Electrical Equipment and Machines: Finite Elements Analysis
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Lecture 13
Whole Domain Approximation

Welcome to the 13th lecture of this course, the previous lecture was little mathematical, but if you understood the content in that lecture, then FEM theory is more or less understood by you. Now the next lectures are more of applications to various PDEs.

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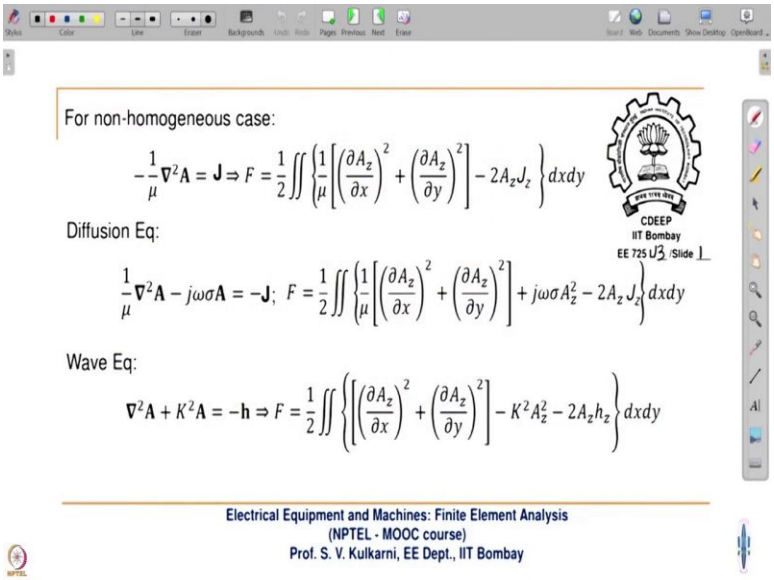

Now, for example, we consider a non-homogeneous case defined by the following equation that we saw in the last slide of the previous lecture.

$$-\frac{1}{\mu}\nabla^2 \mathbf{A} = \mathbf{J} \Rightarrow F = \frac{1}{2}\iint \left\{\frac{1}{\mu}\left[\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right] - 2A_z J_z\right\} dxdy$$

In the previous lecture, we had associated μ with **J** on the right hand side of the equation as given in the following equation.

$$\nabla^2 \mathbf{A} = -\mu\mathbf{J} \Rightarrow F = \frac{1}{2}\iint \left\{\left[\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right] - 2\mu A_z J_z\right\} dxdy$$

This equation will be applicable for a single material case. But, generally μ will vary from point to point, because there could be more than one material in the problem domain.

In case of nonlinear materials like ferromagnetic materials, μ can vary from point to point. That means, when we discretize the whole geometry into a number of elements, in each of the elements we will assume μ as constant. So from element to element, μ will change.

That is why, we cannot associate μ with **J** in any case. So, μ should be associated with the ∇^2 term and it should be inside the integral as given in the previous equation. Now, for example, when we eventually see the finite element formulation, the $dxdy$ term in the integral corresponds to the element level area. But till now we have assumed $dxdy$ for the whole domain.

So, $dxdy$ corresponds to the entire problem domain area, but when we go to finite element formulation, this $dxdy$ will corresponds with every element. Also, you cannot associate μ to **J** because it is current density of some winding.

That is why, that expression has to be recast by taking $1/\mu$ with ∇^2 term. When we solve a finite element problem, we need to consider the following points. First thing is the source term, then the material properties included with the ∇^2 term and the boundary conditions.

So, what actually does the first term which is enclosed in the following integral represent?

$$-\frac{1}{\mu}\nabla^2\mathbf{A} = \mathbf{J} \Rightarrow F = \frac{1}{2}\iint \left\{\frac{1}{\mu}\left[\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right] - 2A_z J_z \right\} dxdy$$

The bracketed term represents the geometry and material properties of the problem domain. This term finally gives element level coefficient matrix and global coefficient matrix when we apply finite element procedure. The global coefficient matrix will have information about material properties and geometry.

The functional for the diffusion equation is given below.

Diffusion Eq:

$$\frac{1}{\mu}\nabla^2\mathbf{A} - j\omega\sigma\mathbf{A} = -\mathbf{J};\;\; F = \frac{1}{2}\iint\left\{\frac{1}{\mu}\left[\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right] + j\omega\sigma A_z^2 - 2A_z J_z\right\}dxdy$$

Theory about this equation also we have seen in basics. In the above expression of *F* you can see the $\nabla^2 A_z$ term in the PDE leads to the same term which we saw in Poisson's equation case in electrostatics.

