Electrical Equipment and Machines: Finite Element Analysis
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Lecture: 17
2D FEM: Problem Definition and Shape Functions

Welcome to this 17 lecture. Now we are ready to go into two-dimensional FE analysis.

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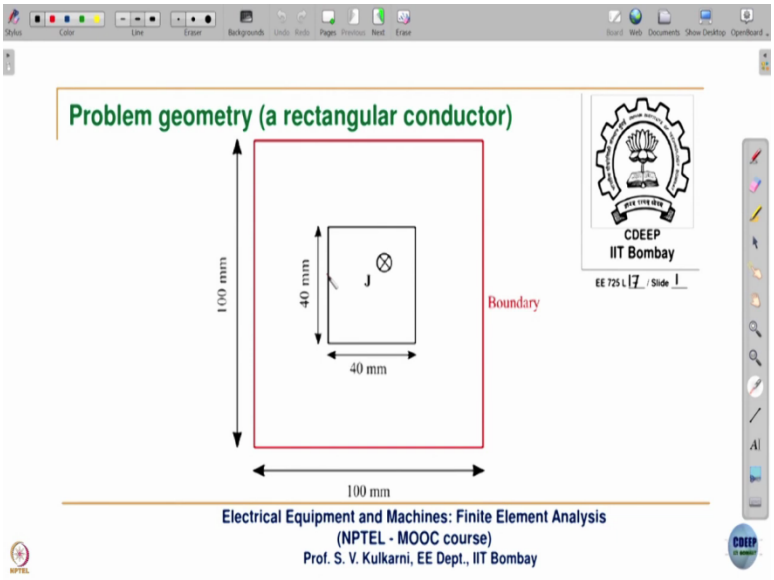

We will start with a magnetostatic problem which involves a rectangular conductor carrying current I. The conductor is enclosed in a rectangular boundary. As the current is going into the plane, the field lines are in the clockwise direction. The dimensions of the problem geometry are also given in the figure shown in the above slide. Please note that this boundary is considerably closer to the conductor.

If you want to analyze the field of an isolated conductor, then the outer boundary has to be far away so that the imposed boundary condition does not affect the field distribution. Here, the FEM formulation and code are demonstrated by taking the boundary which is very close to the conductor, so that we can explain easily. But eventually, when we write a code for this problem, we can take this boundary far off, so that the applied boundary condition does not affect the field distribution. So, why we are saying that boundary condition will affect the solution is because

when you impose magnetic vector potential A = 0, it influences the field distribution as the boundary is closer to the conductor. We will see more about it later.

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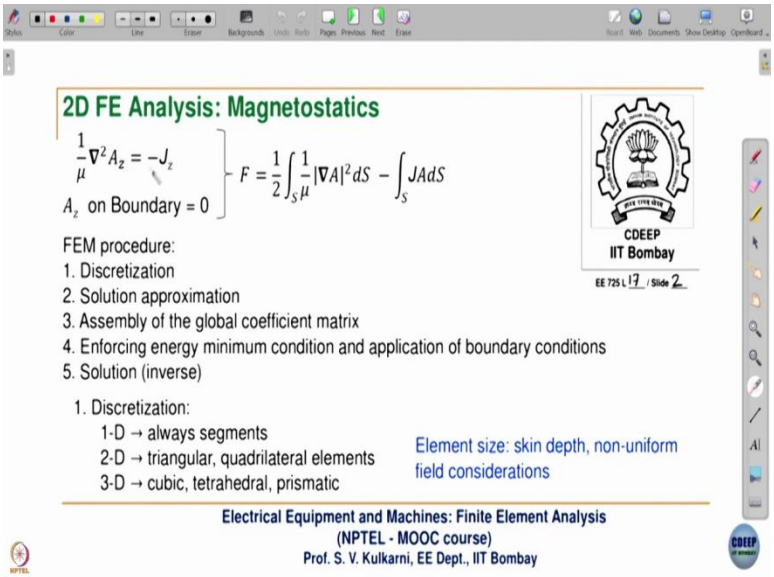

The governing equation for this magnetostatic problem is $\nabla^2 A_z = -\mu J_z$. As we have discussed earlier, in case of a two dimensional problem, we take a cross section which is perpendicular to the direction of the applied current. So the direction of current and therefore current density J is fixed, which is in z direction. The direction of A also gets fixed along z direction. That is why the vector form of **A** and **J** is by replaced A_z and J_z.

