Electrical Equipment and Machines: Finite Element Analysis
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Lecture 14
1D FEM: Problem Definition and Shape Function

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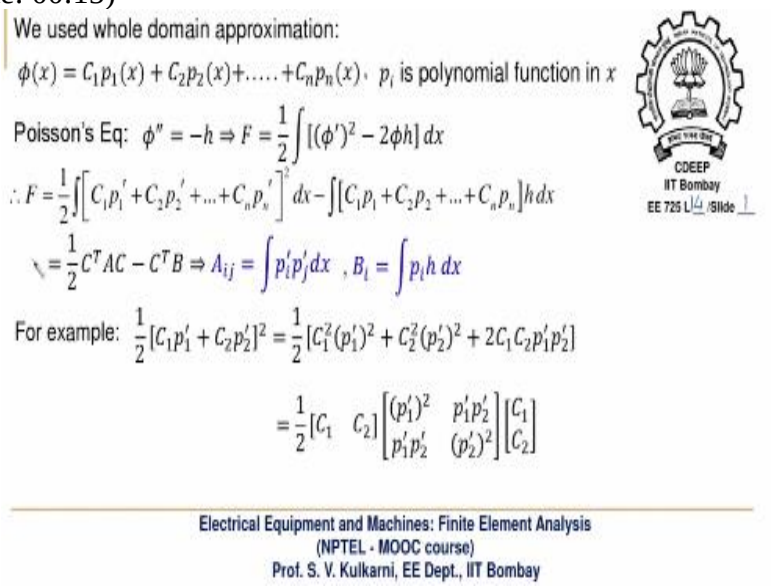

In the previous lecture, we discussed whole domain approximation, in which the potential is approximated using the following equation.

$$\phi(x) = C_1 p_1(x) + C_2 p_2(x) + \ldots + C_n p_n(x)$$

where we have assumed the unknown potential function as $\phi = C_0 + C_1 x + C_2 x^2$. So, you can then generalize this approximation as just $C_1 p_1(x) + C_2 p_2(x) + \ldots + C_n p_n(x)$. Effectively in $C_0 + C_1 x + C_2 x^2$, C_0 was replaced by C_1 to make it uniform in terms of formulation. And also remember it was only C_0 in the previous approximation and here in the above expression it is C_1 times some function of *x*, but that could be written as $C_0 x^0$. Here it is generalized for maintaining the uniformity of the formulation. The first term could be a constant also. Each p_i is a polynomial function of either *x* or x^2 or x^3 .

Consider Poisson's equation $\phi'' = -h$ and then the corresponding functional for Poisson's equation is

$$F = \frac{1}{2}\int [(\phi')^2 - 2\phi h]\, dx$$

Now, we will generalize the whole formulation that we did for the 1 D problem in the previous lecture.

So, what did we do there? In *F*, you can substitute ϕ' and ϕ derived using the above generalized expression. The expression of functional will be simply

$$F = \frac{1}{2}\int\left[C_1 p_1{}' + C_2 p_2{}' + ... + C_n p_n{}'\right]^2 dx - \int\left[C_1 p_1 + C_2 p_2 + ... + C_n p_n\right] h\, dx$$

The two terms of the previous integral are separated and that is why 2 does not appear in the second integral, because $F = \frac{1}{2}\int[(\phi')^2 - 2\phi h]\, dx$, $\frac{1}{2}$ gets cancelled with 2 in $2\phi h$. The above expression of *F* can be written in an elegant form using matrices as given below.

$$F = \frac{1}{2}C^T AC - C^T B \Rightarrow A_{ij} = \int p_i' p_j' dx\,, B_i = \int p_i h\, dx$$

Now, how did we get this? If we take a simple example with two terms as given below and without the integral

$$\frac{1}{2}[C_1 p_1' + C_2 p_2']^2 = \frac{1}{2}[C_1^2 (p_1')^2 + C_2^2 (p_2')^2 + 2C_1 C_2 p_1' p_2']$$

And the above expression can be written in an elegant way consisting of matrices as given below.

$$\frac{1}{2}[C_1^2 (p_1')^2 + C_2^2 (p_2')^2 + 2C_1 C_2 p_1' p_2'] = \frac{1}{2}[C_1 \quad C_2]\begin{bmatrix}(p_1')^2 & p_1' p_2' \\ p_1' p_2' & (p_2')^2\end{bmatrix}\begin{bmatrix}C_1 \\ C_2\end{bmatrix}$$

If you expand the matrix expression, you will get the expression on the left hand side. Now, you have got the above matrix expression of *F* which is functional or the energy which has to be minimized.

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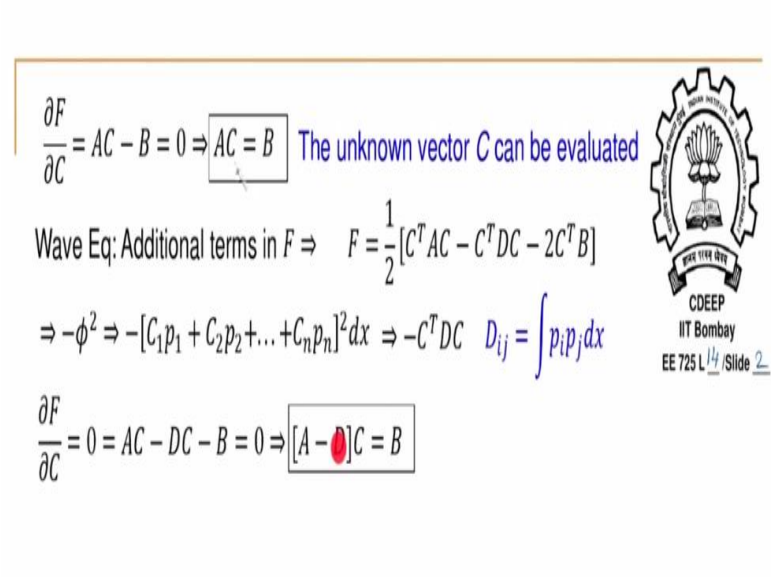