Whatever is on the right hand side of the PDE, for example, for the diffusion equation there is a –**J** term, the corresponding term in the function will also have minus sign. If the $-j\omega\sigma A_z$ term in the PDE is taken on to the right hand side then its sign will be plus and the sign of the corresponding term in the functional is also plus. So, whatever is the sign of the terms (other than $\nabla^2 A_z$ term) on the right hand side of the PDE, that sign repeats in the functional expression. Also those terms get multiplied with the corresponding potential (here, A_z). For example, for diffusion equation, the –J term on the right hand side in the PDE is converted to $-A_z J$ term in the functional. Similarly, the diffusion term on the right hand side of the PDE is $j\omega\sigma A_z$ and it is converted to $j\omega\sigma A_z^2$ in the functional expression. Similarly the functional for the wave equation is given below.

Wave Eq:

$$\nabla^2\mathbf{A} + K^2\mathbf{A} = -\mathbf{h} \Rightarrow F = \frac{1}{2}\iint\left\{\left[\left(\frac{\partial A_z}{\partial x}\right)^2 + \left(\frac{\partial A_z}{\partial y}\right)^2\right] - K^2 A_z^2 - 2A_z h_z\right\}dxdy$$

Now in this equation, *h* will be some source in the wave equation. For this equation also, the corresponding term of $\nabla^2 A_z$ in the functional is same as in case of Poisson's equation.

Now, if you bring the $K^2 A_z$ of the PDE of the wave equation on the right hand side, then it becomes $-K^2 A_z$. In the functional, this term $K^2 A_z$ is multiplied with A_z. So, that is why, $K^2 A_z$ becomes $K^2 A_z^2$ in the functional expression.

By inspection, we can obtain the functional expressions. Similarly, starting with PDE, you can derive the functional. We have covered this derivation in the previous lecture. In the functionals for diffusion and wave equations, you can see that $K^2 A_z$ does not get multiplied by 2, because it is converted to a square term in the functional.You will understand why there is no 2 for $K^2 A_z$ term and 2 gets multiplied to the source term when you actually do the procedure.

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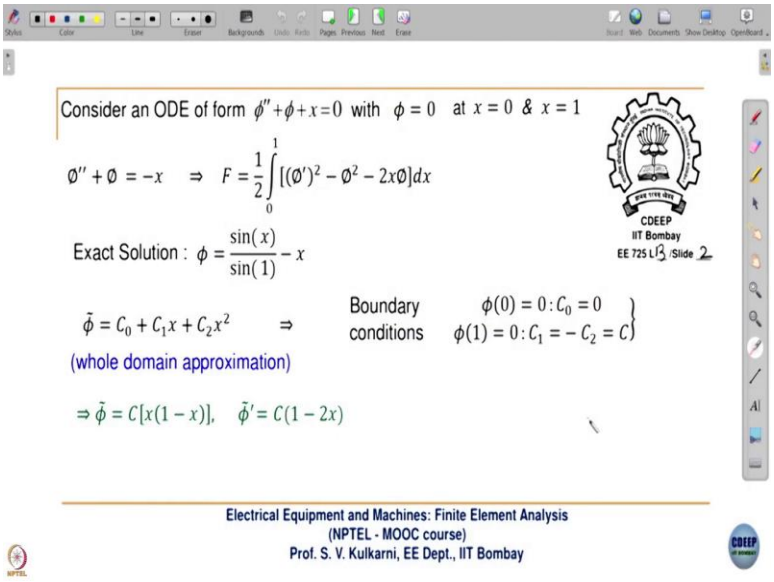

Now, we have finished the complicated theory related to FEM. Later on, we will see complexity in formulations. We will see some simple examples, so that our learning will get consolidated.

Consider an ODE as given below .

$$\phi'' + \phi + x = 0$$

The boundary conditions for this problem are $\phi = 0$ at $x = 0$ and $x = 1$, that means, our one dimensional problem domain is from $x = 0$ to $x = 1$. And then you can recast this equation as given below.

$$\emptyset'' + \emptyset = -x \quad \Rightarrow \quad F = \frac{1}{2}\int_0^1 [(\emptyset')^2 - \emptyset^2 - 2x\emptyset]dx$$

Now if you apply the same logic that I explained to you, by inspection you can write the functional. The ϕ'' in the ODE will become $(\phi')^2$ in the functional. If you bring ϕ in the ODE to right hand side, it will become $-\phi$ and that gets multiplied by the potential (ϕ), so, it becomes $-\phi^2$ and that does not get multiplied by 2, as I explained earlier. And this *x* is the source term, do not consider this *x* as just a geometry variable. What it means is, the source at any *x* which is numerically equal to the *x* at that point.

So, in effect *x* is representing the source and that is why it is multiplied with 2. By inspection, you can derive the functional. If it was simply Laplace equation, the coefficients of other two terms $(\phi \text{ and } x)$ would be 0, you will get only $\frac{1}{2}(\phi')^2$.