Effectively, this reduces to scalar Poisson's equation. Otherwise, it would be a vector Poisson's equation with three components (A_x, A_y, A_z). This concept also we have seen in the basics of electromagnetics. So, by considering a 2D plane perpendicular to **J**, it reduces vector Poisson's equation to a scalar form with unknowns as A_z at various points in the domain. The corresponding functional for this PDE is given below .

$$F = \frac{1}{2}\int_S \frac{1}{\mu}|\nabla A|^2 dS - \int_S JA dS$$

Now you are very familiar with writing a functional for a given PDE. In case of electrostatics, you have ϵE^2 as integrand for the first integral in the above expression, here you have $\frac{1}{\mu}|\nabla A|^2$. In basics

of electromagnetics, we have explained to you that when either μ or ϵ appear singly, that means, if ϵ appears in the numerator, then μ will appear in denominator. So $\frac{1}{2}\epsilon E^2$ got replaced by $\frac{1}{\mu}|\nabla A|^2$.

The $\int_s JAdS$ (source) term will appear on the right hand side. The source term in the PDE gets multiplied by the corresponding potential (here A) in the functional expression. So, the functional expression in the above equation can be directly written based on the theory that we have seen in the previous lectures. The boundary condition is $A_z = 0$ on the outer most boundary. As explained earlier, A_z need not be 0 always. It can be some other number like 50 or 100. If we change the boundary condition value then the values of A at all the nodes in the domain will get shifted by the corresponding value.

Flux flowing between any 2 points in a domain is just a difference between the A values at the corresponding points (A_1 and A_2). So, even if you change the value of the reference potential from 0 to some value, the corresponding field distribution (B_x, B_y) values will not change.

That is why for simplicity, always in the magnetostatic problems or magnetic field problems, we impose boundary condition or reference potential as A = 0. We have already summarized the FEM procedure which is given below.

FEM procedure:
1. Discretization
2. Solution approximation
3. Assembly of the global coefficient matrix
4. Enforcing energy minimum condition and application of boundary conditions
5. Solution (inverse)

The first step of FE procedure is discretization of the geometry. If it is a 2D problem, we will have to use either triangular or quadrilateral elements. Then we approximate the solution. In 1D, we approximated the solution as $a + bx$. In case of 2D, the approximation will be $a + bx + cy$.

Then the next step is assembly of the global coefficient matrix. Now, assembly of the global coefficient matrix effectively represents adding the energies of all elements. So the global coefficient matrix represents the energy of the entire domain. Because the global coefficient matrix gets formed by appending all the element coefficient matrices.

Each of the element coefficient matrices represents the corresponding energy of the element under consideration and the global coefficient matrix represents the energy of the entire system. Remember that the system of equations will also have a source term and that also contributes to the overall energy of the system. The first integral in the above equation will give you K matrix and the second integral will give you B matrix.

In the previous lecture, we used K or A matrix. But here we will call the global coefficient matrix by C, because already we have used the variable A for magnetic vector potential. So, the global coefficient matrix will be denoted by C. The final system of equation will be $CA = B$, where unknowns are magnetic vector potentials (A_z) at all the nodes in the discretized domain.

Then to form the final linear system of equations, we need to minimize the total energy with respect to nodal magnetic vector potentials and apply the boundary conditions. As we have seen in case of 1D, the contributions to the final B matrix will be from boundary condition and source condition. That is why the B matrix will have contributions from J, which is the source, as well as the boundary condition (A = 0) that we are imposing on the outermost rectangular boundary. So, always remember that when we devolop an FEM formulation, the right hand side B matrix will eventually have contributions from source and boundary conditions. When we will get final linear system of equations $CA = B$, you have to take inverse to calculate the unknown vector A which represents magnetic vector potentials at various nodes in the discretized domain.

Let us go into each step of the FE procedure. In the first step, we have to discretize the geometry. In 1D problems, we generally have segments as discretized domains which we have seen earlier. In 2D problems, you can have either triangular or quadrilateral elements and in 3D, you will have cubic, tetrahedral or prismatic elements. How do we decide element size? Earlier we discussed that whenever there is a non-uniform field in a problem domain, you will need a fine discretization.

