Electrical Equipment and Machines: Finite Element Analysis
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Lecture 12
Finding Functional for PDEs

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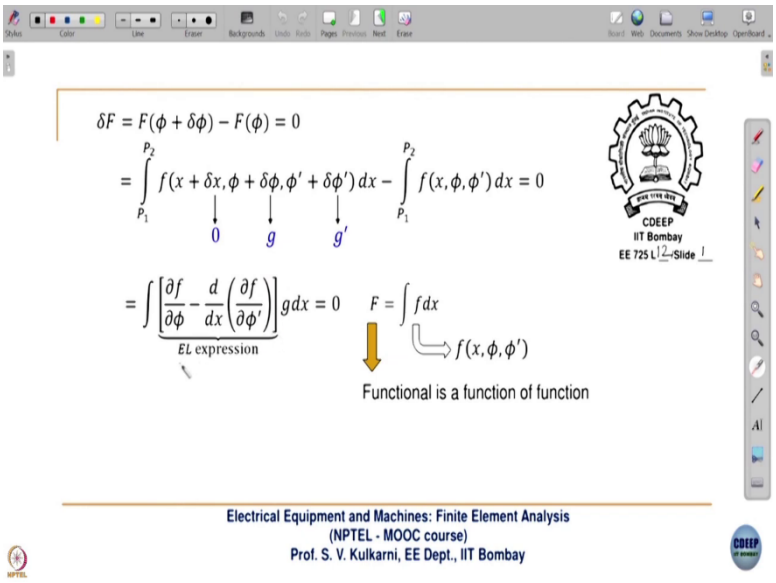

Welcome to this 12th lecture. In the 11th lecture what we saw was that in order to get the solution for any partial differential equation in electromagnetics, we have to minimize the energy. So, when we say minimize the energy, we have to make the variation in *F* equal to 0. This condition will make F as stationary which means that the energy is minimum. The energy minimum condition is given by $\delta F = 0$

$$\delta F = F(\phi + \delta\phi) - F(\phi) = 0$$
$$= \int_{P_1}^{P_2} f(x + \delta x, \phi + \delta\phi, \phi' + \delta\phi')\,dx - \int_{P_1}^{P_2} f(x, \phi, \phi')\,dx = 0$$
$$\delta x \rightarrow 0 \qquad \delta\phi \rightarrow g \qquad \delta\phi' \rightarrow g'$$

And then we also saw that in the expression of $F(\phi + \delta\phi)$, $\delta x = 0$ because we are not varying *x*, but at every *x*, we are varying the potential ϕ. So, we are varying the potential and the corresponding derivative of potential at every point. And we are finding a combination of potentials at various points, which will result in energy minimum condition.

Also, remember the functional (*F*) is the energy or energy functional and that is expressed as

$$F = \int f\,dx$$

$$\Downarrow \quad \hookrightarrow f(x, \phi, \phi')$$

Functional is a function of function

In the above equation, f is a function of x, ϕ, ϕ'. But in case of Laplace equation, it is just a function of ϕ'only. In the case of Poisson's equation, it will be a function of ϕ and ϕ'.

Then in the process of this minimization, we also saw that the $\delta F = 0$ leads to what is known as Euler-Lagrange equation which is given below.

$$\frac{\partial f}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial f}{\partial \phi'}\right) = 0$$

The left hand side of the above equation is known as Euler-Lagrange expression. This equation is important and we will use this later also.

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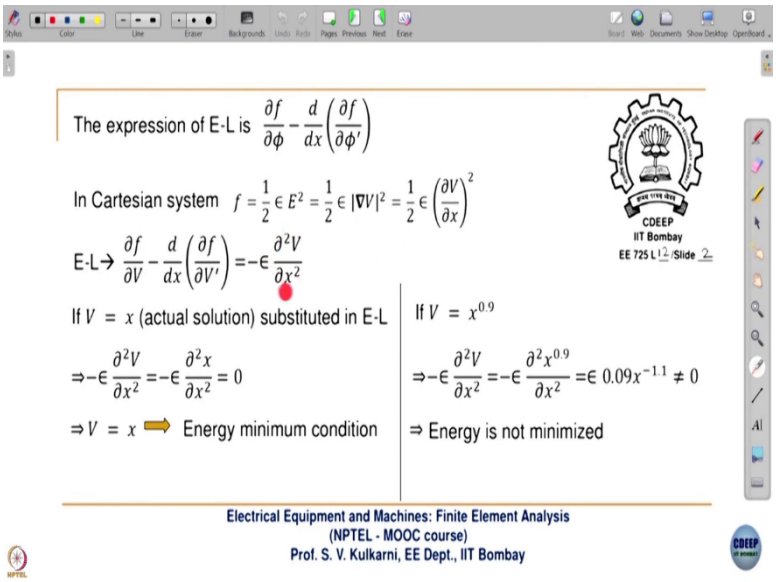

