Electrical Equipment and Machines: Finite Element Analysis
Professor Shrikrishna V. Kulkarni
Department of Electrical Engineering
Indian Institute of Technology, Bombay
Lecture 15
1D FEM: Procedure

In the 14th lecture, we saw finite element procedure through the discretization method and there we saw how to calculate the energy associated with the entire problem domain. In that lecture, we had taken a 3 element example with 4 nodes and we calculated energy stored in the problem domain and then we also saw properties of shape functions.

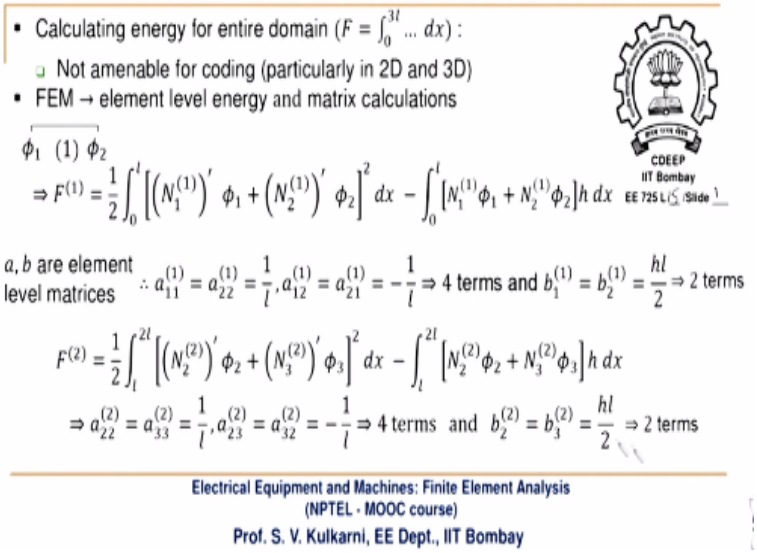
Now if you look at the procedure that we followed for calculating the energy, it was not amenable for coding particularly in 2D or 3D, because we calculated energy for the entire domain at once by evaluating the functional integral over the whole domain as given below.

$$F = \frac{1}{2}\int_0^{3l} [N_1'\phi_1 + N_2'\phi_2 + N_3'\phi_3 + N_4'\phi_4]^2 dx - \int_0^{3l} [N_1\phi_1 + \ \ldots + N_4\phi_4]h\, dx$$

In the above expression, we are integrating from 0 to 3l (total length of the domain) and then depending upon the values of shape function in each element, we have simplified the above equation by either taking some terms or not taking some terms. But this procedure is not good from the point of view of coding.

From the coding point of view, it will be easy if we calculate energy element-wise and form the element coefficient matrices and then combine them to form global coefficient matrix. It is similar to the thing that we saw in the previous lecture where we are calculated the total energy at once, but here we are first calculating the element coefficient matrices because we are calculating the energy of each element separately by using the contributions of shape functions and potentials associated with the element. Then we combine the element coefficient matrices to form the global coefficient matrix. That is what we will do in this lecture.

(Refer Slide Time: 02:09)



In this slide, we are taking only the first element of a problem domain with *n* elements, which is as shown below.

$$\overline{\phi_1 \ (1) \ \phi_2}$$

Now, we will calculate only energy associated with this first element. So, we have taken the limits of the integral as 0 to *l* in the following equation.

$$F^{(1)} = \frac{1}{2}\int_0^l \left[\left(N_1^{(1)}\right)' \phi_1 + \left(N_2^{(1)}\right)' \phi_2\right]^2 dx - \int_0^l \left[N_1^{(1)}\phi_1 + N_2^{(1)}\phi_2\right] h\, dx$$

We know that the above equation is energy functional and we have seen in the previous lecture that ϕ for this first element will be simply $N_1\phi_1 + N_2\phi_2$ because N_3 and N_4 are 0 for element 1. So then the functional expression for this element will have $\left(N_1^{(1)}\right)' \phi_1 + \left(N_2^{(1)}\right)' \phi_2$ and $N_1^{(1)}\phi_1 + N_2^{(1)}\phi_2$ as integrands. Also last time, we discussed that derivative will be valid only for shape functions. Because they are functions of *x* and ϕ_1 and ϕ_2 are not functions of *x* in the variational procedure. In this approach, we are varying the values of ϕ at every *x*. In the second integral of the above equation, *h* stands for the source term. The entries of the element level matrices are formed by using the same procedure. So then we will get the values of entries a_{11} and a_{22} of element 1 as $\frac{1}{l}$. In the previous case, when we calculate a_{22} for node 2, we got the

value as $\frac{2}{l}$ because that node 2 was common to both elements 1 and 2. But here since we are calculating only for the first element, the value of a_{22} will be only $\frac{1}{l}$.