If it is Poisson's equation $(\phi'' = -x)$, then only this ϕ^2 term will go to 0. So, the integrand of functional will be only $(\phi')^2 - 2x\phi$. So the corresponding functional for Poisson's equation is $\frac{1}{2}\int_0^1((\phi')^2 - 2x\phi)\, dx$.

Now, one can easily find the exact solution for this ODE as

$$\text{Exact Solution : } \phi = \frac{\sin(x)}{\sin(1)} - x$$

You can verify this solution by applying boundary conditions, you will substitute $x = 0$ and $x = 1$ in the above expression, you will get $\phi = 0$, that is a simple verification of the exact solution.

Now, we are slowly getting into actual procedures of finite element method. In the finite element method what we have to do? In the above case, the analytical solution does exist, but you can imagine if the ODE or PDE is more complicated, then you do not have the exact solution. In that case, you have to assume something. Let us assume some expression for $\tilde{\phi}$. Henceforth, whenever you see $\tilde{\phi}$ then it is an approximate solution.

Assuming $\tilde{\phi} = C_0 + C_1 x + C_2 x^2$, it may be correct or wrong. But even if it is wrong, since, this is a numerical technique, there will be some error, but that error can be minimized. Only when it is exactly correct, the exact solution occurs, which is generally not possible for practical problems.

Generally in numerical methods, we are not exactly correct, but we are not either too wrong also. There will be some errors because we are approximating the solution and those errors can be minimized by standard procedures in any numerical technique.

So, $\tilde{\phi} = C_0 + C_1 x + C_2 x^2$ is an approximated solution. And now we apply boundary conditions $\tilde{\phi} = 0$ at $x = 0$ and $x = 1$ which we already know. For $\tilde{\phi}(0) = 0$, you substitute $x = 0$ in the expression of $\tilde{\phi}$ which results in $C_0 = 0$.

Now, you use the other boundary condition $\tilde{\phi}(1) = 0$, then you will get $C_1 + C_2 = 0$. And just to make it simple, consider $C_1 = -C_2 = C$.

Now this approximated solution expression reduces to $\tilde{\phi} = C\big(x(1-x)\big)$, by substituting the values of $C_0 = 0$ and $C_1 = -C_2 = C$. So, you will get $\tilde{\phi} = C\big(x(1-x)\big)$, so then $\tilde{\phi}' = C(1-2x)$ which is the derivative of ϕ with respect to x. Then we will substitute the expressions of ϕ and ϕ' in the functional expression.

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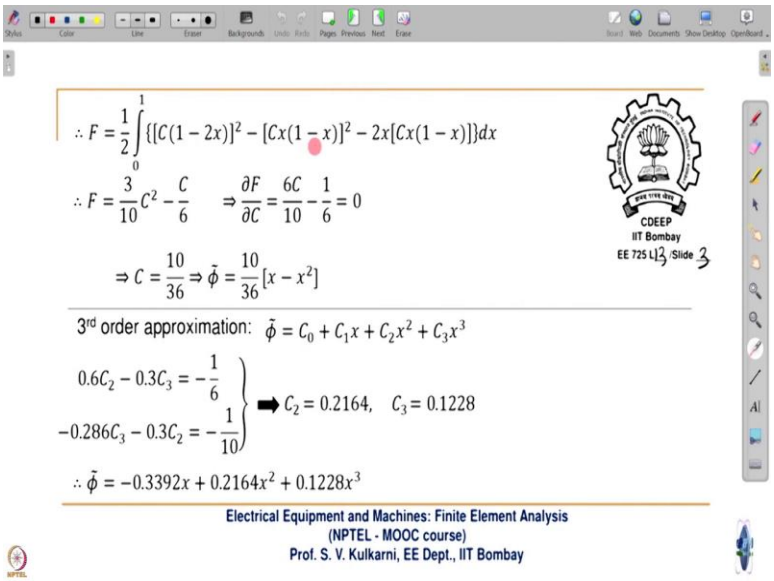

The functional expression reduces to

$$F = \frac{1}{2}\int_0^1 \{[C(1-2x)]^2 - [Cx(1-x)]^2 - 2x[Cx(1-x)]\}dx$$

The above expression represents the total functional and it represents the energy. Now, this energy has to be minimized with respect to which variable? After you evaluate the above integral, *F* will be a function of only *C*.

After evaluating the integral, the functional reduces to

$$F = \frac{3}{10}C^2 - \frac{C}{6}$$

Then you have to differentiate the above expression of *F* with respect to *C* and then you get the *C* as given below.

$$\frac{\partial F}{\partial C} = \frac{6C}{10} - \frac{1}{6} = 0 \Rightarrow C = \frac{10}{36}$$

Now you substitute *C* in the expression of $\tilde{\phi}$ to obtain the solution as given below.

$$\tilde{\phi} = \frac{10}{36}[x - x^2]$$

So here, first we are finding the functional and then differentiating it with the unknown variable *C* to calculate its value. So, this leads to energy minimization which we discussed earlier. Here, we are making $\delta F = 0$. So, $\delta F = 0$ amounts to differentiating *F* with respect to C and equating it to 0.