Now, the Euler-Lagrange expression for the one-dimensional case is given below.

$$\text{E-L} \rightarrow \frac{\partial f}{\partial V} - \frac{d}{dx}\left(\frac{\partial f}{\partial V'}\right)$$

And $f = \frac{1}{2}\epsilon E^2$ which is incidentally the energy density in electrostatics and then if E is replaced by ∇V, you get the energy density function as given below.

$$f = \frac{1}{2} \in E^2 = \frac{1}{2} \in |\nabla V|^2 = \frac{1}{2} \in \left(\frac{\partial V}{\partial x}\right)^2$$

So, if you substitute f in the Euler-Lagrange expression then, it will be simplified as given below.

$$\frac{\partial f}{\partial V} - \frac{d}{dx}\left(\frac{\partial f}{\partial V'}\right) = \frac{\partial}{\partial V}\left(\frac{1}{2} \in \left(\frac{\partial V}{\partial x}\right)^2\right) - \frac{d}{dx}\left(\frac{\partial}{\partial V'}\left(\frac{1}{2} \in \left(\frac{\partial V}{\partial x}\right)^2\right)\right) = -\in \frac{\partial^2 V}{\partial x^2}$$

Now, let us understand a little bit more about the whole concept. We also saw that, if we know the exact solution of *V*, and that is substituted in *f* and that in turn is substituted in Euler-Lagrange equation, we get left hand side of Laplace equation.

If we substitute $f = \frac{1}{2} \in \left(\frac{\partial V}{\partial x}\right)^2$ for 1D electrostatic problem, we get Laplace's equation. And it is quite natural that we get Laplace's equation because we already know that energy density function for the one dimensional electrostatic problem is $f = \frac{1}{2} \in \left(\frac{\partial V}{\partial x}\right)^2$.

Now, if we substitute $V = x$, which is the actual solution in case of a parallel plate capacitor problem. It should be noted that in parallel capacitor problem, the top and bottom plates are having potentials of 10 V and 0 V respectively. Also, consider that the top plate is at x = 10 m and the bottom plate is at x = 0. So the distance between the 2 plates is 10 m, then every equi-potential line say 9 V, 8 V, 7 V, 6 V will be at 9 m,8 m,7 m,6 m. So, effectively we can say that the exact or actual solution for the parallel plate capacitor problem is simply $V = x$. So, at 5 m from the ground plate, you will have 5 V, and likewise. So, $V = x$ is the solution for that simple electrostatic problem without fringing at the ends. Now, if you substitute $V = x$ in the Euler-Lagrange expression, we immediately get the whole thing equal to 0. And that means, Laplace's equation is exactly satisfied. Since we already know the exact solution and then we substituted it in the Euler-Lagrange expression, it is quite natural that we are exactly satisfying Laplace's equation and that is happening because when Euler-Lagrange expression is equals to 0, which means $\delta F = 0$.

The whole theory that we saw is consistent for the solution of Laplace's equation involving the parallel plate capacitor problem. Now, suppose, if we substitute $V = x^{0.9}$ in f and correspondingly in Euler-Lagrange expression, that means, we are not substituting the exact solution that we know, but we are substituting an approximate solution. Now if you substitute $V = x^{0.9}$ in the Euler-Lagrange expression, then it is not coming equal to 0 as given below.

$$\text{If } V = x^{0.9}$$

$$\Rightarrow -\epsilon \frac{\partial^2 V}{\partial x^2} = -\epsilon \frac{\partial^2 x^{0.9}}{\partial x^2} = \epsilon\, 0.09 x^{-1.1} \neq 0$$

$$\Rightarrow \text{Energy is not minimized}$$

That is why Laplace equation also will not get exactly satisfied, because we have assumed some solution, which is not the exact solution. So, this non zero value will be representing the error at every point, but that error can be minimized by, approximating a solution which is more closer to the exact solution by having more terms, whereas here we are just substituted $x^{0.9}$.

