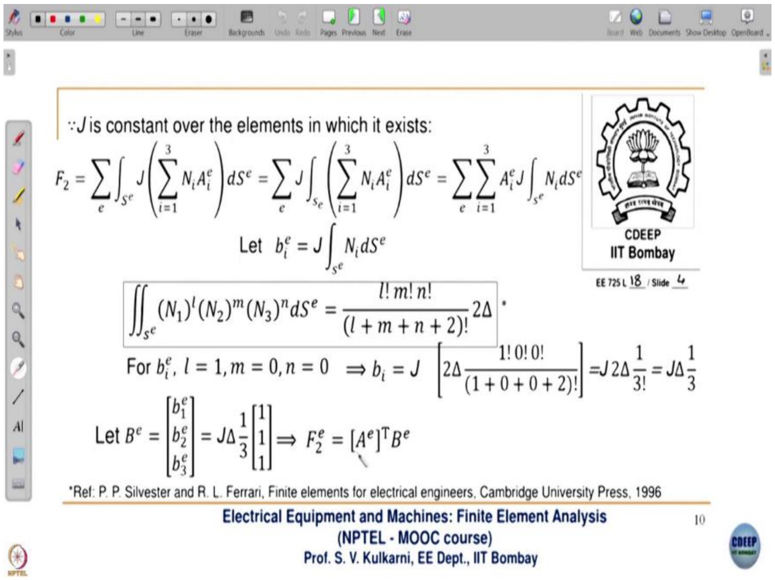

$\because J$ is constant over the elements in which it exists:

$$F_2 = \sum_e \int_{S^e} J\left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e = \sum_e J \int_{S_e} \left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e = \sum_e \sum_{i=1}^{3} A_i^e J \int_{S^e} N_i dS^e$$

$$\text{Let } b_i^e = J \int_{S^e} N_i dS^e$$

$$\boxed{\iint_{S^e} (N_1)^l (N_2)^m (N_3)^n dS^e = \frac{l!\,m!\,n!}{(l+m+n+2)!} 2\Delta}\,^*$$

$$\text{For } b_i^e,\ l = 1, m = 0, n = 0 \ \Rightarrow b_i = J \left[2\Delta \frac{1!\,0!\,0!}{(1+0+0+2)!}\right] = J\,2\Delta\frac{1}{3!} = J\Delta\frac{1}{3}$$

$$\text{Let } B^e = \begin{bmatrix} b_1^e \\ b_2^e \\ b_3^e \end{bmatrix} = J\Delta\frac{1}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow F_2^e = [A^e]^{\mathrm{T}} B^e$$

*Ref: P. P. Silvester and R. L. Ferrari, Finite elements for electrical engineers, Cambridge University Press, 1996

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The expression of F_2 in the discretized domain can be written as given below.

$$F_2 = \sum_e \int_{S^e} J\left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e$$

Going further, J can be brought outside the integral because in most of the problems in magnetic field calculations, J over individual element is considered as constant. In most of the examples we have typically a conductor or winding for which the input current density is constant. In that case, J becomes independent of x and y.

Now, we are just rearranging the above expression by interchanging the summation and the integral operator because the F_2 is a scalar variable or a scalar number which represents energy. Whether you first integrate and then take summation or otherwise it will not matter. So by interchanging the integral and summation operators, we will simplify F_2 as given below.

$$F_2 = \sum_e \int_{S^e} J\left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e = \sum_e J \int_{S_e} \left(\sum_{i=1}^{3} N_i A_i^e\right) dS^e = \sum_e \sum_{i=1}^{3} A_i^e J \int_{S^e} N_i dS^e$$

Let $b_i^e = J\int_{S^e} N_i dS^e$. Now we will use the following expression whose derivation is fairly complicated.

$$\iint_{S^e} (N_1)^l (N_2)^m (N_3)^n dS = \frac{l!\, m!\, n!}{(l+m+n+2)!} 2\Delta \;{}^*$$

So, we will not get into the derivation of the above formula. But we will show a proof of this by applying it to a simple case in the next slide. Remember N_is in the above expressions are functions of x and y. The above expression is a general form. When you have $(N_i)^1$ as for the case of b_i^e, N_2 and N_3 are not there.

But in some other cases, in general you will have N_1, N_2 and N_3 raised to some constant and in that case the above formula reduces to this after some complicated expression. Δ in the above equation is the area of the triangle.

In the next slide, we will see one example for this. Now in the case of $b_i^e, l = 1, m = n = 0$ because $b_i^e = J\int_{S^e} N_i dS^e$. By substituting the values of l, m, n in the above formula you get the value of b_i^e as $\frac{J\Delta}{3}$ after substitution and other simplifications.