When we take the second element, node 2 will correspond to node number 1 (local node 1) of that second element. So in the second element also, it will be $\frac{1}{l}$. And the $\frac{1}{l}$ of node 2 in the first element and $\frac{1}{l}$ in the second element will get added to get $\frac{2}{l}$ and that will be the total contribution of node 2.

For the first element a_{22} is only $\frac{1}{l}$ because the other contribution of potential at node 2 in the second element is yet to be accounted for. That will be accounted when we consider the second element. But this makes it very simple, if you go element-wise when you develop a code. So, first you consider an element and then set up a do loop and go element-wise. In the code, first you calculate element coefficient matrix and later on, you form the global coefficient matrix. So how do we do that, that we will see. The values of a_{12} and a_{21} will remain same as $-\frac{1}{l}$ which we have seen earlier. So the 4 terms of the element coefficient matrix are given below.

$$a_{11}^{(1)} = a_{22}^{(1)} = \frac{1}{l},\ a_{12}^{(1)} = a_{21}^{(1)} = -\frac{1}{l}$$

The entries of element level source matrix $b1$ and b2 will be as given below.

$$b_1^{(1)} = b_2^{(1)} = \frac{hl}{2}$$

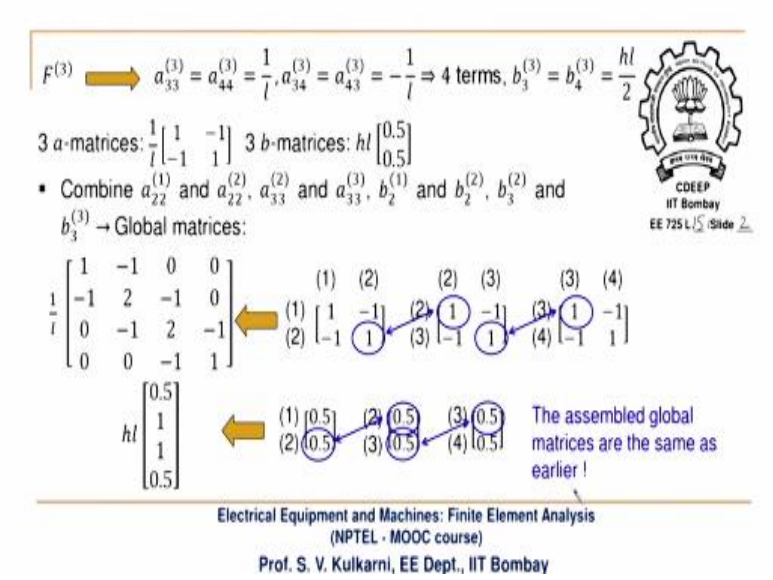
Similarly, the entries of matrices for element 2 are given below

$$\Rightarrow a_{22}^{(2)} = a_{33}^{(2)} = \frac{1}{l},\ a_{23}^{(2)} = a_{32}^{(2)} = -\frac{1}{l} \Rightarrow 4 \text{ terms} \quad \text{and} \quad b_2^{(2)} = b_3^{(2)} = \frac{hl}{2} \Rightarrow 2 \text{ terms}$$

For element 2, local node 1 will be node 2 and local node 2 will be node 3. Also the expression for element 2 is

$$F^{(2)} = \frac{1}{2}\int_l^{2l} \left[\left(N_2^{(2)}\right)'\phi_2 + \left(N_3^{(2)}\right)'\phi_3\right]^2 dx \; - \int_l^{2l} \left[N_2^{(2)}\phi_2 + N_3^{(2)}\phi_3\right] h\, dx$$

(Refer Slide Time: 06:46)



Now, we go further and then similarly for element 3 the entries of matrices are given below.

$$a_{33}^{(3)} = a_{44}^{(3)} = \frac{1}{l},\ a_{34}^{(3)} = a_{43}^{(3)} = -\frac{1}{l} \Rightarrow 4 \text{ terms},\ b_3^{(3)} = b_4^{(3)} = \frac{hl}{2}$$

All the element coefficient matrices are identical in this case which is an advantage. And it becomes very simple to code. So the 3 element coefficient matrices and source matrices are given below.

$$3\ a\text{-matrices: } \frac{1}{l}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \quad 3\ b\text{-matrices: } hl\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$$