Similarly, when the frequency of excitation is high and eddy currents are induced in the conductors and the current is confined to a small surface depth called skin depth. Then you need to have fine elements in that zone so that you can capture the rapid variation of the field as well as the losses. If you have to capture the power loss occurring in the conductor, then you need to have a fine mesh in that region. Such points will be considered while deciding the size of the element.

The type of elements will be decided based on whether the problem is 1D or 2D or 3D and whether we should go for cubic, tetrahedral or prismatic for 3D, triangular, or quadrilateral for 2D, is a matter of convenience and coding. For example in this course, we will always use triangular elements because they are much simpler to code.

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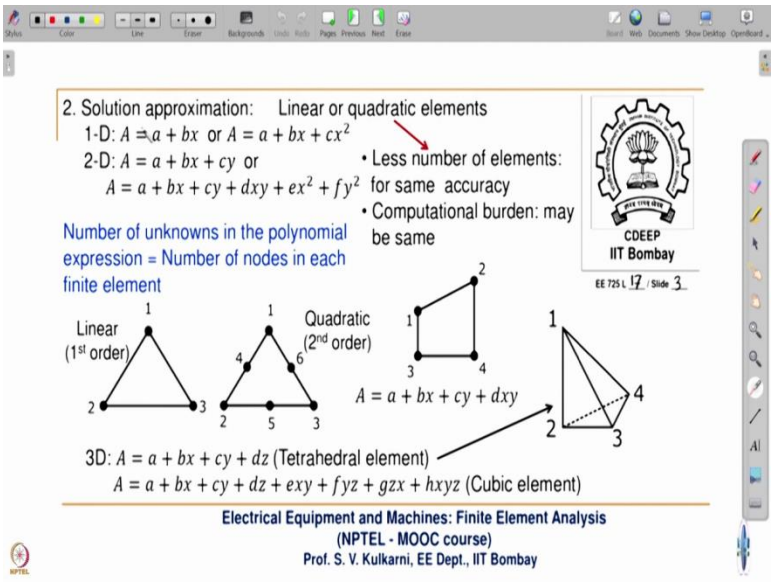
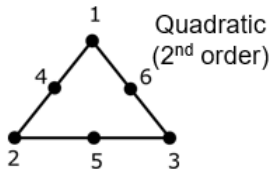


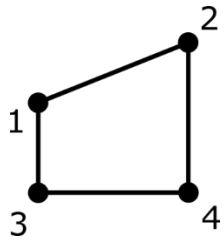
Now let us go further. The second step is solution approximation. In case of 2D, you can have a linear or quadratic element. What is a linear element? Potential approximation for a linear element in 1D analysis is $a + bx$. Whereas the approximation for a quadratic element in 1D analysis is $a + bx + cx^2$. For 2D FE analysis, potential approximation for a linear element will be $a + bx + cy$ whereas the second order approximation will be $a + bx + cy + dxy + ex^2 + fy^2$.

The number of constants in the potential approximation is equal to the number of nodes in the corresponding element. For example, a 1D linear element which is just a segment will have 2 nodes so the potential approximation $(a + bx)$ for this element will have 2 unknown constants (a and b). A triangular element will have 3 nodes. So, the number of constants in the corresponding potential approximation $(a + bx + cy)$ is 3. Now why this is important? Eventually to get an exact solution for the linear system of equations that you solve, the number of unknowns should be equal to the number of equations.

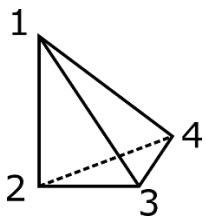
So, the number of nodes in each element equal to the number of constants in the approximation will ensure an exact solution for your problem. For example, approximation for a quadratic 2D element will have 6 constants. So there are 6 nodes in the corresponding 2D quadratic triangular element, 3 of the 6 nodes are at the vertices and the other 3 nodes are at the mid points of the corresponding edges of the triangle as shown in the following figure.