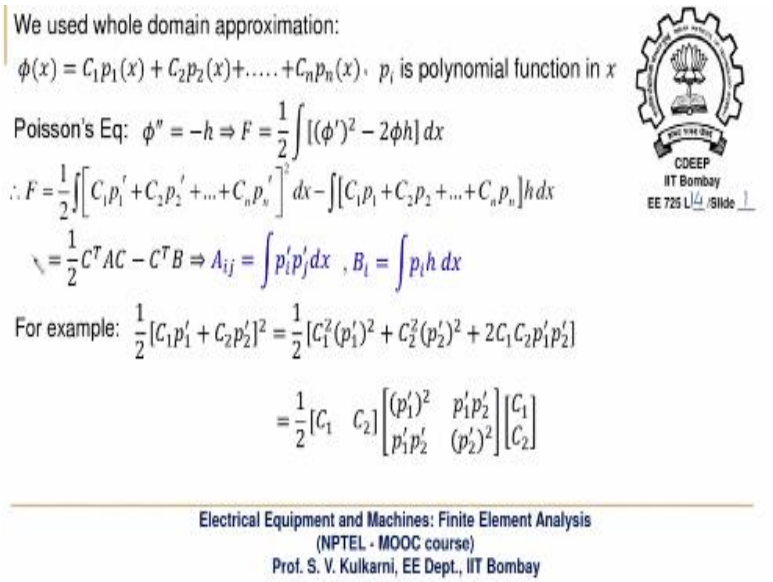

If you evaluate $\frac{\partial F}{\partial C} = 0$, it will simply reduce to $AC - B = 0$. The derivative of $\frac{1}{2}C^T AC$ will be AC because it is effectively $\frac{1}{2}AC^2$. C^2 derivative is $2C$. So, AC will be its derivative and derivative of BC will be simply B. Therefore, the total derivative of F will be $AC - B = 0$ which is the final linear system of equations. C has coefficients of the potential function ϕ and they can be easily evaluated because B is known which depends on source conditions and A can be evaluated using the procedure which we will see. If it was a wave equation, then you would get one more term $(C^T DC)$ along with the two terms as given below. Because for wave equation you have ϕ term.

$$F = \frac{1}{2}[C^T AC - C^T DC - 2C^T B]$$

$\phi'' + \phi + x = 0$ is the wave equation. Poisson's equation is just simply $\phi'' + x = 0$ where x stands for h source. So, when you have ϕ in the equation as mentioned to you earlier it will become ϕ^2 in the functional expression. This again can be written in an elegant matrix multiplication form as $-C^T DC$ where

$$D_{ij} = \int p_i p_j dx$$

Remember p_i and p_j are the corresponding polynomial expressions in x.

Then if you do minimization by calculating $\frac{\partial F}{\partial C} = 0$, you will get the following equation

$$\frac{\partial F}{\partial C} = 0 = AC - DC - B = 0 \Rightarrow [A - D]C = B$$

Again in the above equation, B is known and it is a function of source condition and here A and D matrices are also known. A matrix depends on the material properties and the geometrical dimensions. We will see more of this and what do they mean little later.

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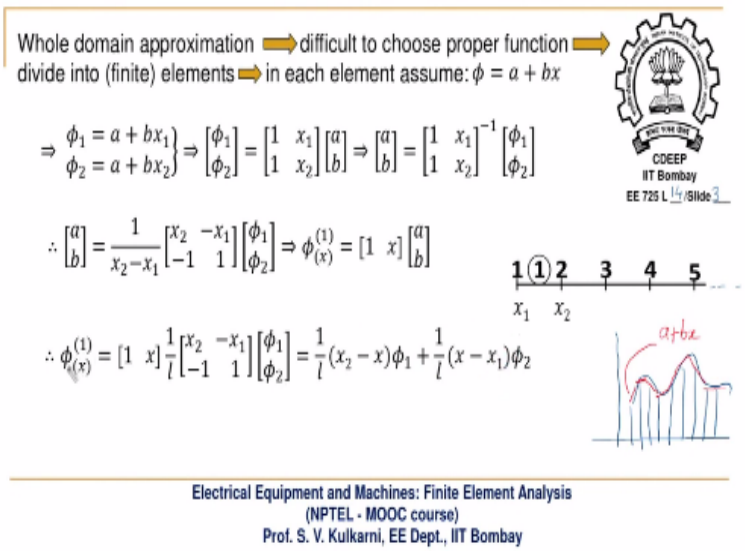
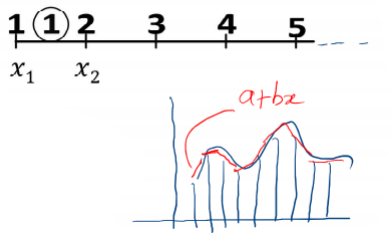


Now we will take the same example of Poisson's equation in 1D and we will see how this generalized formulation can be used to get the final set of equations and the solution. What we

did earlier was whole domain approximation. It is difficult to choose a proper function like $C_0 + C_1x + C_2x^2 + C_3x^3$ for a complicated problem.

If the solution of the problem is a complicated potential distribution, it will be difficult to find a proper polynomial expression which can fit closely to its exact solution. So, what we can do, instead of choosing a polynomial function over the entire domain, we will approximate the solution for each finite element. Now, we are slowly getting into finite element formulation. Now, we will divide that one-dimensional domain as shown in the following figure.