Now, remember this is an example of whole domain approximation that is why here *C* is the unknown variable and we are differentiating with respect to *C* to minimize the energy. When we actually see the finite element procedure which involves discretization of a given problem domain into a number of elements, there the unknown variables will be the potentials at various nodes.

So, there we will not use $\frac{\partial F}{\partial C} = 0$ but it will be $\partial F/$(corresponding potential variables) at various nodes. More details we will see later. Here, $\tilde{\phi} = C_0 + C_1 x + C_2 x^2$ was the second-order approximation. Consider a third order approximation as given below.

$$\tilde{\phi} = C_0 + C_1 x + C_2 x^2 + C_3 x^3$$

In the above expression, one more term $(C_3 x^3)$ is added compared to the second order approximation. If second-order approximation is not good enough then we consider third-order approximation to get more accurate results. And indeed, we will see later that we get accurate results with 3rd order approximation.

For the 3rd order approximation also you follow the same procedure. Using the first boundary condition $\left(\tilde{\phi}(0) = 0\right)$, you will get $C_0 = 0$. Now, it will be more complicated, because after applying the first boundary condition, we will be left with three variables $(C_1, C_2, \text{and } C_3)$. Then you get a more complicated expression for *F* as compared to what we got for the 2nd order approximation . In the previous case, you got *F* with only one variable *C*, but in this case we will get it in two variables C_2 and C_3, because when you impose the second boundary condition, you will replace C_1 in terms of C_2 and C_3. So, C_1 will get eliminated. Eventually, the expression of *F* will have only C_2 and C_3.

When you evaluate the integral of functional, you will get the expression of *F* in terms of C_2 and C_3 and then you will get two equations as given below because there are two unknowns. So, you differentiate *F* by C_2 and then by C_3 to equate it to 0 for minimizing the energy which leads to two linear equations and they are solved to calculate C_2 and C_3.

$$0.6C_2 - 0.3C_3 = -\frac{1}{6}$$

$$-0.286C_3 - 0.3C_2 = -\frac{1}{10}$$

And then you get the following expression of $\tilde{\phi}$.

$$\tilde{\phi} = -0.3392x + 0.2164x^2 + 0.1228x^3$$

Let us go further and see error distribution.

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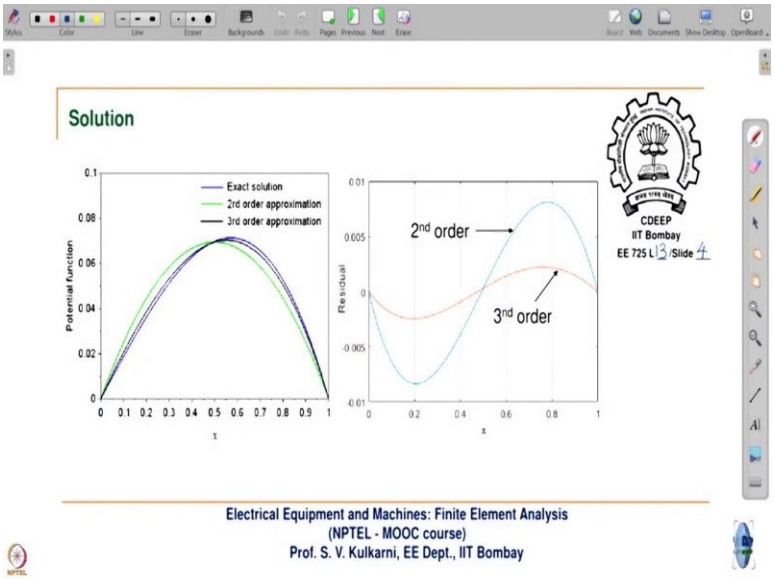
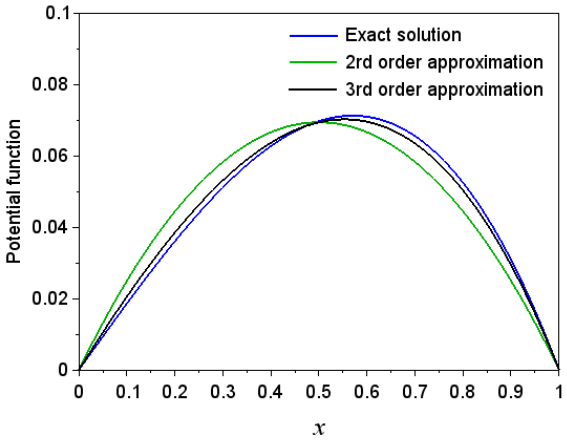


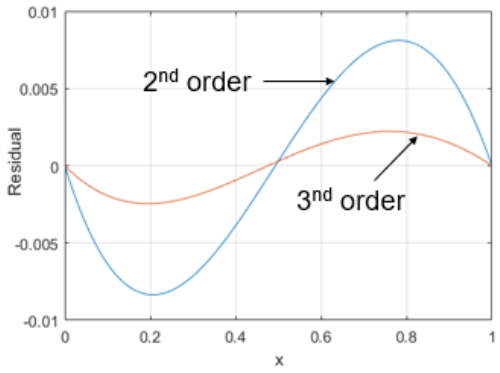
Now, in the following figure, the exact solution is in blue, second-order approximation is in green and black colour represents the third-order approximation.