But for a complicated electrostatic problem (here it is a trivial solution $V = x$, that is why this question does not arise), wherein there is a non-uniform field distribution, then, you will have to assume *V* as some complicated polynomial of *x*. And when you go through the entire procedure and you will get an error which can be very small and that can be further reduced by using FEM procedure.

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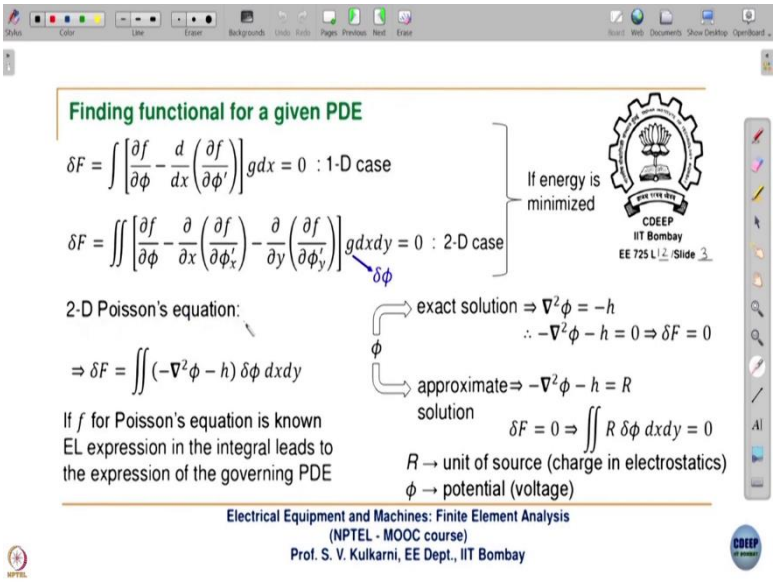


Now going further, the energy minimum condition for a 2D case is given below

$$\delta F = \iint \left[\frac{\partial f}{\partial \phi} - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial \phi'_x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial \phi'_y}\right)\right] g dx dy = 0 \text{ : 2-D case}$$

$\delta\phi$

For the 2D case, you will have 2 independent variables *x* and *y* and the g term in the above equation is nothing but $\delta\phi$ (variation in ϕ).

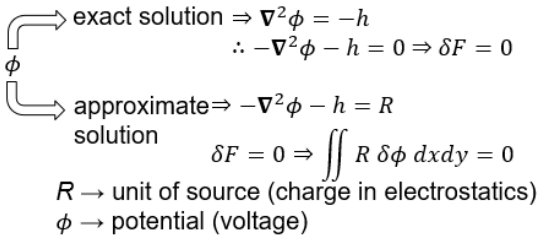
Now, for a one dimensional electrostatic problem, we knew the energy density expression as $\frac{1}{2}\epsilon E^2$, but if you want to find out energy functional for a given PDE, then that is done by using the procedure which is explained in this slide. So, now our goal is to find functional for a given PDE.

So, first we will consider Poisson's equation. When you substitute the unknown *f* , and if that *f* function is exact and when I say f function is exact, the energy density function is known and then when you are substituting the correct potential solution in the above eqution then you will get Poisson's equation in the integrand as shown below.

$$\Rightarrow \delta F = \iint (-\nabla^2\phi - h)\, \delta\phi\, dx dy$$

In the previous case, when we knew the expression of *f* and when we substituted it in the Euler-Lagrange expression (for 1D), we got the corresponding Laplace'sequation.

Similarly if we know *f* for the Poisson's equation, and if you substitute it in the Euler-Lagrange expression it will reduce to $-\nabla^2\phi - h = 0$. Now, the same concept is summarized below.



If this ϕ is the exact solution, then this $\delta F = 0$ gets exactly satisfied and then that is why $-\nabla^2\phi - h = 0$.

But in the expression of *F*, the expression of *f* is known, but in that, if we substitute approximate solution $\tilde{\phi}$ which is not exact, then $-\nabla^2\phi - h \neq 0$ and that will be equal to some residue at every point in the domain. So, that is why when you force $\delta F = 0$ in the process of energy minimization, what effectively you are forcing is $\iint R\,\delta\phi\,dxdy = 0$.

And this in fact, we will see later for a one-dimensional problem when we approximate solution, which is not exact, we get residue and $\iint R\,\delta\phi\,dx = 0$. Also one more point I want to say here is that R has unit of source, it is charge in electrostatics, why? Because in case of electrostatics, h in $\nabla^2\phi = -h$ represents charge because Poisson's equation in electrostatics $\nabla^2 V = -\frac{\rho}{\epsilon_0}$.