Suppose the one dimensional domain is divided into a number of segments which are called as finite elements. Then over each element, for example for element 1 between nodes 1 and 2 we can assume ϕ as $a + bx$ which is much simpler. Earlier in one of the first lectures also I must have mentioned to you this, when we saw the difference between numerical and analytical techniques. Suppose if you have some actual solution like the one shown in the above figure then it is difficult to assume an approximate solution which can closely match the solution over the entire domain. Rather than approximating the entire solution, you could subdivide this into a number of segments. Over each of these segments, you can assume $a + bx$ as the solution.

Because these segments are straight lines, potential can be represented as $a + bx$. Remember the values of a and b will be different for different segments. That means for each segment, the unknowns are a and b, but later on, we will see that we will not solve the problem in terms of a and b. We will eliminate these parameters, a and b, and we will solve it in terms of nodal potentials ϕ_1 and ϕ_2.

Going further, if we agree that it is easier to approximate the solution over each segment using a linear approximation instead of complicated higher-order approximations, for the first element, let $\phi = a + bx$. The value of potentials at the two nodes can be determined by substituting the values of x_1 and x_2 and the corresponding ϕ_1 and ϕ_2 are as given below

$$\phi_1 = a + bx_1$$
$$\phi_2 = a + bx_2$$

So, that will give ϕ_1 and ϕ_2 in matrix form as given below.

$$\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}\begin{bmatrix}a\\ b\end{bmatrix}$$

You can write these equations in a matrix form because ϕ_1 will be simply $a + bx_1$ and ϕ_2 will be $a + bx_2$. Now, we will eliminate these two parameters a and b by rearranging the above matrix equation as given below.

$$\begin{bmatrix}a\\ b\end{bmatrix} = \begin{bmatrix}1 & x_1\\ 1 & x_2\end{bmatrix}^{-1}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$

If you expand the inverse term, the above equation reduces to

$$\begin{bmatrix}a\\ b\end{bmatrix} = \frac{1}{x_2\text{-}x_1}\begin{bmatrix}x_2 & \text{-}x_1\\ \text{-}1 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$

Here, $x_2 - x_1 = l$ and further you get the expressions of a and b. So, the variation of potential in element 1 as a function of x can be written as

$$\phi^{(1)}_{(x)} = [1 \quad x]\begin{bmatrix}a\\ b\end{bmatrix}$$

Now we are replacing the column vector $\begin{bmatrix}a\\ b\end{bmatrix}$ by the previous equation, we obtain

$$\phi^{(1)}_{(x)} = [1 \quad x]\begin{bmatrix}a\\ b\end{bmatrix} = [1 \quad x]\frac{1}{l}\begin{bmatrix}x_2 & \text{-}x_1\\ \text{-}1 & 1\end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix} = \frac{1}{l}(x2 - x)\phi_1 + \frac{1}{l}(x - x1)\phi_2$$

Effectively what we have done is the potential ϕ in the element 1 is expressed in terms of the potential of end nodes. Why we are doing this? Because finally we want to minimize the overall energy of the system using energy for each element. For example, energy is a function of potential at every point in that element. But that would be cumbersome to handle. So we are expressing the energy of an element in terms of potentials of its end nodes and eventually you can imagine that we will get the total energy expression as a function of only the nodal potentials. If we can calculate energy for one element, we can do it for the other elements in the domain using the same procedure. So, that means the total energy will be a function of only the nodal potential values.

And the energy can be minimized by taking derivative of F with respect to each of the potentials and equating it to 0. Then we will get a set of equations which can be solved. So, in nutshell, this is what finite element method is and we will see more of this and we will consolidate this learning as we go ahead.

Now, it is a matter of details. What we are doing is, we are calling $\frac{1}{l}(x_2 - x)$ as N_1 and $\frac{1}{l}(x - x_1)$as N_2. So, $\phi_x^{(1)}$potential in at any x in element 1 can be expressed as

$$\phi_x^{(1)} = N_1\phi_1 + N_2\phi_2$$

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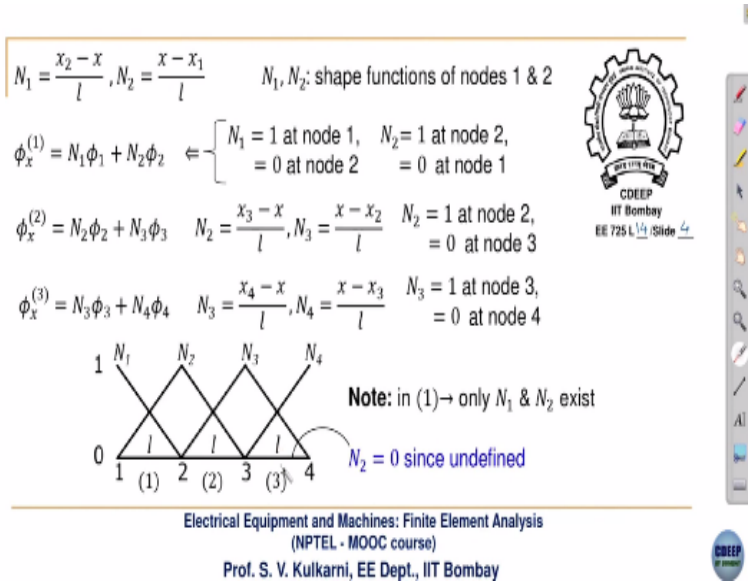


Now, what are these N_1 and N_2? N_1 and N_2 are the shape functions of nodes 1 and 2. What are their properties? You can see $N_1 = 1$ at node number 1 and it is equal to 0 at node number 2. You can verify that by substituting the coordinate of x_1 in the expression of N_1, it will be equal to 1 as given below

$$N_1(x_1) = \frac{x_2 - x_1}{l} = \frac{x_2 - x_1}{x_2 - x_1} = 1$$

But the value of N_1 at node number 2 $(x = x_2)$ will be 0 as given below.

$$N_1(x_2) = \frac{x_2 - x_2}{l} = \frac{0}{x_2 - x_1} = 0$$

Similarly $N_2 = 1$ at node number 2 and equal to 0 at node number 1. This has to be because see the following expression of ϕ in element 1

$$\phi_x^{(1)} = N_1\phi_1 + N_2\phi_2$$

Remember, the number in the bracket represents element number. In the above equation, it is element 1.