So, in the figure, you can see the third-order approximation is better than second order. If you go on increasing the order of approximation, then the numerical solution will match more exactly with the actual solution. But the computational burden will increase. Then

we have to always compromise between the order of approximation, accuracy and the corresponding computational burden.

So, in the following figure, we have plotted residual which is also called as error.



In the above figure, you can see the error is not 0 everywhere except at three points. Somewhere it is positive, somewhere it is negative, and summation of error is also, not equal to 0. Here, the summation of error means $\int error\ dx$ because it is a one-dimensional problem. So, $\int error\ dx \neq 0$. If you take the integral of the error curves with respect to *x*, you will find that the integral is not 0.

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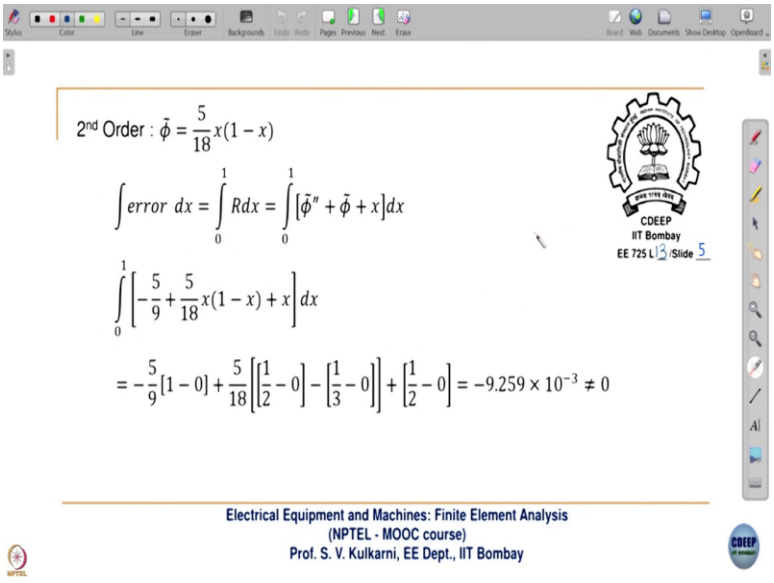


The value of $\int error\, dx$ is calculated in the above slide. This is possible because we have the expression for the approximate solution that we obtained. The approximate solution that we obtained is given below

$$\tilde{\phi} = \frac{5}{18}x(1-x)$$

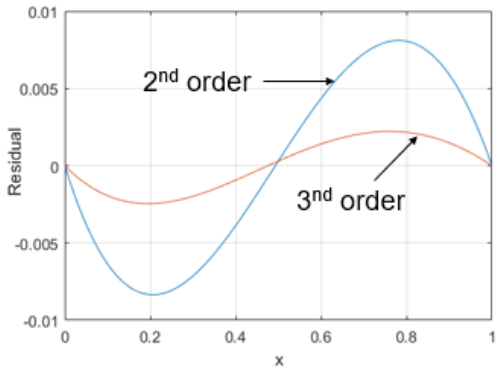
We obtained this solution using the energy minimization procedure and applying boundary conditions. Now, $\int error\, dx$ can be written as given below

$$\int error\ dx = \int_0^1 R dx$$

The residue R in the above equation is

$$R = \tilde{\phi}'' + \tilde{\phi} + x$$

$\tilde{\phi} = \frac{5}{18}x(1-x)$ was our approximate solution . We are substituting $\tilde{\phi}$ and $\tilde{\phi}''$ in the above expression of *R* to calculate the residue. And the value of *R* will not necessarily result into 0 everywhere as shown in the following figure.



In the above figure, we can see that only at one point the error is 0. So, the value of R in general will not be 0 because it is an approximate solution. As shown in the above figure somewhere the error is positive and somewhere, it is negative. Now you integrate the error with respect to dx.

Now, what is the error? Error at every point is given by the above expression of *R*. Now if you evaluate the expression of $\tilde{\phi}''$, you will get $-\frac{5}{9}$ and $\tilde{\phi} = \frac{5}{18}\left(x(1-x)\right)$. So, by substituting these expressions of $\tilde{\phi}''$ and $\tilde{\phi}$ in R, the integral of error reduces to

$$\int error\ dx = \int_0^1 R dx = \int_0^1 \left[-\frac{5}{9} + \frac{5}{18}x(1-x) + x\right] dx$$

And if you evaluate the above integral you will get the result which is not equal to 0.