So, h is a function of charge although it is ρ/ϵ_0, but if ϵ_0 is absorbed into the left hand side then you can take this *h* as simply charge right. So, *h* has the unit of charge. *R* also has the same unit of *h* because $R = -\nabla^2\phi - h$. And charge times potential *V* is work or energy.

δ of residual energy (*R*) integrated over the domain is being equated to 0 so that is the second interpretation of $\delta(\iint R dxdy) = 0$. More about this when we solve a 1D problem, there we will see this being enforced in the formulation.

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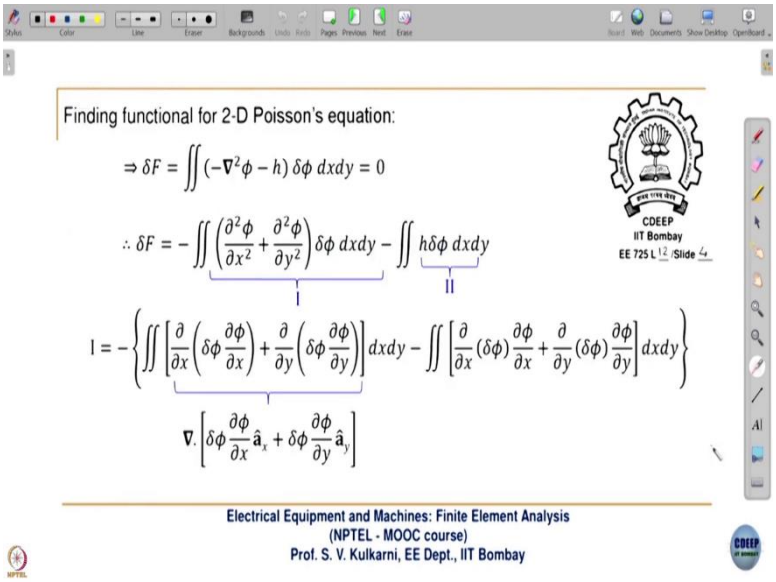

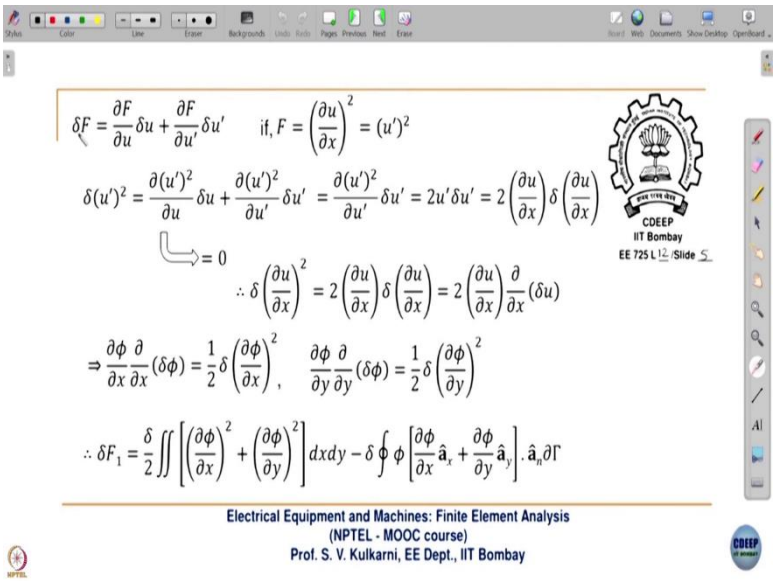

We again start with the following equation.

$$\delta F = \iint (-\nabla^2\phi - h)\,\delta\phi\, dxdy = 0$$

The above equation is expanded as given below

$$\therefore \delta F = -\underbrace{\iint \left(\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2}\right)\delta\phi\, dxdy}_{\text{I}} - \underbrace{\iint h\delta\phi\, dxdy}_{\text{II}}$$

Now, we are splitting the first integral as difference of these 2 terms as given below.