Now, if I calculate $\phi_{x_1}^{(1)}$at node number 1, it will be equal to $N_1(x_1)\phi_1 + N_2(x_1)\phi_2$, but for $\phi_{x_1}^{(1)} = \phi_1$, N_1 has to be 1 and N_2 has to be 0. Similarly, $\phi_{x_2}^{(1)} = N_1(x_2)\phi_1 + N_2(x_2)$. So for $\phi_{x_2}^{(1)} = \phi_2$, N_2 has to be 1 and N_1 has to be 0 at node 2. So, these properties of shape functions are logical.

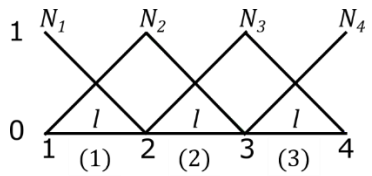
Now, similarly, potential in element 2 can be written as

$$\phi_x^{(2)} = N_2\phi_2 + N_3\phi_3, \qquad N_2 = \frac{x_3 - x}{l}, N_3 = \frac{x - x_2}{l}$$

Because element 2 is between nodes 2 and 3. Then the same set of properties are applicable for N_2 and N_3. Here $N_3 = 1$ at node 3 and $N_3 = 0$ at node 2. Similarly, potential for element 3 is given below

$$\phi_x^{(3)} = N_3\phi_3 + N_4\phi_4 \qquad N_3 = \frac{x_4 - x}{l}, N_4 = \frac{x - x_3}{l}$$

Now we are taking a 3-element example with 4 nodes as shown in the following figure. The variations of shape functions in each element are also indicated. The length of each element is l so the total length of the domain is $3l$



Here N_1 is shape function of node number 1 and it will be 0 at node number 2 and this is indicated in the above figure. N_2 has now 2 sets of expressions, which are given below.

$$N_2 = \begin{cases} \frac{x - x_1}{l}, & x_1 \le x \le x_2 \\ \frac{x_3 - x}{l}, & x_2 \le x \le x_3 \end{cases}$$

N_2 is valid in elements 1 and 2. N_2 is 1 at node number 2 in both elements and it goes to 0 at node number 1 and node number 3. So, that is why N_2 has two expressions. N_2 for the first element is $\frac{x-x_1}{l}$ and N_2 expression for the second element in $\frac{x_3-x}{l}$. So, we have to remember that the expression of N_2 for the first element is not valid for the second element. Although, you can always substitute some value and you will get an answer, but it is not valid. The expression of N_2 of element 1 is valid only in the first element. The expression of N_2 of element 2 is valid only in this second element. So, in element number 1, only N_1 and N_2 exist.

In element 2, N_2 and N_3 exist and in element 3, N_3 and N_4 exist. $N_2 = 0$ in element 3 because it is undefined or invalid. If you substitute any value of *x* of element 3 in N_2, you will get some value, but it is not valid.

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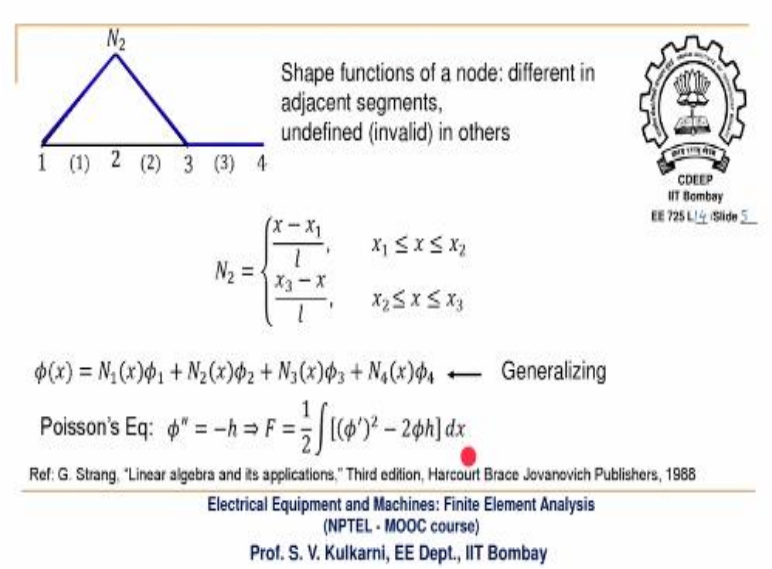
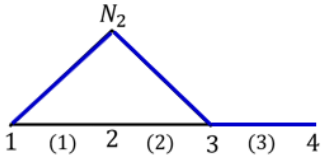


So, the properties that are discussed in the previous slide are explained in the following figure.



The function shown in the above figure is also called as a rooftop function in some textbooks. In the above figure, you can see that the value of N_2 is 1 at node number 2 and it is 0 at all other nodes 1, 3 and 4, and for the entire element 3, N_2 is not defined.

N_2 has two expressions which are given below.