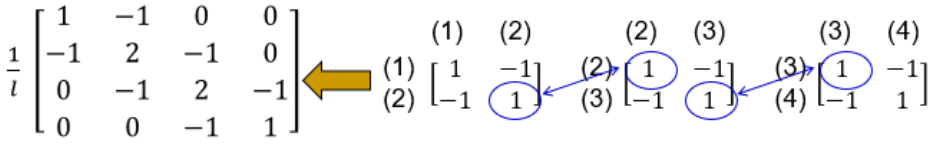
So, we have to combine element coefficient matrices to form the global coefficient matrix. This step will actually add the energies of all the elements and the contributions of the nodal potentials will automatically get added. In the present problem, you have the 3 element coefficient matrices which are given below.

$$\begin{matrix} & (1) & (2) \\ (1) & 1 & -1 \\ (2) & -1 & 1 \end{matrix} \quad \begin{matrix} & (2) & (3) \\ (2) & 1 & -1 \\ (3) & -1 & 1 \end{matrix} \quad \begin{matrix} & (3) & (4) \\ (3) & 1 & -1 \\ (4) & -1 & 1 \end{matrix}$$

Here, $\frac{1}{l}$ is not considered because that is common in these 3 matrices. Now, in the above matrices, what are written in brackets are the global node numbers. Because in the first matrix that corresponds to element 1 is between nodes 1 and 2 so (1), (2) are indicated in that matrix,

element number 2 is between 2 and 3 and element number 3 is between 3 and 4. Then you can see that in element number 1, you have a_{22}, for element number 3, a_{22} is not there.

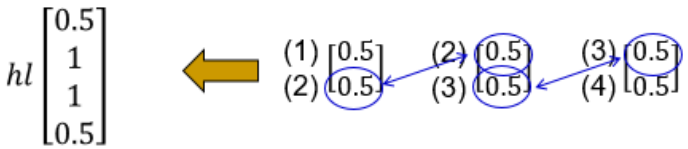
We add a_{22} of element 1 and a_{22} of element 2 to get overall energy as just a number because it is a scalar quantity. The contributions of all elements will get added to give only one number as energy. So all these terms will get added which means you have to collect terms with the same coefficient (which are indicated with blue circles). That is why this gets added and the contribution of node 2 becomes 2 (2nd diagonal entry) in the global coefficient matrix as given below.



Whereas the contribution of node 1 (1st diagonal entry of global matrix which is on the left hand side) remains 1 because a_{11} appears only in the matrix that corresponds to element 1. Similarly, a_{33} appears in the matrices that correspond to elements 2 and 3, so that gives 2 in the global coefficient matrix, a_{44} appears once in element 3 that is why you get 1 in the global matrix and then your off-diagonal elements will remain as that in element coefficient matrices.

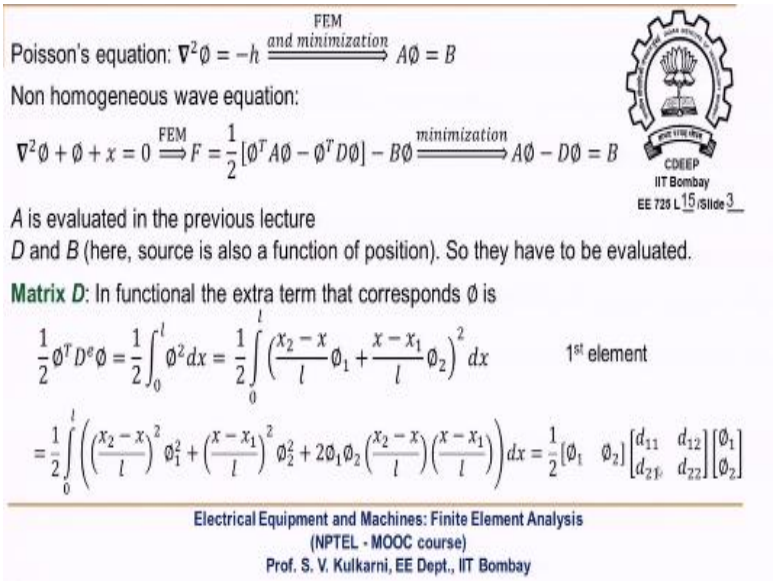
This is how we have formed the global coefficient matrix which is same as the one we have seen in the previous lecture. But it was not so easy if we use whole domain approximation. If we go element-wise, it becomes very easy. So while coding, you have to set up a do loop and calculate the element coefficient matrices element wise and then combine them to calculate the global coefficient matrix.