Similarly, you will get the values of b_1^e, b_2^e and b_3^e as $\frac{J\Delta}{3}$**.** Then you will get the element level source matrix as given below.

$$B^e = \begin{bmatrix} b_1^e \\ b_2^e \\ b_3^e \end{bmatrix} = J\Delta \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

So, B^e is the element level contribution by the source J. Actually, J is distributed over the entire elemental area but in the discretized domain, it is apportioned equally to the 3 nodes. So, J which was distributed over the entire elemental area is apportioned equally as $\frac{J\Delta}{3}$ at the nodal vertices in the discretized domain. Then F_2^e, the energy related to the element due to the source term can be written in matrix form as given below. Now, F_2 has to be added to F_1 to calculate the total energy.

$$F_2^e = [A^e]^{\mathrm{T}} B^e$$

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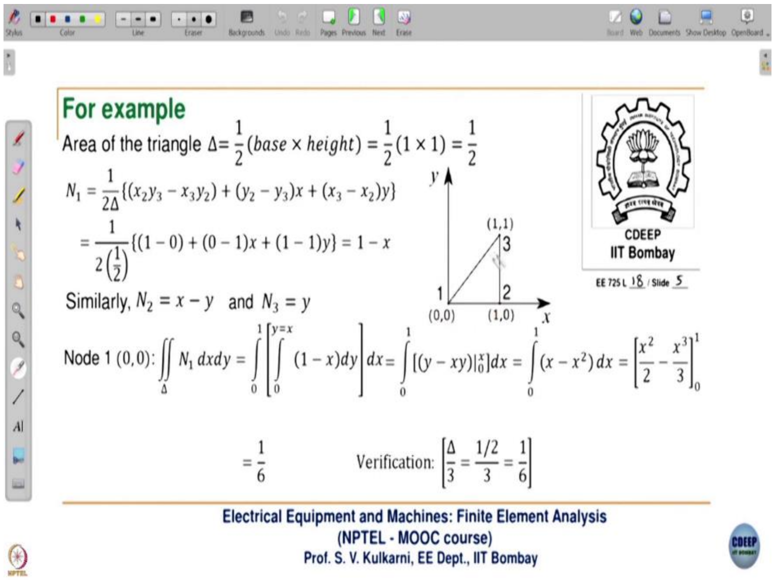

For example

Area of the triangle $\Delta = \frac{1}{2}(base \times height) = \frac{1}{2}(1 \times 1) = \frac{1}{2}$

$$N_1 = \frac{1}{2\Delta}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$

$$= \frac{1}{2\left(\frac{1}{2}\right)}\{(1-0) + (0-1)x + (1-1)y\} = 1 - x$$

Similarly, $N_2 = x - y$ and $N_3 = y$

Node 1 (0,0): $$\iint_\Delta N_1\,dxdy = \int_0^1\left[\int_0^{y=x}(1-x)dy\right]dx = \int_0^1 [(y - xy)|_0^x]dx = \int_0^1 (x - x^2)\,dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{6}$$

Verification: $\left[\frac{\Delta}{3} = \frac{1/2}{3} = \frac{1}{6}\right]$

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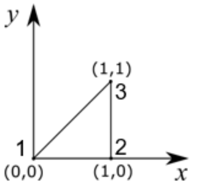
Now, let us consider a basic example to verify this formula. Let us take a triangular element as shown in the following figure.



We have purposely taken a right-angled triangle so that we can get a simplified integral. The vertices of this triangle are $(0,0)$, $(1,0)$, and $(1,1)$. So, the area of the triangle calculated using $\frac{1}{2}(base \times height)$ is simply $\frac{1}{2}$.

We know the expression for N_1 as $N_1 = \frac{1}{2\Delta}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$. Now, you substitute the values of various coordinates in N_1 and then you will get the N_1 expression as $1 - x$. Then you can verify this expression by substituting the coordinates of node 1, $x = 0$ which gives the value of N_1 as 1, which should be the case as per the properties of shape functions. At the other 2 nodes (2 and 3), we will get the value of N_1 as 0. At nodes 2 and 3, $x = 1$. So, $N_1 = 1 - 1 = 0$. So, at the other two nodes, N_1 goes to 0. This is the basic property of shape functions and you can also easily verify these properties.

Similarly, the expressions for N_2 and N_3 for this element can be derived. The expressions of the two shape functions are $N_2 = x - y$ and $N_3 = y$. You can easily verify these expressions.