$$I = -\left\{ \iint \underbrace{\left[\frac{\partial}{\partial x}\left(\delta\phi \frac{\partial \phi}{\partial x} \right) + \frac{\partial}{\partial y}\left(\delta\phi \frac{\partial \phi}{\partial y} \right) \right]}_{\nabla . \left[\delta\phi \frac{\partial \phi}{\partial x} \hat{\mathbf{a}}_x + \delta\phi \frac{\partial \phi}{\partial y} \hat{\mathbf{a}}_y \right]} dxdy - \iint \left[\frac{\partial}{\partial x}(\delta\phi) \frac{\partial \phi}{\partial x} + \frac{\partial}{\partial y}(\delta\phi) \frac{\partial \phi}{\partial y} \right] dxdy \right\}$$

And then the first first term of the above expression is replaced by using divergence of a vector $\left(\delta\phi \frac{\partial \phi}{\partial x} \hat{\mathbf{a}}x + \delta\phi \frac{\partial \phi}{\partial y} \hat{\mathbf{a}}y \right)$, because divergence of this vector is the integrand of the first term in the above expression.

Now, we will further simplify the first term. For that, we need to understand a little more bit about δF, the variation in F is given by the following equation.

$$\delta F = \frac{\partial F}{\partial u}\delta u + \frac{\partial F}{\partial u'}\delta u'$$

Here, as I have mentioned to you early, there is no $\frac{\partial F}{\partial x}\delta x$ term, because $\delta x = 0$ in this variational calculus or the variational procedure.

Now, consider $F = \left(\frac{\partial u}{\partial x}\right)^2 = (u')^2$ and note that $\left(\frac{\partial u}{\partial x}\right)^2$ is nothing but $(u')^2$. Then $\delta(u')^2$ is given by the following formula.

$$\delta(u')^2 = \underbrace{\frac{\partial (u')^2}{\partial u}\delta u}_{\hookrightarrow = 0} + \frac{\partial (u')^2}{\partial u'}\delta u'$$

And now, the first term is 0, because $(u')^2$ is not a function of u, so, the first term is equal to 0, only the second term in the right hand side of above summation remains as written below.

$$\delta(u')^2 = \frac{\partial (u')^2}{\partial u'}\delta u' = 2u'\delta u'$$

Now u' is replaced by $\frac{\partial u}{\partial x}$ as given below.

$$\delta(u')^2 = 2\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)$$

$$\delta\left(\frac{\partial u}{\partial x}\right)^2 = 2\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right)$$

Now, the term on the right hand side can be further simplified as

$$\delta\left(\frac{\partial u}{\partial x}\right)^2 = 2\left(\frac{\partial u}{\partial x}\right)\delta\left(\frac{\partial u}{\partial x}\right) = 2\left(\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x}(\delta u)$$

The δ operator can be taken inside the $\frac{\partial}{\partial x}$, because by the definition of δ it does not involve variation with *x*. Now the above expression can be rearranged as given below.

$$\frac{\partial u}{\partial x}\frac{\partial}{\partial x}(\delta u) = \frac{1}{2}\delta\left(\frac{\partial u}{\partial x}\right)^2$$

Similarly, the following two terms can be rewritten as given below.

$$\frac{\partial \phi}{\partial x}\frac{\partial}{\partial x}(\delta\phi) + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial y}(\delta\phi) = \frac{1}{2}\delta\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\delta\left(\frac{\partial \phi}{\partial y}\right)^2$$

Now, the integral *I* in the previous slide which is nothing but δF_1 can be written as

$$\delta F_1 = \frac{\delta}{2}\iint\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right]dxdy - \delta\oint \phi\left[\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial \phi}{\partial y}\hat{\mathbf{a}}_y\right].\hat{\mathbf{a}}_n\partial\Gamma$$

And the first term of the integral *I* in the previous slide is replaced by $\boldsymbol{\nabla}.\left[\delta\phi\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}x + \delta\phi\frac{\partial \phi}{\partial y}\hat{\mathbf{a}}y\right]$ as given below.

$$\iint\left[\frac{\partial}{\partial x}\left(\delta\phi\frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(\delta\phi\frac{\partial \phi}{\partial y}\right)\right]dxdy = \iint\boldsymbol{\nabla}.\left[\delta\phi\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}x + \delta\phi\frac{\partial \phi}{\partial y}\hat{\mathbf{a}}y\right]dxdy$$

By invoking the divergence theorem in 2D that we discussed in one of the previous lectures, the surface integral of divergence of a vector is replaced by the corresponding contour integral as given below.

$$\iint\boldsymbol{\nabla}.\left[\delta\phi\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}_x + \delta\phi\frac{\partial \phi}{\partial y}\hat{\mathbf{a}}_y\right]dxdy = \delta\oint \phi\left[\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial \phi}{\partial y}\hat{\mathbf{a}}_y\right].\hat{\mathbf{a}}_n\partial\Gamma$$

This we have seen in basics of electromagnetics, when divergence theorem is invoked for 3D, the volume integral got converted into a closed surface integral.