Later on, when we see an FE code, we will see how easy it is, so just wait till that time. In fact after this discussion, we are going to show you an FE code written in Scilab and this will make things more clear. Similarly, the *b* matrix which represents source will be added to form its global level matrix as given below.



The 3 b matrices are added as shown above. Again, hl is not considered here because it is common. b_2 is in elements 2 and 3 and b_3 is in elements 3 and 4. So 0.5 in each of the two matrices are added to give 1. The contribution becomes 1 for node numbers 2 and 3, because nodes 2 and 3 are common and that is how you get the global b matrix as shown in the left hand side of the above figure.

(Refer Slide Time: 11:14)



Till now this procedure was to solve Poisson's equation using FEM. Now we will see the procedure to solve the following non-homogenous 1D wave equation that we have seen in the previous lectures.

$$\nabla^2\emptyset + \emptyset + x = 0$$

Compared to Poisson's equation in non-homogenous wave equation, you have an extra ϕ term and x represents the source. Earlier we were calling h as the source term. In the previous example, h was constant that means everywhere in the domain the source is same.

Now in the present example, the source is a function of x and it is not constant over the entire domain. After applying FEM procedure, the expression of the functional for the above differential equation in terms of global matrices is given below.

$$F = \frac{1}{2}[\emptyset^T A\emptyset - \emptyset^T D\emptyset] - B\emptyset$$

After minimizing the above functional, the final system of linear equations is given below.

$$A\emptyset - D\emptyset = B$$

In whole domain approximation, the final matrix equation is $AC - DC = B$ where the entries of C are the unknown coefficients in that polynomial expression.

Here we have gone from the whole domain approximation to elemental level approach. So the coefficients (C) are replaced by the potentials (ϕ) at nodes. So, that is why the final matrix equation becomes

$$A\emptyset - D\emptyset = B$$

In the above equation, *A* matrix is already evaluated in the previous lecture because $\nabla^2\phi$ is the same whether it is in wave equation or in Poisson's equation or even in Laplace's equation. When you follow the FEM procedure, $\nabla^2\phi$ term will result into the same global coefficient matrix if the geometry and material properties are the same. So what is the new thing that we need to form is the matrix corresponding to ϕ term because we have already worked out the $\nabla^2\phi$ term that is the *A* matrix in the previous 2-3 slides.

Now we have to worry about this ϕ term and then about *x* because earlier the source was constant and it was not varying with *x*. But now the source is a function of *x*. So now how do we form the *D* matrix? In the functional, the extra term that corresponds to ϕ term (*D* matrix) is $\frac{1}{2}\phi^2 dx$. Remember, this term for the functional can be written by taking the ϕ term to the right hand side, then you multiply by ϕ and it is then multiplied by $\frac{1}{2}$.

Now this ϕ^2 for element 1 is replaced by $(N_1\phi_1 + N_2\phi_2)^2$ of that element. So we have to just evaluate the following integral.

$$\frac{1}{2}\emptyset^T D^e \emptyset = \frac{1}{2}\int_0^l \emptyset^2 dx = \frac{1}{2}\int_0^l \left(\frac{x_2 - x}{l}\emptyset_1 + \frac{x - x_1}{l}\emptyset_2\right)^2 dx$$

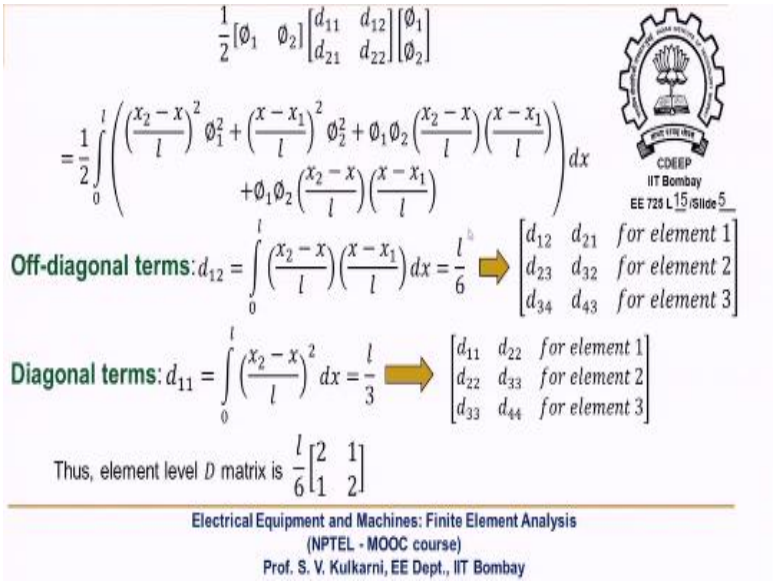
So if you expand the square in the integrand, we will get the following equation.