Similarly, you will have 2D quadrilateral element like the one shown in the following figure. Then the corresponding potential approximation for this element will be $a + bx + cy + dxy$ with 4 unknown constants which are to be determined for each element. Remember that we directly do not determine the unknown constants a b c, because we eliminate those constants with the nodal potentials as we have seen in the case of 1D. Effectively, we convert the problem in which constants need to be determined to a problem wherein we directly determine the nodal potential values because we eliminate the constants a, b, c, and so on. We will see this procedure in this lecture.



A 3D tetrahedral element will have 4 nodes as shown in the following figure. So the corresponding potential approximation $a + bx + cy + dz$ will have 4 unknown constants.



A cube element will have 8 nodes. So, there are 8 constants in the potential approximation $a + bx + cy + dz + exy + fyz + gzx + hxyz$. Now you have to know when to use which element. Here, the question is when do we use linear elements or when do we use quadratic elements?

If you are using linear elements, then you have to use a higher number of elements for problems with non-uniform field distribution. When you are using quadratic elements, you can use a lower number of elements. But one cannot say confidently which one would be better. Because the computational burden may be more or less same. So, it is a matter of convenience to choose a higher number of first-order elements compared to a lower number of second-order elements.

Because even though the number of second order elements is less in number, the expression for potential approximation is more complex. Also the number of nodes are increasing because of increase in the order of approximation. So, the complexity of the formulation is higher in the case of a quadratic element, although the number of elements is less and therefore the computational burden maybe more or less the same.

Choosing the element type and the number of elements depends on the convenience of the person who is developing the code. That is why in commercial FE software both options are generally available. The user has to choose whether he wants to use linear or quadratic elements in the simulation.

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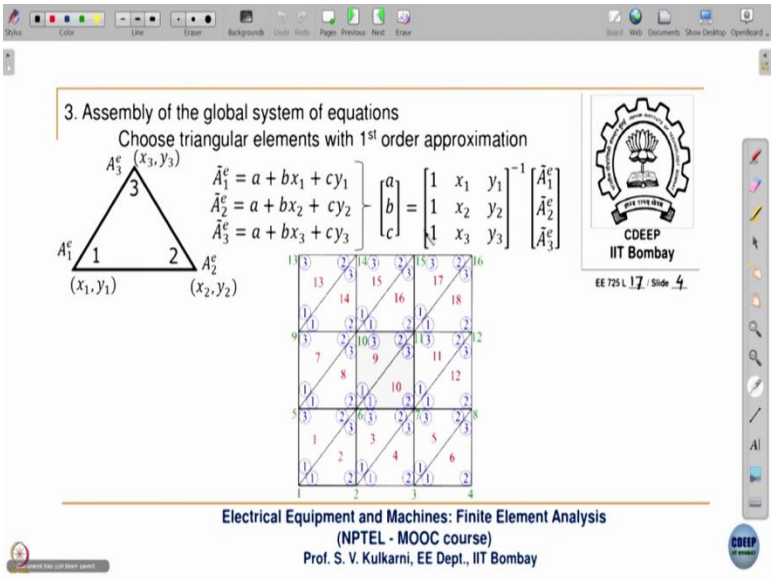
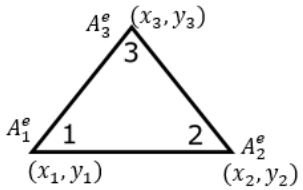


Going further, the third step is the assembly of the global system of equations. For that, first we have to find element coefficient matrices, and then combine all these matrices to form the global coefficient matrix. Now we will start with the triangular element shown in the following figure.



For this element, the nodes are numbered in the anti-clockwise direction because then only the area of the triangle (Δ) calculated using the following expression will be positive.

$$\Delta = \frac{1}{2}\begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

The above expression for calculating the area of a triangle is a standard formula where $(x_1, y_1), (x_2, y_2),$ and (x_3, y_3) are the coordinates of the vertices of the triangle under consideration. If you do not number these nodes in anti-clockwise fashion and if you number them in the clockwise direction, then the area calculated will be negative. To avoid this issue, we always number the nodes in the anti-clockwise direction.