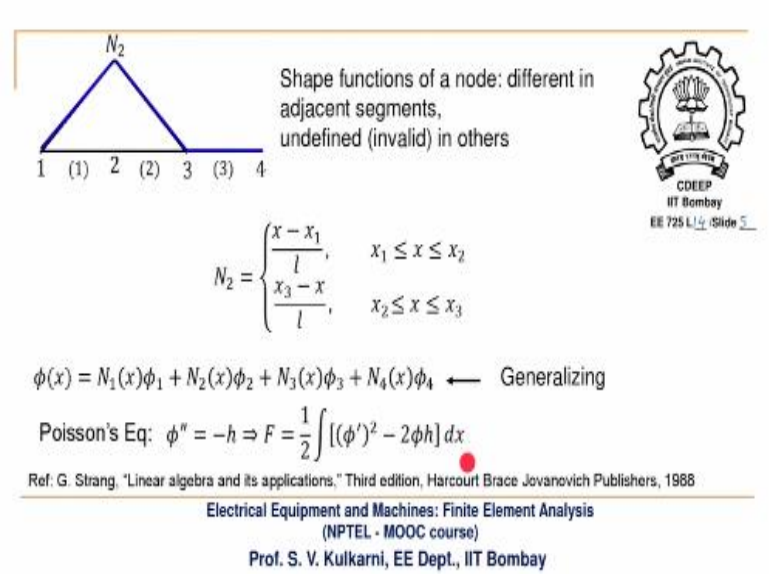
$$N_2 = \begin{cases} \dfrac{x - x_1}{l}, & x_1 \leq x \leq x_2 \\ \dfrac{x_3 - x}{l}, & x_2 \leq x \leq x_3 \end{cases}$$

The first equation is valid for the first element between x_1 and x_2 and the other equation is valid for the element between x_2 and x_3. Remember, at node 2 whether you evaluate the first expression or the second expression, both will give you 1, that is why, it is less than or equal to is defined for the limits of both the equations. So, now we can generalize that the ϕ at any x in the entire domain with 3 elements can be written as

$$\phi(x) = N_1(x)\phi_1 + N_2(x)\phi_2 + N_3(x)\phi_3 + N_4(x)\phi_4$$

For element 1, it will be just $N_1(x)\phi_1 + N_2(x)\phi_2$ because N_3 and N_4 are equal to 0. Similarly, for element 2, it will be $N_2(x)\phi_2 + N_3(x)\phi_3$. So the above expression is a generalized expression of ϕ.

Now, let us solve Poisson's equation and see the FEM procedure in 1D. So first we will see 1D and then later we will see FE procedure for 2D problems. So, again we are solving Poisson's equations $(\phi'' + h = 0)$ where h represents a source and we have seen the energy functional to be minimized in this problem is as given below.



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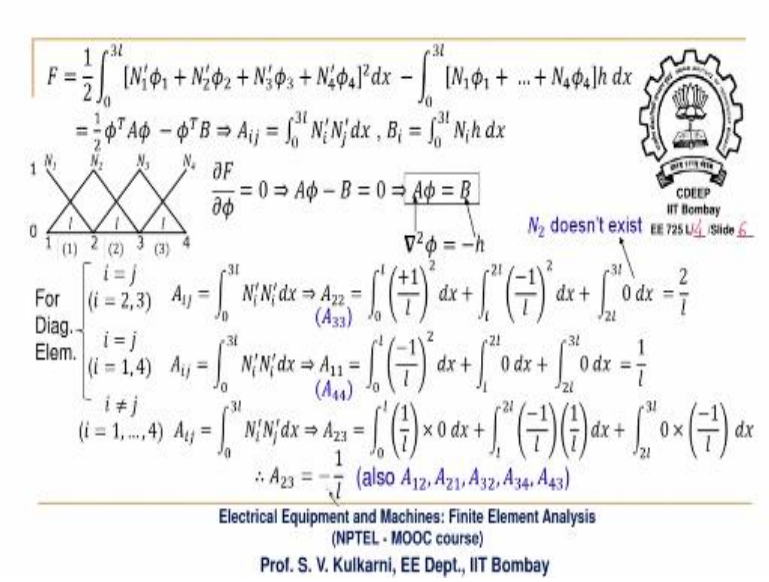


The functional for Poisson's equation is modified as given below by substituting the general expression for ϕ.

$$F = \frac{1}{2}\int_0^{3l} [N_1'\phi_1 + N_2'\phi_2 + N_3'\phi_3 + N_4'\phi_4]^2 dx - \int_0^{3l} [N_1\phi_1 + \ldots + N_4\phi_4]h\, dx$$

Remember one important thing in the expression of $(\phi')^2$, the derivative sign does not appear on ϕ_1, ϕ_2 and all that, because they are not varying with *x* and *y*. N_1, N_2, N_3 and N_4 vary with *x*, because they are shape functions. As discussed in the previous lecture, if you remember in

the variational formulation of FEM, we have not varied potentials at every point. So ϕ at every point is not a function of *x* in the variational procedure. In this procedure, we are changing the potential values at every *x* and we will see which combination of potentials at various nodes is going to give the minimum energy.

So, ϕ_i potentials at various points are not made to vary with *x* in the energy minimization procedure. Here only the shape functions are functions of *x*. So, that is why, we are taking derivatives only for shape functions in the above expression and ϕ_1, ϕ_2, ϕ_3 and ϕ_4 are not varying with *x* in the variational formulation.