$$\frac{1}{2}\int_0^l \emptyset^2 dx = \frac{1}{2}\int_0^l \left(\left(\frac{x_2 - x}{l}\right)^2 \emptyset_1^2 + \left(\frac{x - x_1}{l}\right)^2 \emptyset_2^2 + 2\emptyset_1\emptyset_2\left(\frac{x_2 - x}{l}\right)\left(\frac{x - x_1}{l}\right)\right) dx$$

$$= \frac{1}{2}[\emptyset_1 \quad \emptyset_2]\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix}$$

This integral expression can be further elegantly written in the matrix form. Now we will see the expressions of entries of these matrices.

(Refer Slide Time: 15:17)



The above integral is rewritten below with one change: instead of $2\phi_1\phi_2$ we have to just split that into 2 terms as $\phi_1\phi_2 + \phi_1\phi_2$.

$$\frac{1}{2}[\emptyset_1 \quad \emptyset_2]\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix}$$

$$= \frac{1}{2}\int_0^l \left(\left(\frac{x_2 - x}{l}\right)^2 \emptyset_1^2 + \left(\frac{x - x_1}{l}\right)^2 \emptyset_2^2 + \emptyset_1\emptyset_2\left(\frac{x_2 - x}{l}\right)\left(\frac{x - x_1}{l}\right) + \emptyset_1\emptyset_2\left(\frac{x_2 - x}{l}\right)\left(\frac{x - x_1}{l}\right) \right) dx$$

The d_{12} entry in the above matrix equation is given by the following expression.

$$d_{12} = \int_0^l \left(\frac{x_2 - x}{l}\right)\left(\frac{x - x_1}{l}\right) dx = \frac{l}{6}$$

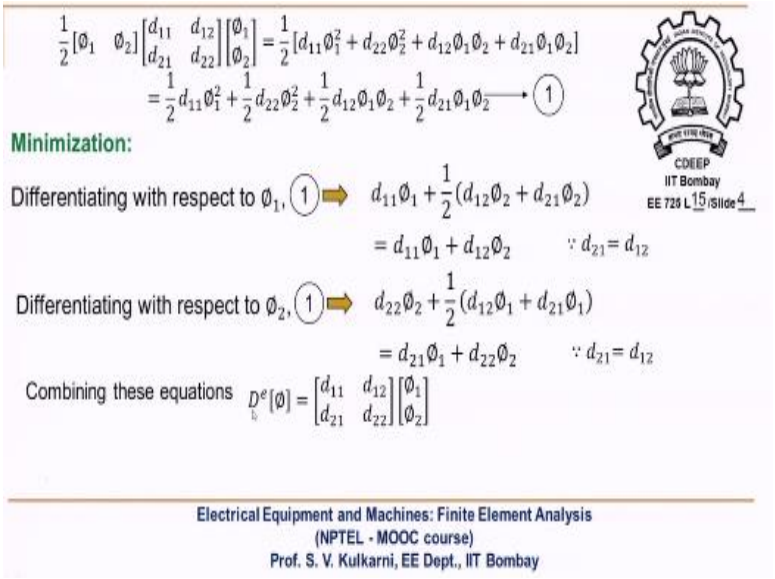
So, it is obvious because if you expand the above matrix expression, you can easily correlate that the off diagonal term d_{12} will be the above integral. Now if you evaluate this integral, you will get the value of d_{12} as $\frac{l}{6}$. Off diagonal terms for various elements are d_{12}, d_{21} for element 1, d_{32}, d_{23} for element 2, and d_{34}, d_{43} for element 3. The diagonal terms for example, d_{11} will be given by this following expression.