Here it is a two dimensional problem, but effectively it is a 3D problem with $dxdy \times 1$, that means, it is a per unit dimension in z direction. So, in 2D approximation, the open surface integral gets replaced by a closed contour integral.

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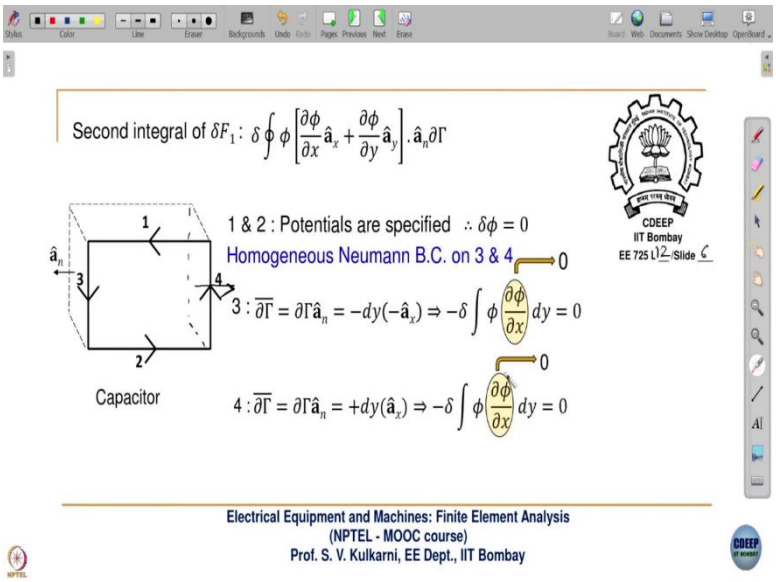
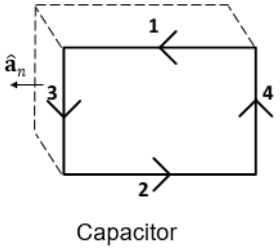


Now, we consider this second integral of δF_1 which is given by the following expression.

$$\text{Second integral of } \delta F_1 : \; -\delta \oint \phi \left[\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial \phi}{\partial y}\hat{\mathbf{a}}_y\right].\hat{\mathbf{a}}_n \partial\Gamma$$

Now for understanding how to evaluate this integral we will consider our standard parallel plate capacitor problem geometry as shown in the following figure.



For this capacitor problem, potentials are specified on the segments 1 and 2 and homogeneous Neumann boundary conditions are specified on the two vertical segments (3 and 4).

I will explain open and closed surfaces a little later. First, let us understand, this closed contour in the above figure which is made up of 4 segments 1, 2, 3 and 4. On segments 1 and 2, potentials are specified, that means $\delta\phi = 0$ because there is no variation in ϕ on these segments. So, $\delta\phi = 0$ on segments 1 and 2 of this closed contour of 1, 3, 2, 4.

Now, when it comes to segment 3 whose direction is vertically down then $\overline{\partial\Gamma} = \partial\Gamma\hat{\mathbf{a}}_n = -\partial y\hat{\mathbf{a}}_n$. And $\hat{\mathbf{a}}_n$ for this segment 3 is basically $-\hat{\mathbf{a}}_x$, why? Because remember, we obtained this equation by invoking divergence theorem which is applied to convert a volume integral to a closed surface integral. But since we are having a 2D approximation of the problem, the open surface integral gets converted to a closed contour integral. So, now that is why you have the closed contour in the above expression, but we should remember in actual 3D case this is a closed surface as shown in the above figure. So, it is a closed cubic surface. Now, the normal to the segment 3 is effectively normal to the surface formed by the segment and the dotted lines in the 3rd dimension of the above figure.