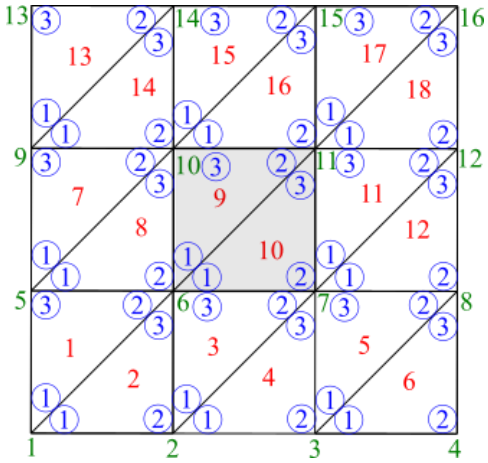
Now, using the potential approximation for a linear element $A^e = a + bx + cy$ and following the procedure that we did for the 1D case, we will develop the 2D FE formulation. The magnitudes of potentials at the three nodes A_1, A_2 and A_3 are as given below.

$$\left.\begin{aligned} \tilde{A}_1^e &= a + bx_1 + cy_1 \\ \tilde{A}_2^e &= a + bx_2 + cy_2 \\ \tilde{A}_3^e &= a + bx_3 + cy_3 \end{aligned}\right\} \rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} \tilde{A}_1^e \\ \tilde{A}_2^e \\ \tilde{A}_3^e \end{bmatrix}$$

The ~ represents that the potential functions are approximated solutions. As we did in the 1D formulation, we eliminate the constants a, b, and c and express them in terms of nodal potential values which are given in the above equation.

Effectively, the total energy of the system which is a function of a, b, c constants of each element is represented as a function of the nodal potential values in the discretized system. This conversion makes the formulation simpler. In the variational procedure that we have been seeing in this course, the corresponding minimization procedure involves varying the potentials at each of the nodes in the system and we see for which combination of potentials we get the minimum energy condition.

It is very logical to eliminate these a, b, c constants from the formulation and express the energy of each element in terms of the potentials of the three vertices. Then we have the discretized problem domain with 18 elements as shown in the following figure.



Now the elements in the above figure are uniformly distributed because using this example we want to explain the FEM procedure on a paper. But if you use some software then the elements will not be so less in number like the way shown in the above figure, because in the above figure there is only one layer of elements between the conductor and the outermost boundary. So the solution obtained using the above discretization is going to be very approximate. The expression of magnetic vector potential for any element is simply $a + bx + cy$. The expression for B will be calculated by using $\mathbf{B} = \nabla \times \mathbf{A}$. When we expand $\mathbf{B} = \nabla \times \mathbf{A}$, we will get derivatives of the components of A. We will see the expression of B using the linear approximation in one of the further lectures.

If $A = a + bx + cy$ and if you take the derivative of A with respect to x, it will be only b. Derivative with respect to y will be only c. So the components of B, B_x and B_y will be constant and hence the overall magnitude of **B** will be constant in the corresponding element. That means

effectively we are approximating B as a constant which will lead to errors in the solution if we use the mesh as shown in the above figure. Later on, when we complete the FE code, we will increase the number of elements in the problem domain and we can observe how the solution improves.

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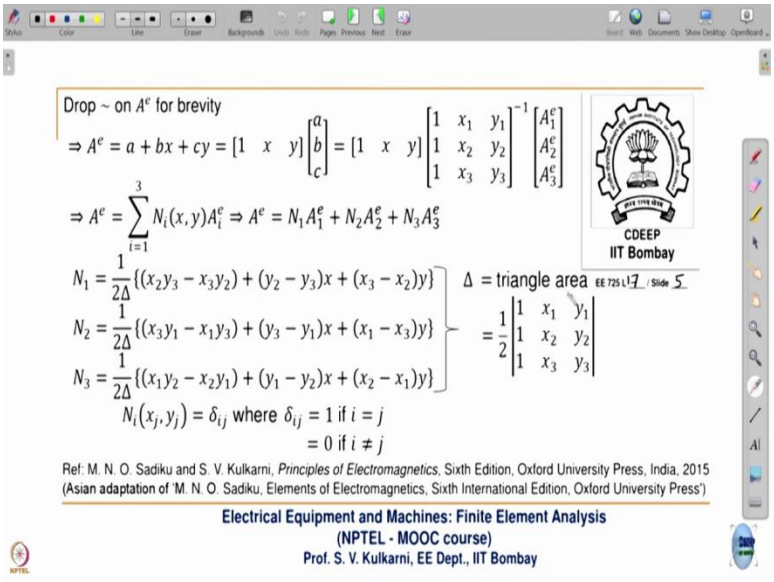