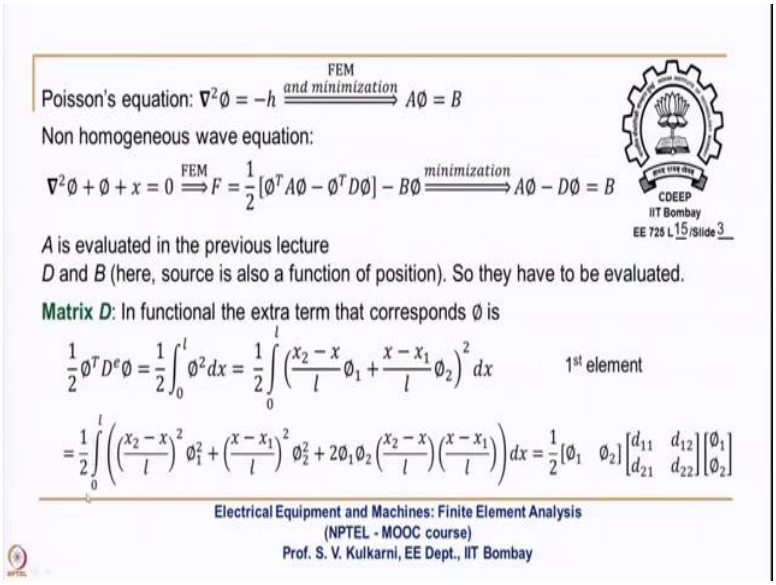
$$d_{11} = \int_0^l \left(\frac{x_2 - x}{l}\right)^2 dx = \frac{l}{3}$$

If we evaluate the above integral, its value will be equal to $\frac{l}{3}$. And if you evaluate it for various elements then you will get d_{11} and d_{22} for element 1, d_{22} and d_{33} for element 2, and d_{33} and d_{44} for element 3. Remember that this is a 3 element example with 4 nodes. Thus the element level D matrix will be as given below.

$$[D]^e = \frac{l}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(Refer Slide Time: 17:44)





Now let us again go back to the following expression that we saw in the previous slide.

$$\frac{1}{2}[\emptyset_1 \quad \emptyset_2]\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix} = \frac{1}{2}[d_{11}\emptyset_1^2 + d_{22}\emptyset_2^2 + d_{12}\emptyset_1\emptyset_2 + d_{21}\emptyset_1\emptyset_2]$$

This whole expression involving the product of 3 matrices can be expanded to give the expression on right hand side. Now, these 4 terms are split as given below.

$$\frac{1}{2}[\emptyset_1 \quad \emptyset_2]\begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix}\begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix} = \frac{1}{2}d_{11}\emptyset_1^2 + \frac{1}{2}d_{22}\emptyset_2^2 + \frac{1}{2}d_{12}\emptyset_1\emptyset_2 + \frac{1}{2}d_{21}\emptyset_1\emptyset_2$$

Remember the above expression is a part of the functional expression and it corresponds to element number 1. So this is representing the part of energy of element 1 and split into 4 terms as shown in the above expression. Now in the process of FEM method we are minimizing the energy. Then we will have to differentiate the above energy term with respect to ϕ_1. If you differentiate this with ϕ_1, you will get the following expression since $d_{12} = d_{21}$

$$\text{Differentiating with respect to } \emptyset_1 \Rightarrow \quad d_{11}\emptyset_1 + \frac{1}{2}(d_{12}\emptyset_2 + d_{21}\emptyset_2)$$
$$= d_{11}\emptyset_1 + d_{12}\emptyset_2 \qquad \because d_{21} = d_{12}$$

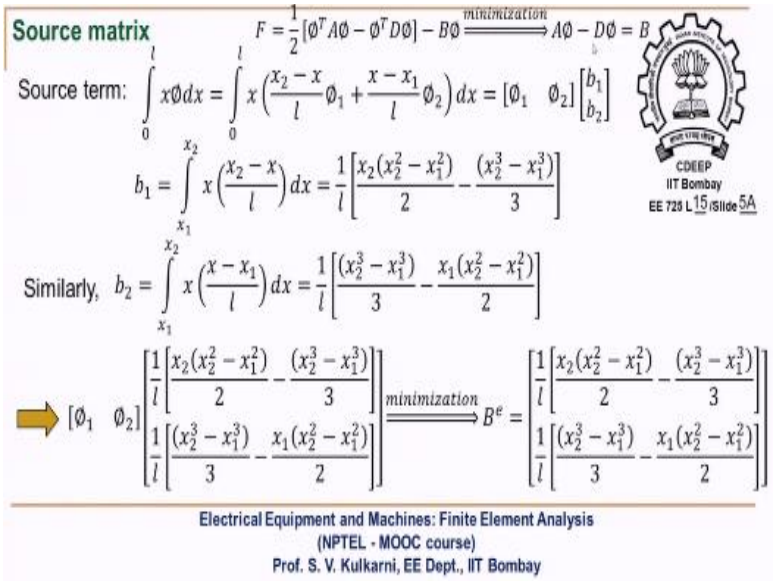
Similarly, differentiating the energy term with respect to ϕ_2, you will get the following expression.