We know that normal for a face of any closed surface is always in the outward direction. Therefore, for segment 3 the outward normal is $-\hat{\mathbf{a}}_x$ as shown in the figure. So, the above expression gets simplified to as given below.

$$-\delta \oint \phi \left[\frac{\partial\phi}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial\phi}{\partial y}\hat{\mathbf{a}}_y\right].\hat{\mathbf{a}}_x dx = -\delta \int \phi \, \frac{\partial\phi}{\partial x}\, dy$$

The $\frac{\partial\phi}{\partial y}$ in the right hand side of the above equation will not contribute because $\hat{\mathbf{a}}_y \cdot \hat{\mathbf{a}}_x = 0$. So, only the $\frac{\partial\phi}{\partial x}$ term will contribute. So, that is why only $\frac{\partial\phi}{\partial x}\, dy$ will remain for this contour integral.

Similarly for segment 4, the corresponding surface is also shown in the figure. This surface is also formed by segment 4 and the dotted lines in the third direction. The corresponding outward normal will be $\hat{\mathbf{a}}_x$ and then the direction of $\partial\bar{\Gamma}$ be in vertically

up direction. So, in this case also the contour integral reduces to the equation given below.

$$-\delta \oint \phi \left[\frac{\partial \phi}{\partial x}\hat{\mathbf{a}}_x + \frac{\partial \phi}{\partial y}\hat{\mathbf{a}}_y\right].\hat{\mathbf{a}}_x dx = -\delta \int \phi \, \frac{\partial \phi}{\partial x}\, dy$$

So, again you get the same equation and then here since $\frac{\partial \phi}{\partial x}$ in both these cases is 0 because , we imposed homogeneous Neumann condition on the two vertical segments. Therefore, we have got these equipotential lines which are normal to the vertical surfaces that means that $\frac{\partial \phi}{\partial x} = 0$.

So, that is why the contribution of $\frac{\partial \phi}{\partial x}$ term is 0 for both segments 3 and 4. So, that is why the integrations on these 3 and 4 segments reduce to 0. That is why this whole closed contour integral does not contribute in this case of the capacitor. In this problem, it does not contribute anything, but in case you have some boundary, wherein some non-zero homogeneous Neumann condition is imposed, then this closed contour integral will be non-zero and this will be an additional term in the energy expression and correspondingly in the final FEM equation (which is a linear system of equations), you will get an additional matrix term corresponding to this integral and that has to be taken into account. More details about how do we form matrices we will see later as we go further in this course.

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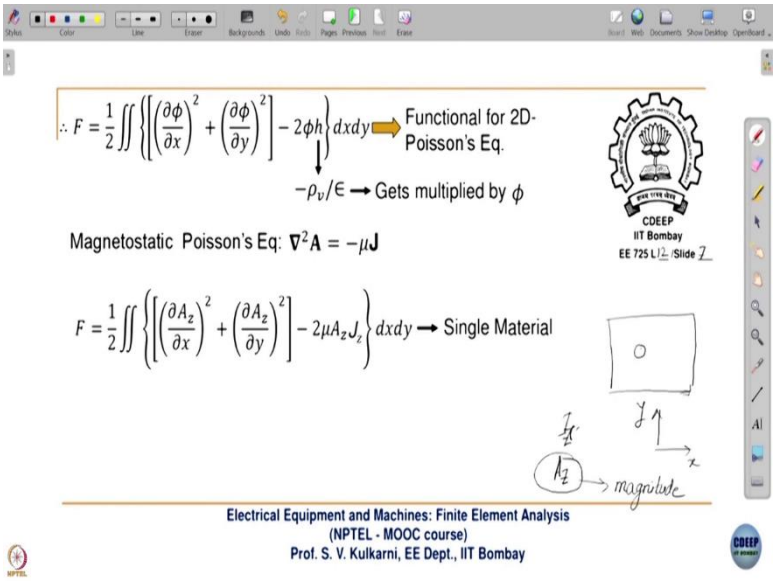
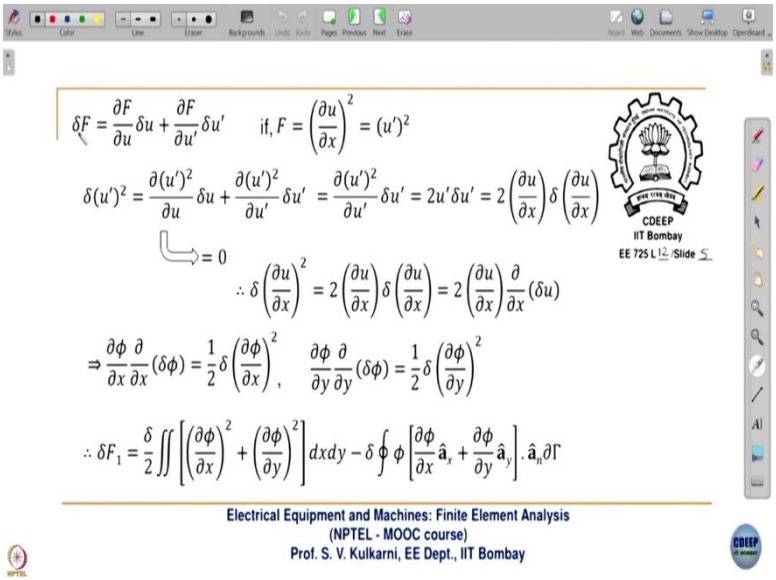