For node 1, we are trying to evaluate $\int_{s^e} N_i dS^e$. Now we evaluate the integral over the element as given below.

$$\iint_{\Delta} N_1\, dxdy = \int_0^1 \left[\int_0^{y=x} (1-x)dy\right] dx$$

For x, we are integrating from 0 to 1 because x varies from 0 to 1 as shown in the above figure and y varies from 0 to whatever is the y on the hypotenuse line joining the nodes at (0,0) and (1,1).

So, the limits of y is from 0 to $y = x$. Because the equation of the line is $y = x$. So, the limits of our integration is 0 to 1 for x and 0 to $y = x$ for y. The evaluation of the integral is given below.

$$\iint_{\Delta} N_1\, dxdy = \int_0^1 \left[\int_0^{y=x} (1-x)dy\right] dx = \int_0^1 [(y - xy)|_0^x] dx = \int_0^1 (x - x^2)\, dx = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \frac{1}{6}$$

You finally get the value of the integral as $\frac{1}{6} = \frac{1/2}{3} = \frac{\Delta}{3}$. The area of triangle (Δ) is half. So, we verified the formula $\frac{\Delta}{3}$ and it worked for this right-angled triangle. If it can work for this right angled triangle, it will work for any arbitrary triangle also.

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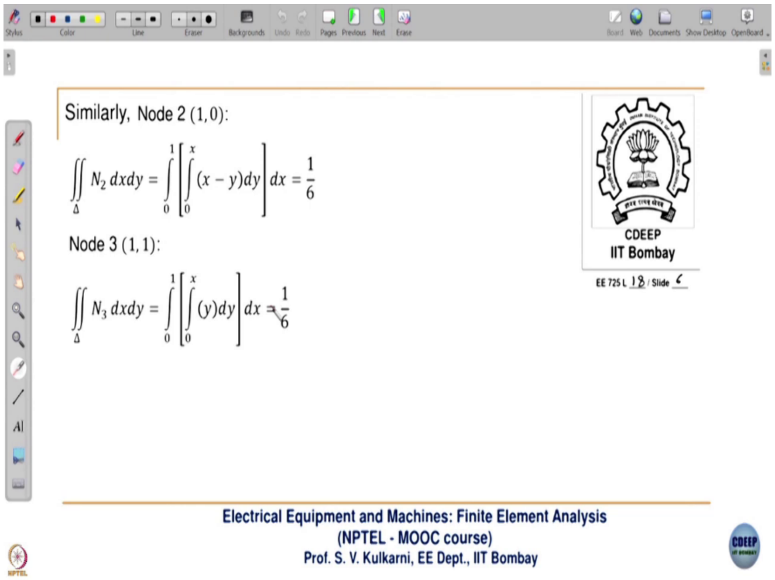

Similarly, Node 2 $(1,0)$:

$$\iint_{\Delta} N_2\,dxdy = \int_0^1 \left[\int_0^x (x-y)dy\right] dx = \frac{1}{6}$$

Node 3 $(1,1)$:

$$\iint_{\Delta} N_3\,dxdy = \int_0^1 \left[\int_0^x (y)dy\right] dx = \frac{1}{6}$$

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Similarly you can verify for node 2. We already got the expressions for $N_2(= x - y)$ and $N_3(= y)$ and then if you do the integration $\iint_{\Delta} N_2\,dxdy$ and $\iint_{\Delta} N_3\,dxdy$ you will get $\frac{1}{6}$.

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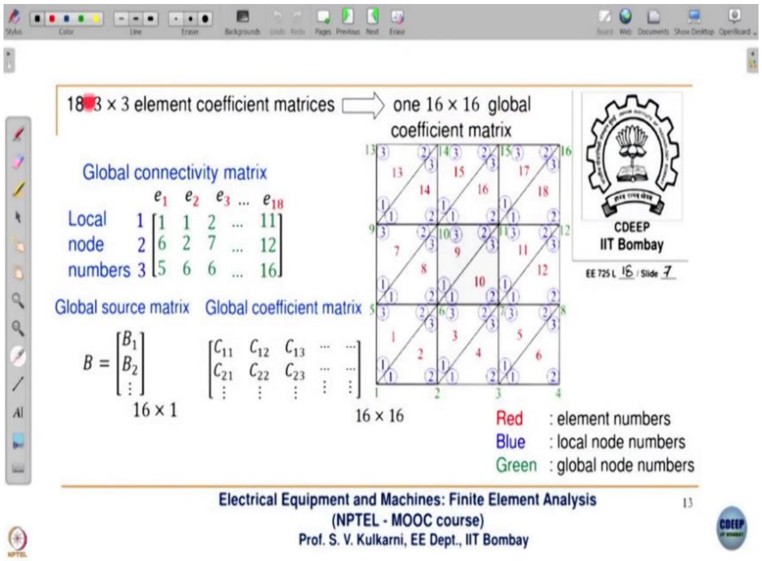