$$\text{Differentiating with respect to } \emptyset_2 \Rightarrow \quad d_{22}\emptyset_2 + \frac{1}{2}(d_{12}\emptyset_1 + d_{21}\emptyset_1)$$
$$= d_{21}\emptyset_1 + d_{22}\emptyset_2 \qquad \because d_{21} = d_{12}$$

Combining these two expressions, after minimization of this whole term involving 3 matrices it will reduce to the following expression.

$$D^e[\emptyset] = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} \begin{bmatrix} \emptyset_1 \\ \emptyset_2 \end{bmatrix}$$

(Refer Slide Time: 19:54)



Now let us see the source matrix. Just to summarize, we started with the following the expression for functional *F*

$$F = \frac{1}{2}[\emptyset^T A\emptyset - \emptyset^T D\emptyset] - B\emptyset$$

and after minimization, we would get

$$A\emptyset - D\emptyset = B$$

We have already seen the derivation of *A* which is a standard global coefficient matrix and we just saw how to derive 2×2 coefficient matrix *D* for element 1 and similarly it can be done for other elements.

Now we will see how to evaluate the source (*B*) matrix which represent the source. The source term for the first element is given below.

$$\int_{x_1}^{x_2} x\emptyset dx = \int_{x_1}^{x_2} x\left(\frac{x_2 - x}{l}\emptyset_1 + \frac{x - x_1}{l}\emptyset_2\right) dx$$

Since it is for element 1, the limits of the integral are from 0 to *l*. Again just going back, the source term in this example is *x*. When you take the source term on right hand side it will become $-x\phi$ so integral will be $\int_{x_1}^{x_2} x\emptyset dx$ and $\phi = N_1\phi_1 + N_2\phi_2$ where N_1 is the shape function for node 1 and N_2 is the shape function for node 2 and then this can be written elegantly in the matrix form as given below.

$$\int_{x_1}^{x_2} x\emptyset dx = \int_{x_1}^{x_2} x\left(\frac{x_2 - x}{l}\emptyset_1 + \frac{x - x_1}{l}\emptyset_2\right) dx = [\emptyset_1 \quad \emptyset_2]\begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

So now this b_1 can be calculated by using the following expression.

$$b_1 = \int_{x_1}^{x_2} x\left(\frac{x_2 - x}{l}\right) dx = \frac{1}{l}\left[\frac{x_2(x_2^2 - x_1^2)}{2} - \frac{(x_2^3 - x_1^3)}{3}\right]$$

Now instead of 0 to *l*, the limits are taken as x_1 to x_2 so that we can generalize. Although we are doing this for the first element, this is applicable for the second or third element for the corresponding first nodal coordinate. So instead of 0 to *l* (because 0 to *l* will be valid for the first element), we are generalizing it as x_1 to x_2.

Now b_2 can be calculated by using the following expression.

$$b_2 = \int_{x_1}^{x_2} x\left(\frac{x - x_1}{l}\right) dx = \frac{1}{l}\left[\frac{(x_2^3 - x_1^3)}{3} - \frac{x_1(x_2^2 - x_1^2)}{2}\right]$$

Anyway we are forming the source term in matrix form as given below.

$$[\emptyset_1 \quad \emptyset_2]\begin{bmatrix} \frac{1}{l}\left[\frac{x_2(x_2^2 - x_1^2)}{2} - \frac{(x_2^3 - x_1^3)}{3}\right] \\ \frac{1}{l}\left[\frac{(x_2^3 - x_1^3)}{3} - \frac{x_1(x_2^2 - x_1^2)}{2}\right] \end{bmatrix} \xrightarrow{minimization} B^e = \begin{bmatrix} \frac{1}{l}\left[\frac{x_2(x_2^2 - x_1^2)}{2} - \frac{(x_2^3 - x_1^3)}{3}\right] \\ \frac{1}{l}\left[\frac{(x_2^3 - x_1^3)}{3} - \frac{x_1(x_2^2 - x_1^2)}{2}\right] \end{bmatrix}$$