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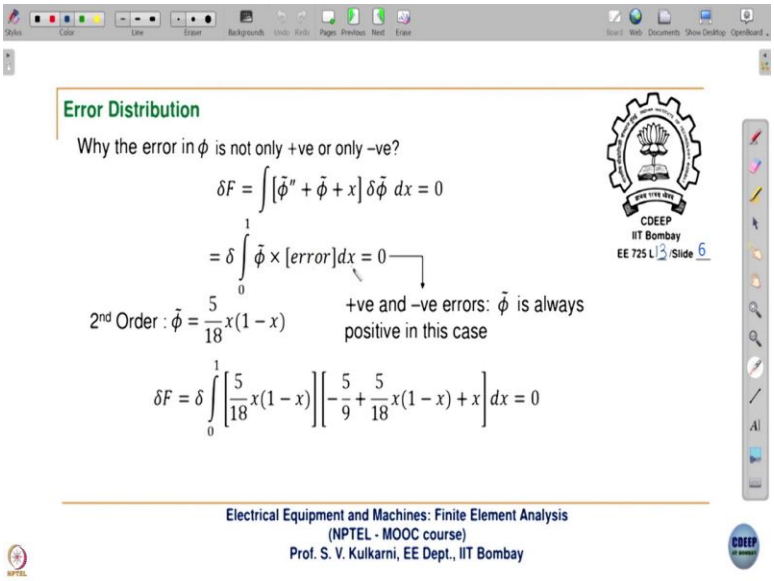

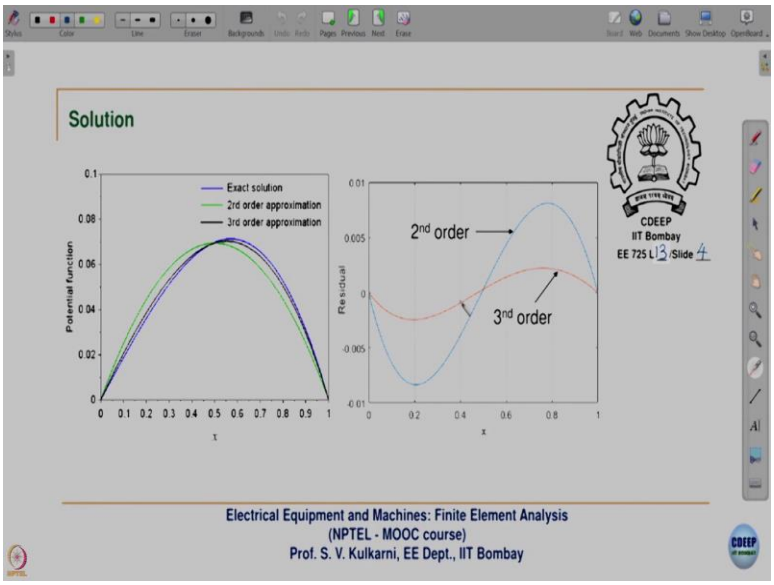

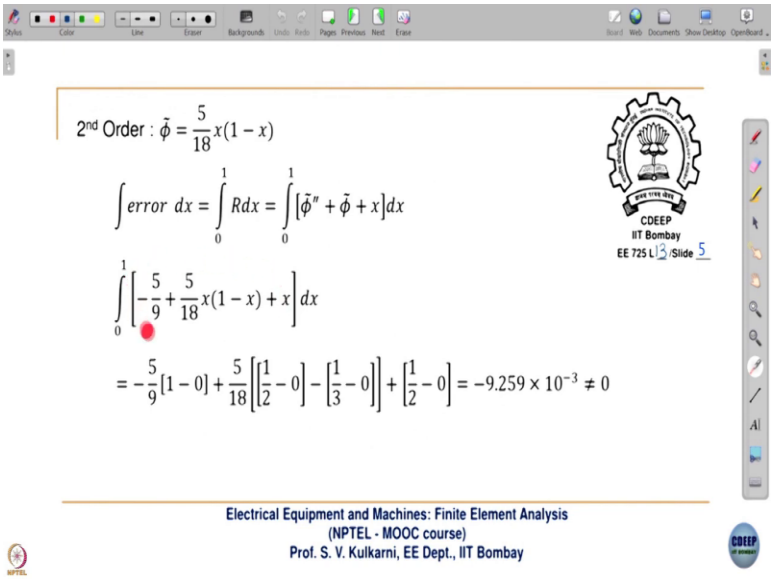

Now, we will evaluate the following integral of $\tilde{\phi}$ error dx. It will be equal to 0 as given below.