18 3 × 3 element coefficient matrices $\Rightarrow$ one 16 × 16 global coefficient matrix

Global connectivity matrix

$$\begin{array}{ll} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \dots & e_{18} \end{array} \\ \begin{array}{r} \text{Local} \quad 1 \\ \text{node} \quad 2 \\ \text{numbers} \quad 3 \end{array} & \begin{bmatrix} 1 & 1 & 2 & \dots & 11 \\ 6 & 2 & 7 & \dots & 12 \\ 5 & 6 & 6 & \dots & 16 \end{bmatrix} \end{array}$$

Global source matrix

$$B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix}$$

16 × 1

Global coefficient matrix

$$\begin{bmatrix} C_{11} & C_{12} & C_{13} & \cdots & \cdots \\ C_{21} & C_{22} & C_{23} & \cdots & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

16 × 16

Red : element numbers
Blue : local node numbers
Green : global node numbers

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We have derived the entries for element level coefficient matrices and source matrices. How many element-level matrices are there in this example? 18, because there are 18 elements as shown in the figure on the above slide. Remember in this geometry there is a rectangular conductor at the center of the domain. Now in this example there are 18 elements which are indicated with numbers written in red colour. Numbers written in blue colour (1, 2, 3) are the local node numbers and numbers in green colour are the global node numbers (1, 2, 3, 4, ………, 15, 16). Using 18, 3×3 element coefficient matrices which we have derived in the previous slide, we have to form one 16×16 global coefficient matrix.

The dimensions of the global coefficient matrix is 16×16 because there are 16 global nodes and eventually there are 16 potential variables with respect to which we have to minimize the energy. Out of these 16 nodes, potentials at some of the nodes are known. For example, here, we are going to impose A = 0 on the whole outer boundary. So, the potentials of the boundary nodes are known.

Eventually, we will impose the boundary condition A = 0. So, there are 16 nodes in our geometry and the matrix size will be 16×16. As we saw in 1D example, here also we will have a connectivity matrix. So, here we will have the connectivity matrix which is given below.

Global connectivity matrix

$$\begin{array}{cc} & \begin{array}{ccccc} e_1 & e_2 & e_3 & \dots & e_{18} \end{array} \\ \text{Local node numbers} \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 1 & 2 & \dots & 11 \\ 6 & 2 & 7 & \dots & 12 \\ 5 & 6 & 6 & \dots & 16 \end{bmatrix} \end{array}$$

In the above matrix, we have e_1 to e_{18} (18 elements) and the corresponding global node numbers are given in each column of the matrix. The global node numbers for element number 1 are 1, 6, 5. For element number 2, global node numbers are 1, 2, 6.

Similarly, you can do for the rest of the elements. The global source matrix with entries B_1 to B_{16} is given below.

$$B = \underset{16 \times 1}{\begin{bmatrix} B_1 \\ B_2 \\ \vdots \end{bmatrix}}$$

We have evaluated the element level contributions (b_1^e, b_2^e, b_3^e for each element) in the previous slide. Using element level source matrices, we have to form the global source matrix given in the above equation. How do we do that? We will see in the further slides.

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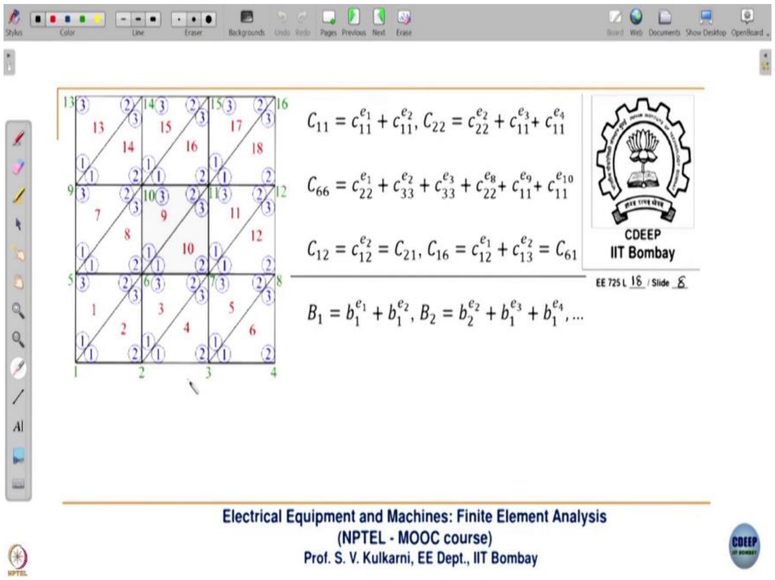