The limits of the integral are from 0 to $3l$ because $3l$ is the total length of the domain under consideration. Now this can be written in matrix form as given below.

$$F = \frac{1}{2}\phi^T A\phi \ - \phi^T B \Rightarrow A_{ij} = \int_0^{3l} N_i' N_j' \, dx, B_i = \int_0^{3l} N_i h \, dx$$

And then when you evaluate $\frac{\partial F}{\partial \phi} = 0$, you will get $A\phi - B = 0$ that will give you $A\phi = B$.

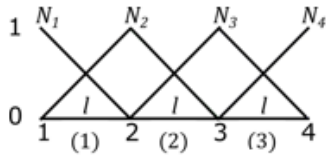
Now, see the difference, earlier it was $AC = B$ in the previous case when we had done the whole domain approximation in terms of $\phi = C_0 + C_1 x + C_2 x^2$. The unknowns were the coefficients C_0, C_1 and C_2 and that is the matrix C and that is the whole domain approximation and over the entire domain we approximated or we assumed some potential distribution using a polynomial function. But in the present case, we have not done that, we have approximated the potential over each segment or element using $\phi^{(1)} = a + bx$.

And then we eliminated the parameters a and b and we made the potential approximation as a function of the end node potential values and that is the reason why our unknown variables become the nodal potentials at all the nodes 1, 2, 3, and 4. So, from the case of coefficients being unknown, we have come to the case where nodal potentials are unknown variables. Earlier, the coefficients (C_0, C_1, C_2) were variables.

Now, here ϕ_0, ϕ_1, ϕ_2 are unknowns and then when we minimize the energy with respect to ϕ_is, and we will finally get potentials at various nodes in the problem domain. Of course, there will be some boundary conditions, in terms of some known potentials, to be applied; we will

see that later. Now, finally we have got the set of linear equations ($A\phi = B$) after applying the FEM procedure.

So, the FEM procedure involves discretization of the problem domain into segments as given below.



After the discretization procedure, we got $A\phi = B$. Earlier ϕ was a continuous variable when it was a whole domain approximation or if it was some analytical solution, then ϕ would be a continuous variable.

But in the FE formulation, ϕ is not a continuous variable because it is defined only at the nodes of the discretized domain. So, from the continuous domain, we have come to a discretized domain. A_{ij} of the final matrix equation can be written as

$$A_{ij} = \int_0^{3l} N_i' N_i' dx$$

Now how do we evaluate this integral? When $i = j$, and there are 2 cases here, one is for nodes 2 and 3 and the other is for nodes 1 and 4. You can write A_{ii} as

$$A_{ij} = \int_0^{3l} N_i' N_i' dx \Rightarrow \underset{(A_{33})}{A_{22}} = \int_0^{l} \left(\frac{+1}{l}\right)^2 dx + \int_l^{2l} \left(\frac{-1}{l}\right)^2 dx + \int_{2l}^{3l} 0\, dx = \frac{2}{l}$$

In the above expressions, $i = j = 2$ and $N_i = N_j = N_2$. So, now we have to split this 0 to $3l$ integral into three integrals 0 to l, l to $2l$, and $2l$ to $3l$. The N_2 in the first integral with limits 0 to l will be simply $\frac{1}{l}$ and it basically defines the slope of the shape function.

In the second element, from l to $2l$, N_2' is the slope $-\frac{1}{l}$ and in the third integral i.e, the third element N_2 is not defined and it is equal to 0. So, if you evaluate the two integrals, their sum will be $\frac{2}{l}$ as given above. The same thing will be true for A_{33} also with $N_i = N_j = N_3$. In the

integral for A_{33}, in the first element will be 0 because N_3 is not defined. Also the value of A_{33} is equal to $\frac{2}{l}$.

But A_{11} and A_{44} are different from A_{22} and A_{33} because nodes 2 and 3 are common to 2 elements, whereas 1 and 4 are appearing only in one element.

Now we will calculate A_{11}. $N_1' = -\frac{1}{l}$ because N_1 is reducing in element 1 so it is reducing slope which is equal to $-\frac{1}{l}$ and N_1 is 0 in the other two elements. In the expression of A_{11}, the values of two integrals will be zero as given below.

$$A_{ij} = \int_0^{3l} N_i' N_i' dx \Rightarrow \underset{(A_{44})}{A_{11}} = \int_0^{l} \left(\frac{-1}{l}\right)^2 dx + \int_l^{2l} 0\, dx + \int_{2l}^{3l} 0\, dx = \frac{1}{l}$$

And N_1 appears only in the first element and for the other two integrals it will be 0. So, the value of $A_{11} = \frac{1}{l}$ and now let us consider the off diagonal elements that is for $i \neq j$. Now, in the expression of A_{ij}, you will get N_i' and N_j' as given below for A_{23}.

$$\underset{(i=1,\ldots,4)}{i \neq j}\quad A_{ij} = \int_0^{3l} N_i' N_j' dx \Rightarrow A_{23} = \int_0^{l} \left(\frac{1}{l}\right) \times 0\, dx + \int_l^{2l} \left(\frac{-1}{l}\right)\left(\frac{1}{l}\right) dx + \int_{2l}^{3l} 0 \times \left(\frac{-1}{l}\right) dx$$
$$\therefore A_{23} = -\frac{1}{l} \quad (\text{also } A_{12}, A_{21}, A_{32}, A_{34}, A_{43})$$

Let us consider the A_{23} entry and its value is equal to $\frac{1}{l}$. The contributions of the first and second integrals are equal to 0 because N_3' is 0 (or not defined) in element 1 and N_2' is 0 (or not defined) in element 3. So, $A_{23} = -\frac{1}{l}$. Only in segment 2, both N_2 and N_3 are non zero and will contribute to A_{23}. And that is why N_2' and N_3' exist is this segment and the slope of N_3 is positive and it is equal to $1/l$ and the slope of N_2 is negative and it is equal to $-1/l$. and that is why you get the value of A_{23} as simplify $-1/l$. And this will be valid for other off diagonal entries where there is connectivity like 12, 21, 32, 23, 34, and 43. So, where there is a direct connectivity between nodes, you will get $-1/l$ and where there is no connectivity, you will get the off diagonal entry as 0.