The approximate function of magnetic vector potential is written in matrix form as given below.

$$A^e = a + bx + cy = [1 \quad x \quad y]\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

And then we replace the constants with the expression in terms of nodal potentials as given below.

$$A^e = a + bx + cy = [1 \quad x \quad y]\begin{bmatrix} a \\ b \\ c \end{bmatrix} = [1 \quad x \quad y]\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1}\begin{bmatrix} A_1^e \\ A_2^e \\ A_3^e \end{bmatrix}$$

When we expand the inverse in the above expression, you can express A as

$$A^e = \sum_{i=1}^{3} N_i(x,y)A_i^e \Rightarrow A^e = N_1A_1^e + N_2A_2^e + N_3A_3^e$$

wherein N_1, N_2, and N_3 are the shape functions which are expressed as given below.

$$N_1 = \frac{1}{2\Delta}\{(x_2y_3 - x_3y_2) + (y_2 - y_3)x + (x_3 - x_2)y\}$$
$$N_2 = \frac{1}{2\Delta}\{(x_3y_1 - x_1y_3) + (y_3 - y_1)x + (x_1 - x_3)y\}$$
$$N_3 = \frac{1}{2\Delta}\{(x_1y_2 - x_2y_1) + (y_1 - y_2)x + (x_2 - x_1)y\}$$

In the previous lecture, we have derived the same expression for 1D formulation. We derived N_1 and N_2 for the local coordinates 1, 2 of each element. In the present problem, we have 3 nodes for every element, so we will have 3 shape functions, N_1, N_2, and N_3. The Δ in the above expression is the area of the triangle. The following property of shape functions that we saw in case of 1D holds for 2D also.

$$N_i(x_j, y_j) = \delta_{ij} \text{ where } \delta_{ij} = 1 \text{ if } i = j$$
$$= 0 \text{ if } i \neq j$$

From the above expression, the value of shape function N_1 at node 1 will be equal to 1, and 0 at nodes 2 and 3. Similarly, the value of N_2 will be 1 at node number 2, and 0 at nodes 1 and 3. For example, in the expression of N_1, if you substitute x as x_1 and y as y_1, we will get the value of the shape function as 1. If you substitute x as x_2, and y as y_2 in the expression of N_1, you will get its value as 0. Similarly, you can verify for N_2 and N_3. We will discuss why this property is important.

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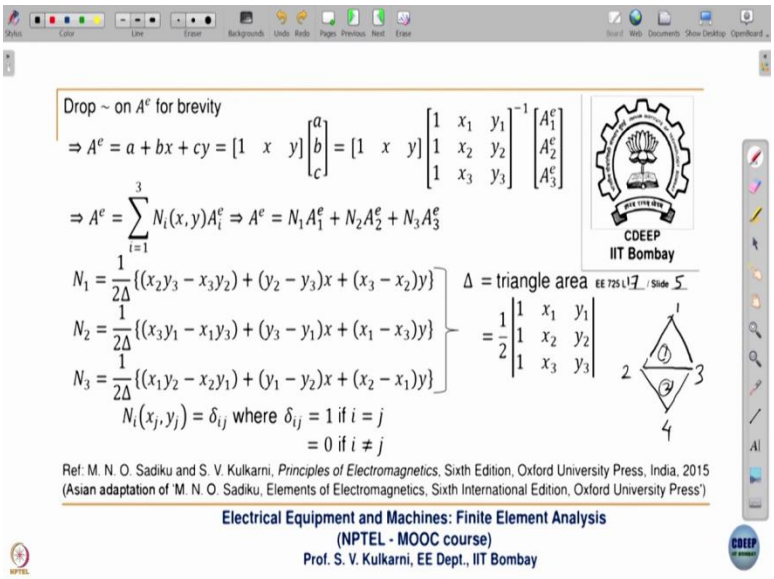

Let us take 2 adjacent or contiguous elements as shown in the following figure.