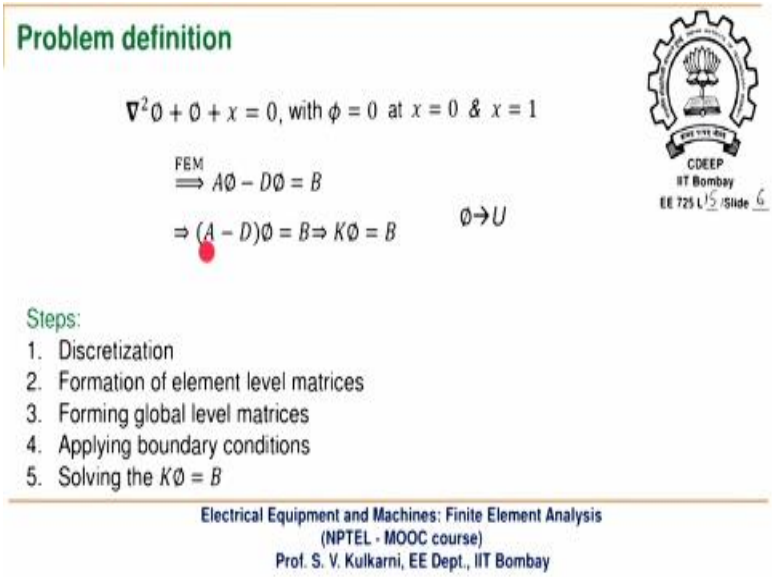
So b_1 is given by evaluating its corresponding integral and we will get the first term of B^e matrix. Similarly, b_2 is calculated by solving the previous integral, which will give the second term in B^e matrix.

Now you can write the energy due to source term as multipliclation of two matrices as given above. The expressions for b_1 and b_2 are written as given in the above matrix equation. After minimization, that means when we differentiate the energy term with respect to ϕ_1 and ϕ_2, you will get the element level b matrix. Thus we have understood how do we calculate the entries of matrices A, D, and B. And we have already seen the derivation of matrix A in case of Poisson's equation, but this D matrix would be required when you are solving either wave equation or diffusion equation.

Now we will start with the explanation of a 1D FE code and my PhD student B. Sairam has developed that code. So he will explain it to you.

As we discussed in the previous slide, we will be developing a 1-dimensional FE code to solve a 1-dimensional partial differential equation that we have seen in the previous lectures.

(Refer Slide Time: 25:07)



The 1-dimensional partial differential equation and the corresponding boundary conditions are given below.

$$\nabla^2\emptyset + \emptyset + x = 0, \text{ with } \phi = 0 \text{ at } x = 0 \ \& \ x = 1$$

It is the same equation that we have solved manually using whole domain approach and FE procedure with 3 elements in the previous lectures. Now we will solve the same equation using a code. After applying FE procedure, the above partial differential equation will be converted into a linear system of equations or a matrix equation of the form which is given below.

$$A\emptyset - D\emptyset = B$$

And then the above equation will again be simplified as given below.

$$(A - D)\emptyset = B \Rightarrow K\emptyset = B$$

In the above equation, considering ϕ as common, will result into $(A - D)\emptyset$ and $A - D$ is taken as K and ϕ is represented as U. As you have seen in the previous lecture, the partial differential equation will be converted into a matrix form and then it is simplified into $AX = B$ form. Here A is nothing but K and the unknown matrix X is nothing but ϕ and B is the source matrix. In the previous lecture, we have seen a 3-element example. In that example, we have 3 elements and 4 nodes using which we formed a simple 4×4 matrix. Using three 2×2 element level matrices we have formed one 4×4 matrix and it is reduced to a 2×2 matrix after applying boundary conditions. Since the number of elements are less it is obvious and simple to do it manually, but if we increase the number of elements then it will be a very tough job to form those matrices manually. Let us say that if we have chosen some 5 elements with 6 nodes then the global coefficient matrix will be of size 6×6. After applying boundary conditions, the size of the matrix reduces to 4×4 then how you can take the inverse of that matrix. Manually doing that is a very tough job. We can easily do such kind of operations by using scientific computing software like Scilab, MATLAB, python, etc. Here in this course, we will be developing codes on Scilab only. We have chosen Scilab because most of the syntax of Scilab and MATLAB are similar.

Most of the parts of the code that is being demonstrated can be used in MATLAB directly with only one or two changes. The codes you developed in Scilab can be simulated in MATLAB. We will now see what are all the steps involved in developing a 1D FE code.

Since the first lecture of the FEM module we have seen FEM starts with choosing a geometry, discretizing the geometry, formulating element coefficient matrices and then joining them to form global coefficient matrix. After applying boundary conditions, the resulting matrix

equation is solved. These are the only steps to solve any partial differential equation by using FEM.

Only in the case of transient analysis, we will be solving the matrix equations every time, but the procedure will be same. The same procedure can be used to solve various differential equations which we will see in this course. In next lecture, we will see the complete coding procedure in Scilab. We will stop here and continue in next class. Thank you.

(Refer Slide Time: 29:35)