$C_{11} = c_{11}^{e_1} + c_{11}^{e_2}, C_{22} = c_{22}^{e_2} + c_{11}^{e_3} + c_{11}^{e_4}$

$C_{66} = c_{22}^{e_1} + c_{33}^{e_2} + c_{33}^{e_3} + c_{22}^{e_8} + c_{11}^{e_9} + c_{11}^{e_{10}}$

$C_{12} = c_{12}^{e_2} = C_{21}, C_{16} = c_{12}^{e_1} + c_{13}^{e_2} = C_{61}$

$B_1 = b_1^{e_1} + b_1^{e_2}, B_2 = b_2^{e_2} + b_1^{e_3} + b_1^{e_4}, \ldots$

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In 1D code, we saw how do we combine various element coefficient matrices to calculate the entries of the global coefficient matrix. We will do the same thing in this 2D example also. We will see how to find out $C(1,1)$ entry of the global coefficient matrix. In this discretization, 2 triangles are incident on global node number 1. Because this node is a vertex of element number 1 and element number 2. So $C_{11} = c_{11}^{e_1} + c_{11}^{e_2}$. Effectively, the potential at node number 1 contributes in deciding the energy for both elements 1 and 2.

Similarly, $C_{22} = c_{22}^{e_2} + c_{11}^{e_3} + c_{11}^{e_4}$ because node 2 is common to elements 2, 3, 4 and will have corresponding 3 contributions. The global node number 6 $\left(C_{66} = c_{22}^{e_1} + c_{33}^{e_2} + c_{33}^{e_3} + c_{22}^{e_8} + c_{11}^{e_9} + c_{11}^{e_{10}}\right)$ has contributions from 6 elements. Because it is common to 6 elements. Also, you have to remember that some diagonal entries will have addition of multiple terms and some off diagonal entries will have addition of 2 terms or contributions from two elements.

For example, $C_{16} = c_{12}^{e_1} + c_{13}^{e_2} = C_{61}$. In the case of 1D, off-diagonal elements contributions from different elements was not there because in 1-dimensional discretization only nodes were common between two adjacent elements. But in 2D along with nodes, edges are also common to 2 elements.

Also, note that $C_{13} = 0$ because there is no direct connection between nodes 1 and 3. Similarly, for the global source matrix $B_1 = b_1^{e_1} + b_1^{e_2}$. We have already calculated the element level source matrices in the previous slide. Similarly, $B_2 (= b_2^{e_2} + b_1^{e_3} + b_1^{e_4})$ will have 3 contributions from 3 elements and so on. So at the end of this step, you would have got one global coefficient matrix of size 16×16 and one global source matrix of size 16×1, by combining all element coefficient matrices and element level source matrices.

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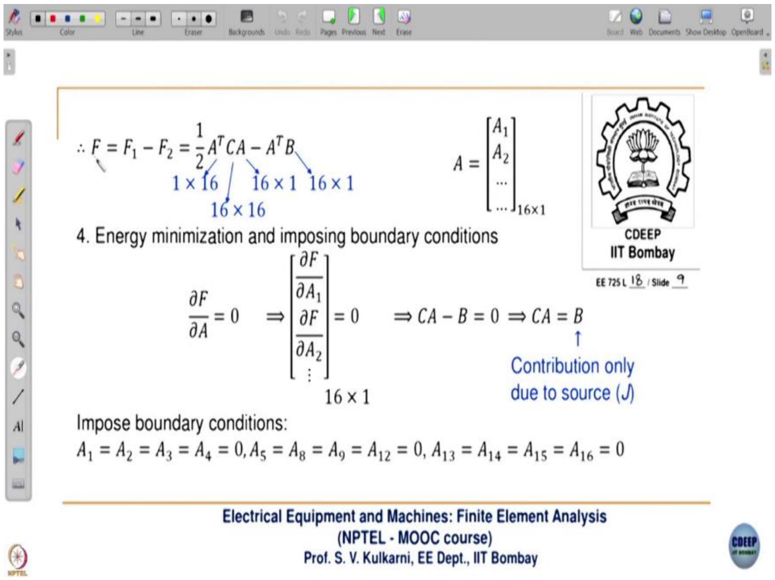

$$\therefore F = F_1 - F_2 = \frac{1}{2}\underset{1\times 16}{A^T}\,\underset{16\times 16}{C}\,\underset{16\times 1}{A} - A^T \underset{16\times 1}{B} \qquad A = \begin{bmatrix} A_1 \\ A_2 \\ \cdots \\ \cdots \end{bmatrix}_{16\times 1}$$