$$\delta F = \delta \int_0^1 \tilde{\phi} \times [error] dx = \int [\tilde{\phi}'' + \tilde{\phi} + x]\, \delta\tilde{\phi}\ dx = 0$$

And then we will also understand why error in ϕ is not only positive or not only negative. In the following figure, you can see somewhere the error is positive and somewhere, it is negative. Now, if we calculate this $\int_0^1 \tilde{\phi} \times [error] dx$. Now, we have seen this equation

while deriving the functional in which we forced that this integral should be equal to 0 which results into energy minimization. Now, let us see whether $\delta F = \delta \int_0^1 \tilde{\phi} \times [error] dx = 0$ get satisfied for this solution.

So, we have the solution with second-order approximation as $\tilde{\phi} = \frac{5}{18} x(1-x)$. Now, you substitute $\tilde{\phi}$ and $\tilde{\phi}''$ in δF . The expressions of $\tilde{\phi}$ and $\tilde{\phi}''$ are derived in the previous slide, the expression of the error is $R = -\frac{5}{9} + \frac{5}{18} x(1-x) + x$. So, we are integrating $\tilde{\phi} \times [error]$ and if you evaluate this, you will find that it comes identically equal to 0 as given below. So, it means that energy minimization condition is indeed getting satisfied.

$$\delta F = \delta \int_0^1 \left[\frac{5}{18} x(1-x)\right]\left[-\frac{5}{9} + \frac{5}{18} x(1-x) + x\right] dx = 0$$

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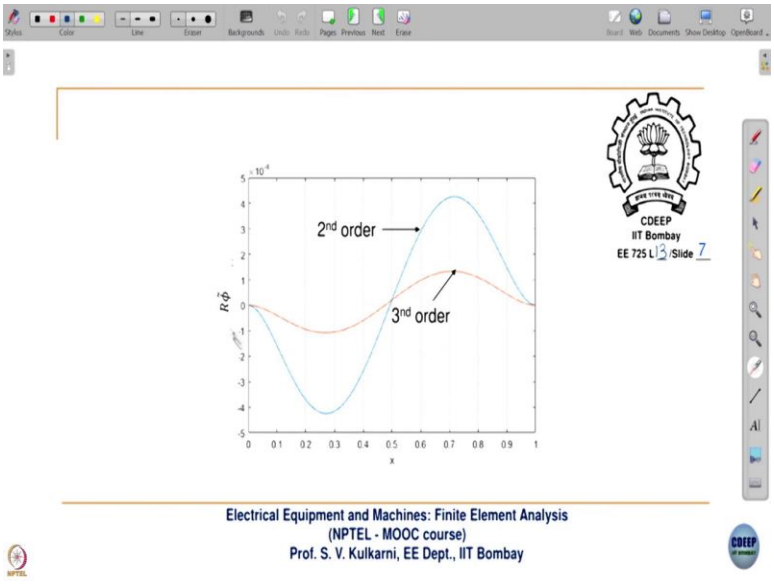

In the above figure, you can see the variation of $R \times \tilde{\phi}$ and then if you integrate this over *x*, the positive and negative areas in the above figure will exactly be equal and they will get cancelled.

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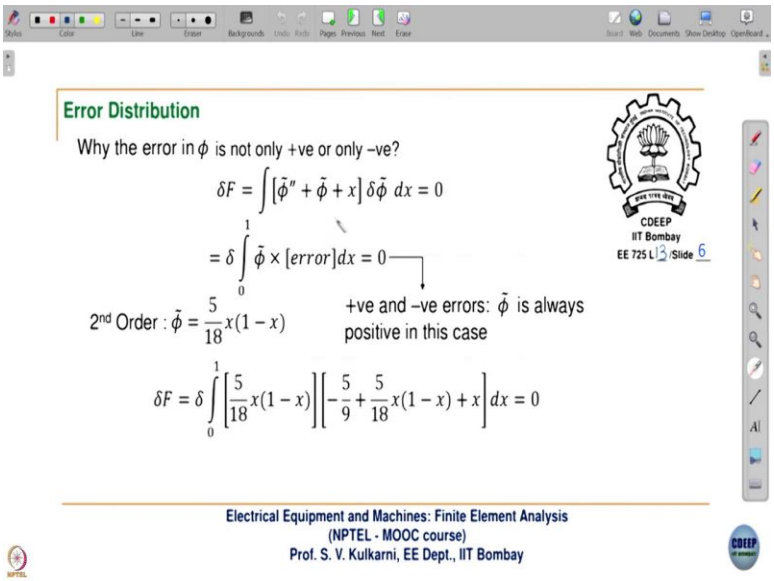

Here, we are getting $\delta \int_0^1 \tilde{\phi} \times [error] dx = 0$ because we are forcing this equal to 0 to derive the expression of the functional.