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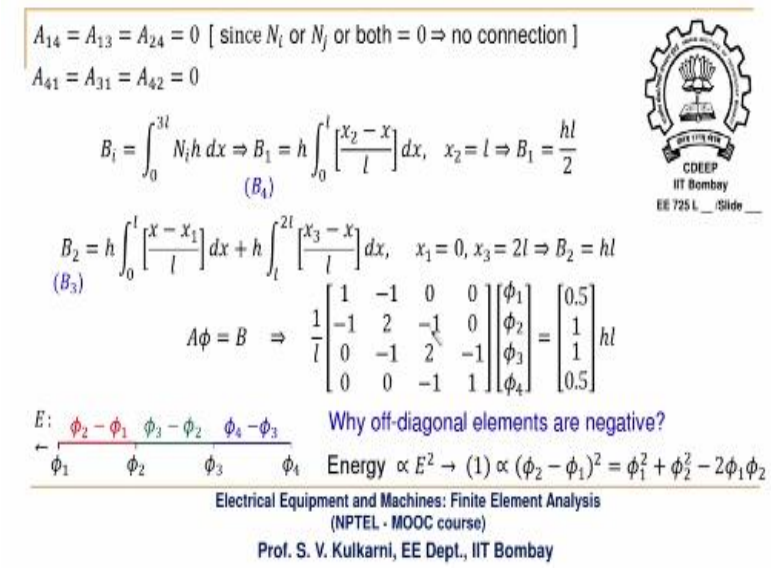
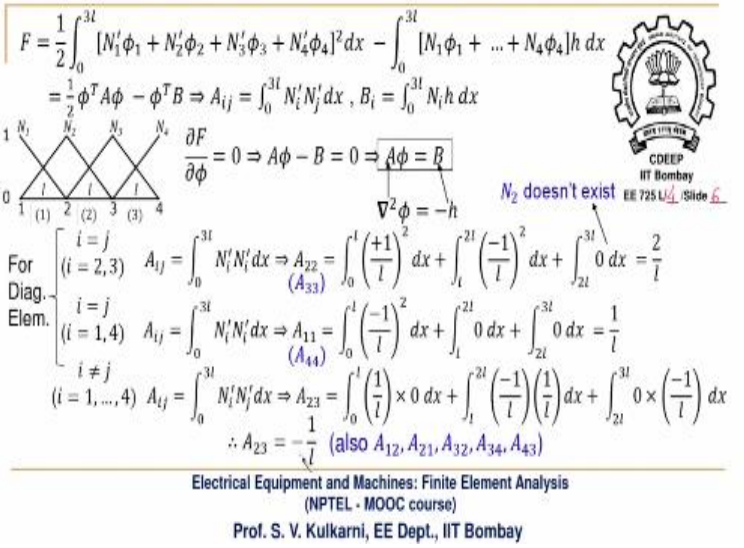




There are 6 off diagonal elements where there is no connectivity, like between nodes 1 and 4 because either N_1 is 0 or N_4 is equal to 0 in the three integrals. So when there is no connectivity then one of them will be 0. For example if we are calculating A_{14}, you will find that in every segment either $N_1 = 0$ or $N_4 = 0$ or both are 0. For example, in segment 2 both N_1 and N_4 are 0. So, effectively what we have done? We have evaluated all the entries of A matrix. Similarly we can evaluate B matrix which is defined as given below

$$B_i = \int_0^{3l} N_i h\, dx$$

B matrix is representing the source. For example if you evaluate B_1, then the above integral is valid only for element between 0 and l because N_1 is 0 or not valid in the other two segments.

So, the integral of the above integral reduces to 0 to l instead of 0 to $3l$ because in the second and third segments, N_1 is 0. You substitute the expression of N_1 in the formula of B_1 as given below.

$$B_1 = h\int_0^l \left[\frac{x_2 - x}{l}\right] dx, \quad x_2 = l$$

Then you can evaluate the integral and you will get

$$B_1 = \frac{hl}{2}$$

The same thing will be for B_4.

Now, nodes 2 and 3 are common to two segments that is why you get two terms in the corresponding source term as given below

$$B_2 = h\int_0^l \left[\frac{x - x_1}{l}\right] dx + h\int_l^{2l} \left[\frac{x_3 - x}{l}\right] dx, \quad x_1 = 0,\ x_3 = 2l \Rightarrow B_2 = hl$$

(B_3)

Because N_2 and N_3 are non zero for over two segments so that is why you get two terms in the above expression and the corresponding expression for B_2 is hl.

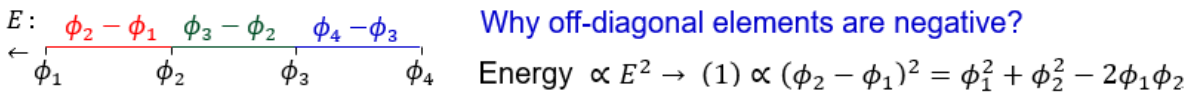
Effectively, for nodes 2 and 3, both elements are contributing because they are common. So the final $A\phi = B$ equation can be written as given below.

$$A\phi = B \quad \Rightarrow \quad \frac{1}{l}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 1 \\ 1 \\ 0.5 \end{bmatrix} hl$$

Then the above matrix equation can be solved after substituting the boundary conditions and the unknowns in the above equation is the ϕ matrix and the right hand side matrix (B matrix) is known because h is known and it represents source. So, then you can calculate the unknown

vector which is $[\phi_1 \quad \phi_2 \quad \phi_3 \quad \phi_4]^T$, some of the potentials of the unknown vector would be known.