4. Energy minimization and imposing boundary conditions

$$\frac{\partial F}{\partial A} = 0 \quad \Rightarrow \underset{16\times 1}{\begin{bmatrix} \frac{\partial F}{\partial A_1} \\ \frac{\partial F}{\partial A_2} \\ \vdots \end{bmatrix}} = 0 \quad \Rightarrow CA - B = 0 \Rightarrow CA = \underset{\substack{\uparrow \\ \text{Contribution only} \\ \text{due to source } (J)}}{B}$$

Impose boundary conditions:

$A_1 = A_2 = A_3 = A_4 = 0, A_5 = A_8 = A_9 = A_{12} = 0, A_{13} = A_{14} = A_{15} = A_{16} = 0$

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Now, you have the total energy $F = F_1 - F_2$ written in matrix form as

$$F = F_1 - F_2 = \frac{1}{2}\underset{1\times 16}{A^T}\,\underset{16\times 16}{C}\,\underset{16\times 1}{A} - A^T \underset{16\times 1}{B}$$

The dimensions of all the matrices are also given in the above equation. Remember, always all these A and B matrices are column vectors and A^T will become a row vector. So, the size of A^T will be 1×16, C will be 16×16 and A will be 16×1. The whole product will be 1×1.

Similarly, in the second term, the size of A^T is 1×16 and B is 16×1. So again, the size of the product $A^T B$ will be 1×1, it should be, because this F is the energy of the whole domain and it is a scalar. So, it will be 1×1. Then the fourth step of the whole FEM procedure is energy minimization and then imposing the boundary conditions.

To minimize the energy (F) with respect to the potential variables, we have to evaluate the following equations.

$$\frac{\partial F}{\partial A} = 0 \quad \Longrightarrow \begin{bmatrix} \frac{\partial F}{\partial A_1} \\ \frac{\partial F}{\partial A_2} \\ \vdots \end{bmatrix}_{16 \times 1} = 0$$

Now this matrix A is a 16×1 column vector. That means minimization of energy with respect to 16 potentials amounts to 16 equations. Those 16 equations are written as given in the above equation. So, when you differentiate F by individual A_is, you will get the matrix equation of the form $CA - B = 0 \implies CA = B$. But here the matrix B has contribution due to source only.

Now there is an additional contribution due to boundary conditions. We already discussed the point that the right hand side matrix B will have contributions from two things, one is the source and the other is the boundary condition. So now, we will impose the boundary conditions and modify the system of equations. The boundary conditions have to be imposed to all the nodes on the outermost rectangle. All the potentials on the outermost boundary are equal to 0. This we have seen earlier. Here we are taking A = 0, as you know, instead of 0, you could take something else also, but the answer (field distribution) will not change because flux is $\oint A \cdot dl$ and for a 2D case, it will be simply the difference of potentials at 2 points. So only the difference matters. If you scale the boundary condition to some other value, the difference between any two potentials is going to remain the same.

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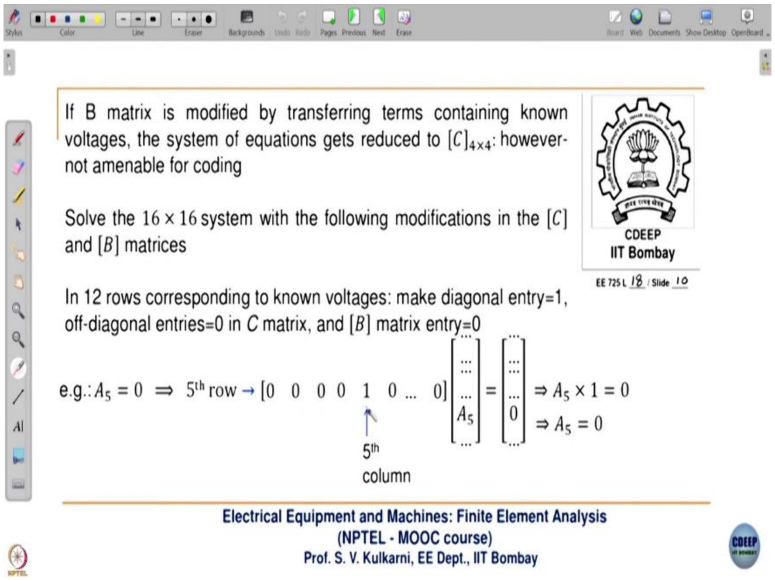