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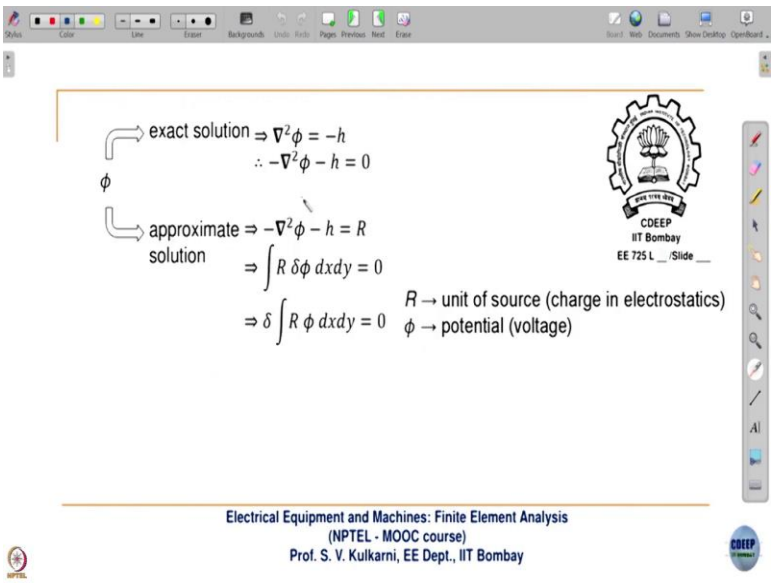

Do you remember the above slide? Here you can see if the solution is exact, then $\boldsymbol{\nabla^2\phi} = -h$ at every point and $\delta F = 0 \Rightarrow \iint R\, \delta\phi\, dxdy = 0$ will be satisfied. If it is an approximate solution, then $-\boldsymbol{\nabla^2\phi} - h$ will be equal to some residual or error.

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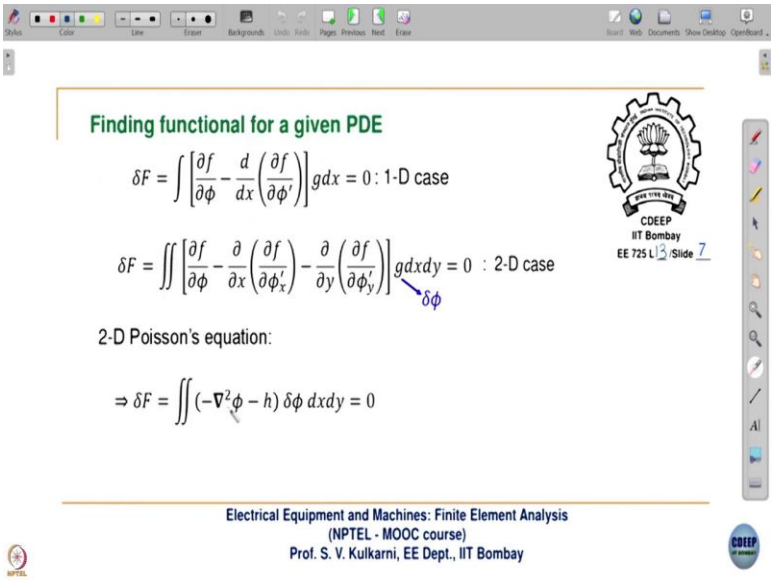

For example, if you see in this slide, the following expression is the starting step to derive functional from a given PDE.

$$\delta F = \iint (-\nabla^2 \phi - h)\, \delta\phi \, dx dy = 0$$

Now, here ϕ in the integrand is an approximate solution and $R = -\nabla^2 \phi - h$ is the residue. So we are making the integral of residue times ϕ as equal to 0. Here, we are forcing this condition, if ϕ is not the exact solution. If ϕ is the exact solution, then $-\nabla^2 \phi - h$ is anyway equal to 0 and $\delta F = 0$. If ϕ is not exact, then there will be some residue R and that integral of residue multiplied by ϕ over the domain is forced to 0 as given in the above expression.

So, if the solution is exact then $\delta F = 0$ gets exactly satisfied. If it is an approximate solution there will be an error, but that integral error multiplied with ϕ, in this our case $\tilde{\phi}$, the δ will anyway come outside the above integral as given below,

$$\delta F = \delta \iint (-\nabla^2 \phi - h)\, \phi \, dx dy = 0$$

which means the variation of integral residue times ϕ is equal to 0. So, in our case, it is only dx in the integral, since it is one dimension. So, this exactly confirms that we are minimizing the functional or energy and equating $\delta F = 0$ to determine the solution. If it is an approximate solution, it will amount to making $\iint R\,\phi\,dxdy = 0$. In our case, it will be *dx* since this is a 1-D case. So, we will stop here and we will go to further theory in the next lecture.

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