Going further, the expression of *F* for Poisson's equation is given below.

$$F = \frac{1}{2}\iint\left\{\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] - 2\phi h\right\}dxdy$$

Functional for 2D-Poisson's Eq.

$-\rho_v/\in$ → Gets multiplied by ϕ

It should be noted that the whole expression of δF_1 reduces to the first term in the above equation. Because the boundaries are with either homogeneous Neumann conditions or Dirichlet conditions as explained in the previous slide.

Remember, if there are non homogeneous Neumann condition somewhere, then there will be an additional term in F1, but in most of the cases what we generally have, for example, in the capacitor problem case we will have either homogeneous Neumann condition where boundary conditions are not defined or generally we enclose the entire problem domain in a box or a rectangle and we specify reference potential as $\phi = 0$. So, in that process, the contour integral will go down to 0 because there is a $\delta\phi$ term. Generally whenever a boundary is there, in most of the problems, we either have Dirichlet condition wherein ϕ is specified or homogeneous Neumann condition and that is why this contour integral will reduce to 0 and only the first term of the above integral will remain.

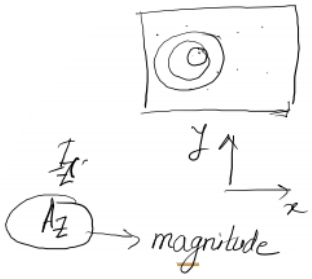
So, now in this δF_1 you have only one term remaining (given in the above expression) and now we take the second term which is simply $-2\phi h$. This will be twice because

here we have taken $\frac{1}{2}$ as common in the above equation. So, remember this *h* represents the source, in case of electrostatic problems it will be simply ρ_v/ϵ .

Also remember that in the functional expression, the terms involving *h* or source or any constant term or any other term which is not having ϕ, suppose here in this case, *h* was on the right hand side of Poisson's equation $(\nabla^2\phi = -h)$ and $-h$ gets multiplied by ϕ in the energy expression.

Now we will apply the same logic to derive functional for the magnetostatic Poisson's equation. For this, the governing partial differential equation is $\nabla^2\mathbf{A} = -\mu\mathbf{J}$. And now, if we are doing 2 dimensional approximation then we will take the 2D cross section perpendicular to the direction of the current. So, in any two dimensional problem, when you are analyzing some electrical machine or transformer or any other device with some current distribution, you will always find that we take a cross section for 2D analysis which is perpendicular to the direction of the current.

The direction of current will be shown by dot or cross. Always we take a cross-section perpendicular to the current and then we approximate it as a 2D problem. So for a current carrying conductor whose cross section is in xy plane, the current is in z direction as shown in the following figure.



So, the current (I) has only z component. Similarly, magnetic vector potential (**A**) will also have only z component because the direction of A is the direction of I. Effectively what is happening, the direction of **A** gets fixed, so only the magnitude of **A** needs to be determined at every point in the domain. This now becomes a scalar formulation because the direction of unknown vector potential is already known, so we have to find out only the magnitude of **A** at every point.

That is the advantage of using the magnetic vector potential formulation in 2 dimensional magnetic field problems, because the direction of current and therefore, direction of magnetic vector potential gets fixed, because we take the cross section perpendicular to the current and the direction of **A** becomes fixed in the direction of current.

So, only the magnitude of **A** needs to be determined at all points in the domain. Once the magnitude of A_z is known, you can find out B_x and B_y. As the current is in z direction, **B** will be in *xy* plane, because the flux lines are going to be as indicated by the contours in the above figure. So, *x* and *y* components of **B** can be determined from **A** by using the expression $\mathbf{B} = \nabla \times \mathbf{A}$. How do you do that we will see later when we get into finite element analysis. Thank you.

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