If B matrix is modified by transferring terms containing known voltages, the system of equations gets reduced to $[C]_{4\times4}$: however-not amenable for coding

Solve the 16×16 system with the following modifications in the $[C]$ and $[B]$ matrices

In 12 rows corresponding to known voltages: make diagonal entry=1, off-diagonal entries=0 in C matrix, and $[B]$ matrix entry=0

$$\text{e.g.: } A_5 = 0 \Rightarrow 5^{th}\text{ row} \rightarrow [0 \quad 0 \quad 0 \quad 0 \quad \underset{\substack{\uparrow \\ 5^{th} \\ \text{column}}}{1} \quad 0 \; \dots \; 0]\begin{bmatrix} \vdots \\ \vdots \\ \dots \\ A_5 \\ \dots \end{bmatrix} = \begin{bmatrix} \vdots \\ \vdots \\ \dots \\ 0 \\ \dots \end{bmatrix} \Rightarrow A_5 \times 1 = 0 \quad \Rightarrow A_5 = 0$$

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Boundary conditions can be imposed by transferring terms containing known voltages to the right hand side matrix and modifying B matrix and the system of equations get reduced to 4×4. But, it is not amenable for coding. In the previous code, Mr. Sairam has already explained you both ways to apply boundary conditions like reducing the system to 4×4 and how we can continue with the original matrix equations. So, both things were explained with respect to the 1D code.

So, in this example, we will continue to solve the 16×16 system because that is easy from the point of coding rather than reducing it to 4×4. Because if you want to reduce the matrix, we have to do some row operations like adding or subtracting some row from the other and then simplify it. So, this procedure involves a lot of extra operations. Instead of doing that you can operate the whole 16×16 matrix, although it looks like a bigger matrix.

Now, for example, if you want to impose the condition $A_5 = 0$ where node 5 is on the outermost boundary. How will we do that? For the fifth row of $CA = B$, you make all the off-diagonal entries as 0 and make $C(5,5)$ as 1 and then make the corresponding fifth entry of B matrix as 0.

After imposing boundary conditions, if you expand $CA = B$, you will get $A_5 = 0$. So, effectively you have imposed boundary condition $A_5 = 0$. Now on the 12 boundary nodes, you impose the same condition. Here, there are 12 boundary nodes. So, 12 rows of the C matrix will get modified as explained for the case of node 5. So, in 12 rows you will make all off diagonal entries as 0 and the corresponding diagonal entry as 1 and make the corresponding B entry as 0. Then you invert the matrix C and calculate $A = C\backslash B$. Finally, by taking C^{-1} and multiplying it by B, you will get A.

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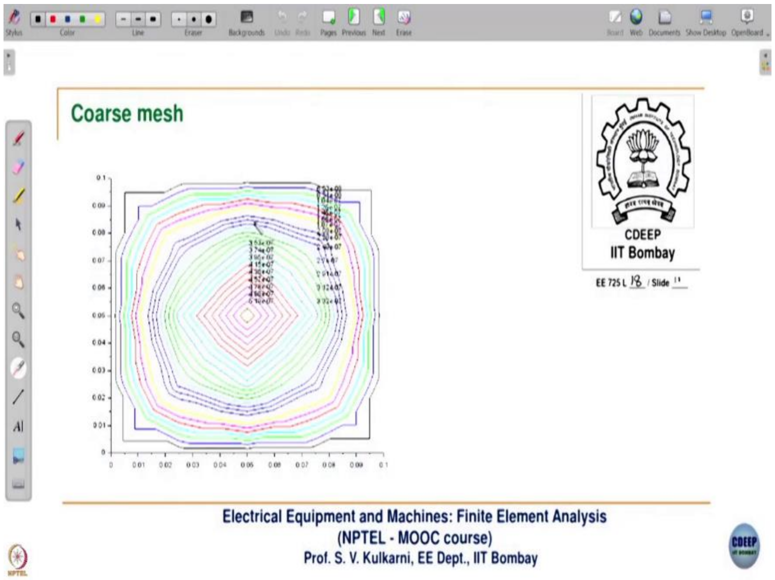

Coarse mesh

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Then we will get the solution (nodal potentials). The solution shown in the above slide is for coarse mesh. In this figure, you can see the field lines are not smooth. In one of the very first lectures, it was mentioned that by just looking at the field distribution, you can judge whether your meshing is good enough or not. In the figure, the field contours are not smooth or circular, that means there is a scope for improvement in the mesh.