Now, one question that may arise is why non zero off diagonal entries of A are negative? Intuitively, you can understand using the following explanation.



Our basic thing is the electrostatic energy calculated by using $\frac{1}{2}\epsilon E^2$. So, energy for each element is proportional to the corresponding square of the magnitude of electric field intensity. So, now in the above example, there are three elements and later we are going to take $\phi_1 = 0$ and $\phi_4 = 1$ as boundary conditions. In this case, the electric field intensity will be directed from nodes 4 to 1.

So, that means electric field intensity magnitude will be proportional to $\phi_2 - \phi_1$, E^2 for element number 1 will be proportional to $(\phi_2 - \phi_1)^2$ and if we expand this you get $(\phi_2 - \phi_1)^2 = \phi_2^2 + \phi_1^2 - 2\phi_1\phi_2$. In this expression, the square terms are diagonal entries which will be positive and $-2\phi_1\phi_2$ represent off diagonal terms. This is just a logical explanation of why we are getting a negative sign for off diagonal entries.

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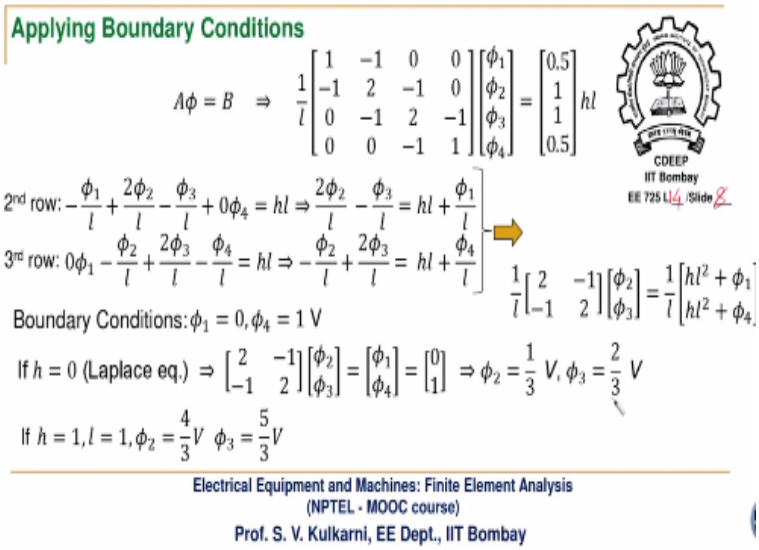

$A\phi = B$ is the final matrix equation. Now, we impose the boundary conditions which are in terms of $\phi_1$and ϕ_4. The boundary conditions for this problem are $\phi_1 = 0$ and $\phi_4 = 1\,V$. Now, to apply these boundary conditions, we are expanding the second and third rows of the matrix equation $A\phi = B$ as given below.

$$2^{nd}\text{ row: } -\frac{\phi_1}{l} + \frac{2\phi_2}{l} - \frac{\phi_3}{l} + 0\phi_4 = hl \Rightarrow \frac{2\phi_2}{l} - \frac{\phi_3}{l} = hl + \frac{\phi_1}{l}$$
$$3^{rd}\text{ row: } 0\phi_1 - \frac{\phi_2}{l} + \frac{2\phi_3}{l} - \frac{\phi_4}{l} = hl \Rightarrow -\frac{\phi_2}{l} + \frac{2\phi_3}{l} = hl + \frac{\phi_4}{l}$$

Now the two terms ϕ_1 and ϕ_4 are transferred to the right hand side as given in the above equations. Because the right hand side in any linear system of equations like $AX = B$ represents known quantities.

In the 2nd row of the equation, the term corresponding to ϕ_4 is getting multiplied by 0 so that is why, it is $0\phi_4$, whereas in third row 0 gets multiplied by ϕ_1 so that is why $0\phi_1$ and then we transfer the known potentials (ϕ_1 and ϕ_4) in both these equations on the right hand side. So, the right hand side is completely known, because *h, l,* ϕ_1 and ϕ_4 are known.

When we impose boundary conditions, effectively the 4×4 matrix equation gets converted to 2×2 main matrix equation as given below.

$$\frac{1}{l}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \frac{1}{l}\begin{bmatrix} hl^2 + \phi_1 \\ hl^2 + \phi_4 \end{bmatrix}$$

If you solve this matrix equation, you will immediately get the solution. So, for example, if h = 0, then it is Laplace's equation and then in the above equation if we substitute $h = 0$ it will be only ϕ_1 and ϕ_4 on the right hand side and that is equal to 0 and 1 respectively as given below.

$$\text{If } h = 0 \text{ (Laplace eq.)} \Rightarrow \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} \phi_2 \\ \phi_3 \end{bmatrix} = \begin{bmatrix} \phi_1 \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

If you solve this above equation, you will get $\phi_2 = \frac{1}{3}$ and $\phi_3 = \frac{2}{3}$ which is obvious because the whole domain is between 0 and 1 and we considered the nodes as equi-spaced points. So,

the potential at second node will be $\frac{1}{3}$ V and the third node will have $\frac{2}{3}$ V. If $h = 1$ then it is Poisson's equation and you will get ϕ_2 and ϕ_3 as given below

$$\text{If } h = 1, l = 1, \phi_2 = \frac{4}{3}V \quad \phi_3 = \frac{5}{3}V$$

Presence of some charge throughout the region will change the potentials as compared to Laplace's equation. We will stop here and continue next time. Thank you.

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