Now by the property of shape functions, the potential at nodes 2 and 3, as well as at any point on the segment joining nodes 2 and 3 will be decided by the potentials at nodes 2 and 3 only. Because the value N_1 is 0 at nodes 2 and 3 and at any point on the segment 2, 3. This can be verified by substituting expressions of x and y that represents any point on the segment 2-3 in N_1, you will find that the value of N_1 on this segment will be 0. So in the expression $A^e = N_1A_1^e + N_2A_2^e + N_3A_3^e$, N_1 is zero for any point on segment 2-3 and the value of A at any point on this segment is given by $A^e = N_2A_2^e + N_3A_3^e$. This can be proved for the two elements in the above figure.

So, the value of the potential at any point on the segment 2-3 will be decided by the potentials of nodes 2 and 3 only. At any of point on segment 2-3, if N_1 is 0, then potential at that point will be decided by N_2 and N_3. This should be the case because if it is not true and if the potentials at nodes 1 and 4 were deciding potentials on segment 2-3, then when you calculate the potential at any (x, y) on the segment with reference to elements 1 and 2, the two calculated potentials will be different and mathematically it will be absurd. So, it is very logical that the potential at any point on the segment 2-3 is decided by the potential values of nodes 2 and 3 only. In other words, the continuity of potential across element boundaries is ensured by this property of shape functions and this is called as ensuring continuity of potential. That means the variation of potential in the two elements is continuous at the interface. Otherwise, the formulation will be mathematically absurd.

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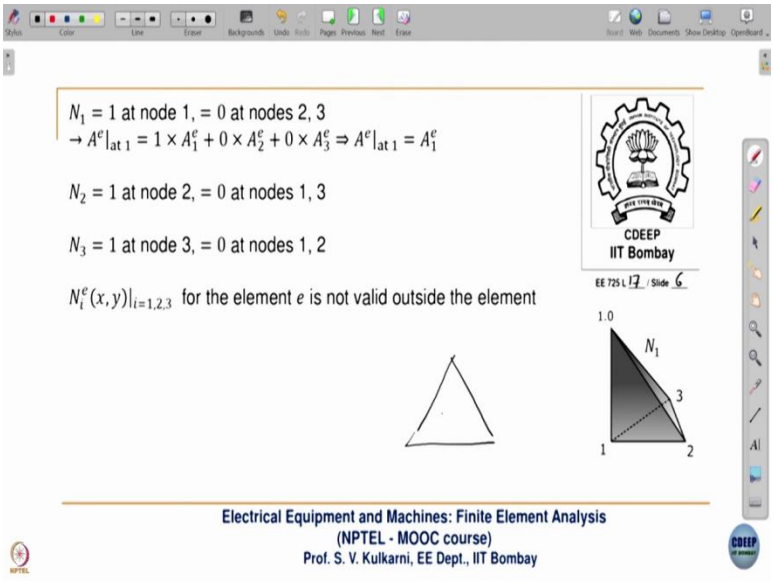
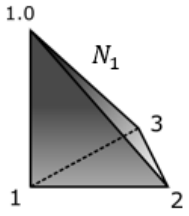


As explained earlier, N_1 should be equal to 1 at node 1 and 0 at nodes 2 and 3. The value of magnetic vector potential at node 1 will be A_1^e as derived below.

$$A^e(x_1, y_1) = N_1(x_1, y_1)A_1^e + N_2(x_1, y_1)A_2^e + N_3(x_1, y_1)A_3^e = A_1^e$$

Similarly, for the other two also $A^e(x_2, y_2) = A_2^e$ and $A^e(x_3, y_3) = A_3^e$. The variation of shape function properties are represented in the following figure.



In the above figure, the triangular element is formed by nodes 1, 2, and 3. Then the value of N_1 will be equal to 1 at node number 1 and that will reduce to 0 linearly at nodes 2 and 3.

Somewhere inside the triangle the values of N_1, N_2, and N_3 will exist and they will not be equal to zero. If the point is shifted towards node number 1, the contribution of N_1 will increase and the contributions of N_2 and N_3 will reduce in deciding the potential at that point. It is more intuitive, of course, this can be proved mathematically.

When the point coincides with node 1, then potential of that point is solely decided by node 1. Suppose the point is at the centroid of this triangle, then the values of N_1, N_2, and N_3 will be equal to $1/3$. That means potentials at the three vertices of the triangle will have equal contributions in deciding the potential at the point under consideration. In other words, the potential at the centroid is the average of potentials at the three vertices of the triangle. We will stop here and continue in the next lecture.

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