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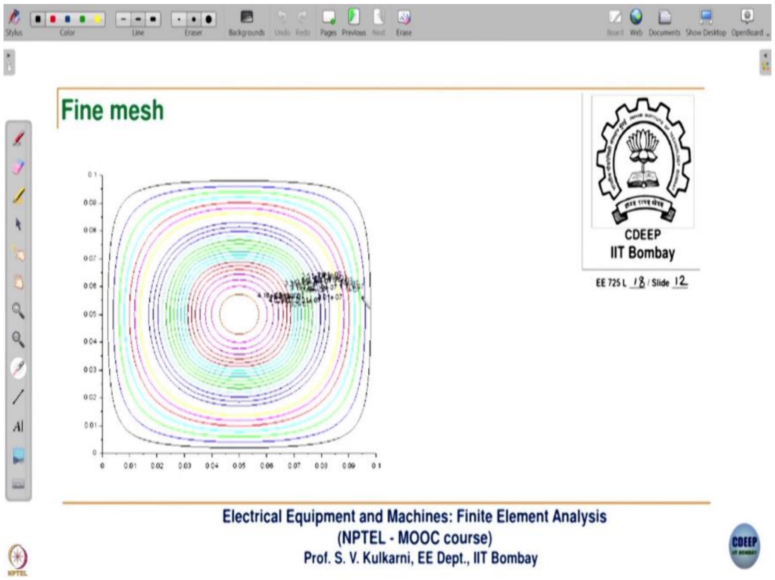

Fine mesh

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After making the mesh very fine, you will get a solution whose field contours are quite smooth as shown in the above figure. But in the above figure, you can see that the boundary is quite close to the current-carrying conductor and you can see that the contours suddenly become flat near the boundary although they are smooth. Ideally, if it was an isolated conductor, all these flux contours particularly the ones near the boundary should be circular.

For the present problem to become an isolated conductor, the boundary should be far off so that the boundary conditions do not affect the field distribution. Since, we have not put that boundary far off, we can see that suddenly, we are forcing the potential to go to 0. Otherwise, the flux contours would have been circular.

In fact, later on, we will see by using a closer boundary, we will not get the correct answer when we calculate inductance of the bar. We will find that the error is high when the boundary is closer. When we take the boundary far off, the calculated inductance is more correct.

So, with this we will end the 18th lecture. I hope you have now understood the finite element procedure because we have seen 1D and 2D examples. In the next lecture, we will study a 2D code, but not in details because we have already seen a 1D code. Thank you.

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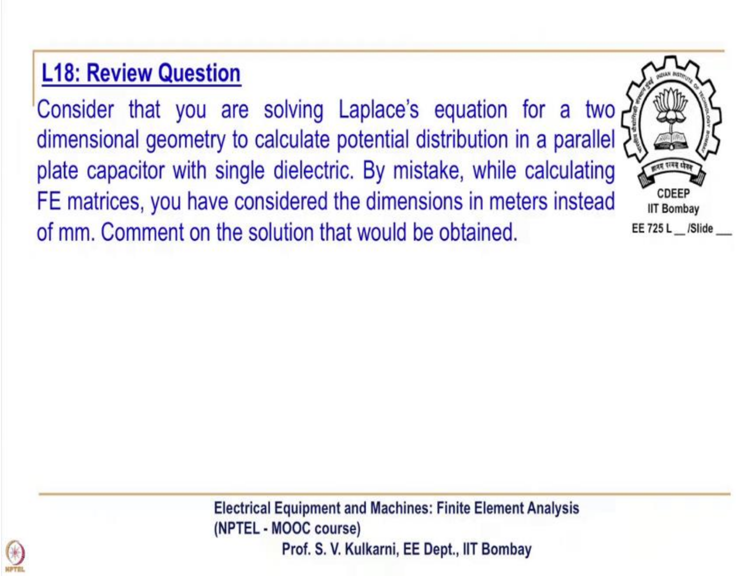

L18: Review Question

Consider that you are solving Laplace's equation for a two dimensional geometry to calculate potential distribution in a parallel plate capacitor with single dielectric. By mistake, while calculating FE matrices, you have considered the dimensions in meters instead of mm. Comment on the solution that would be obtained.

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