Electrical Equipment and Machines: Finite Element Analysis
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Lecture 11
FEM – Variational Approach

Welcome to 11th lecture. And from this lecture we are getting into finite element analysis.

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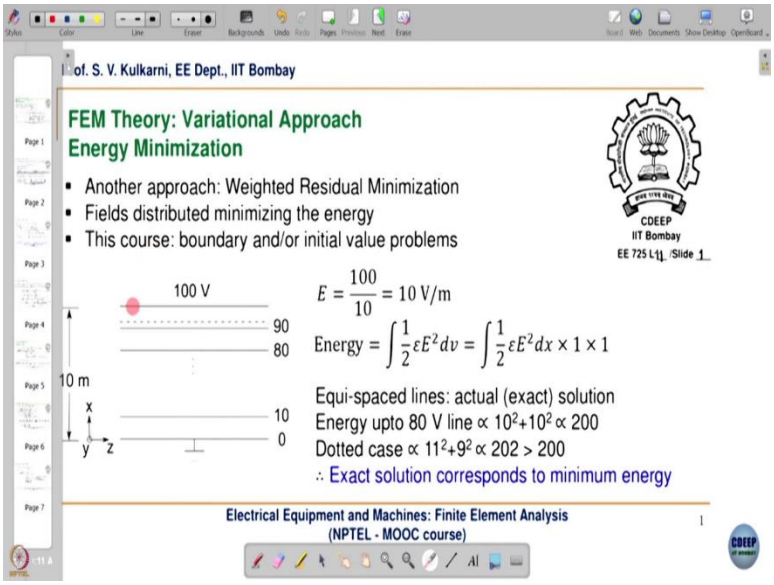
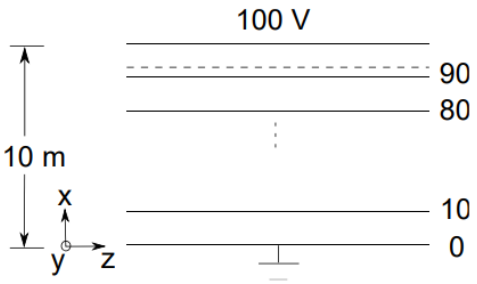


There are two distinct approaches in finite element method, one is the variational approach, which is an energy minimization approach and it is more physical because we are minimizing the energy to determine the solution. And why we minimize energies? Basically, fields would always get distributed in a given problem domain such that the corresponding energy stored in the domain is minimized. That is one of nature's laws.

So, the fields will get distributed such that the corresponding energy content is minimized. Weighted residual method is another approach, wherein we minimize errors because of approximating unknown potentials. We will discuss the weighted residual approach little later. So, in the first two sets of lectures, we will concentrate on the variational method, which is an energy minimization approach.

Now consider the parallel plate capacitor problem (shown in the following figure) to understand what is this energy minimization and how the field distribution corresponding to energy minimum condition occurs. So, as discussed earlier, even though we are considering this problem in electrostatics, you will find that we are not considering the charges on the plate.



Always remember in electromagnetics the basic entity of source is charge only, when it is moving, then it is current and it can be DC or AC. So, the basic source is charge or current. But here in the above figure, you may wonder that we are not representing charges, but they are represented by the corresponding boundary conditions. In the above figure, the top plate of the capacitor is at 100 V and the bottom plate is at 0 V.

Now again for simplicity, we are neglecting the end effects as discussed in one of the previous lectures. We model the capacitor as a rectangular geometry and the homogeneous Neumann conditions $\left(\frac{\partial V}{\partial z} = 0\right)$ on the two vertical boundaries. With these conditions, a uniform field condition can be achived because end effects are neglected and everywhere the field is uniform.

So, electric field intensity can be simply calculated as $\frac{100\ \text{V}}{10\ \text{m}}$because 10 m is the gap between the two plates whose potential difference is 100 V. So, the magnitude of E field everywhere inside the above capacitor is 10 V/m . And the energy is calculated by using the following expression.

$$\text{Energy} = \int \frac{1}{2}\varepsilon E^2 dv = \int \frac{1}{2}\varepsilon E^2 dx \times 1 \times 1$$

And now here the potentials are varying with x. Remember when we impose homogeneous Neumann condition, effectively we are making the plates as infinite in the z direction. In the y direction, i.e, into the paper, it is infinite. So, effectively we are reducing the dv as $dx \times 1 \times 1$. Like we saw in basics, when we did 2D approximation, we always said $dS \times 1$ which represents per metre depth. And also we saw the parameters are calculated for per metre depth. So, here, you are assuming infinite in y and z, and considering per metre depth in both directions.

So that is why this problem will be essentially reduced to 1D. And then we also know for this problem, the solution (potential distribution) is the equi-spaced, equi-potential lines as shown in the above figure. This solution should lead to the minimum energy condition. Now, we need to prove that the equi-spaced, equipotential lines indeed lead to the minimum energy condition.

We will consider a counter argument and then prove it. Let us assume that the solution given in the above figure does not lead to the energy minimum condition. Suppose, if you shift the 90 V line in the above figure to the dotted line as shown, then the solution is not equi-spaced lines. But now we will calculate the energy in the region up to the 80 V line.

In the previous case, when it is the exact solution, the energy is proportional to $10^2 + 10^2 = 200$, because E is 10 V/m in the whole capacitor region. But in the dotted line case, since this dotted line is taken above the position of 90 V line (exact solution), the electric field intensity will be more.

And I have assumed that as 11 V/m . And in this case the energy in the region between 80 V and 90 V, the electric field intensity will be lower, which I am considering as say 9 V/m. Now you can see the energy is proportional to $11^2 + 9^2 = 202$. So, that is why the dotted case would represent higher energy than the exact solution. In a way, we have proved that the equi-spaced equipotential lines lead to the minimum energy condition.

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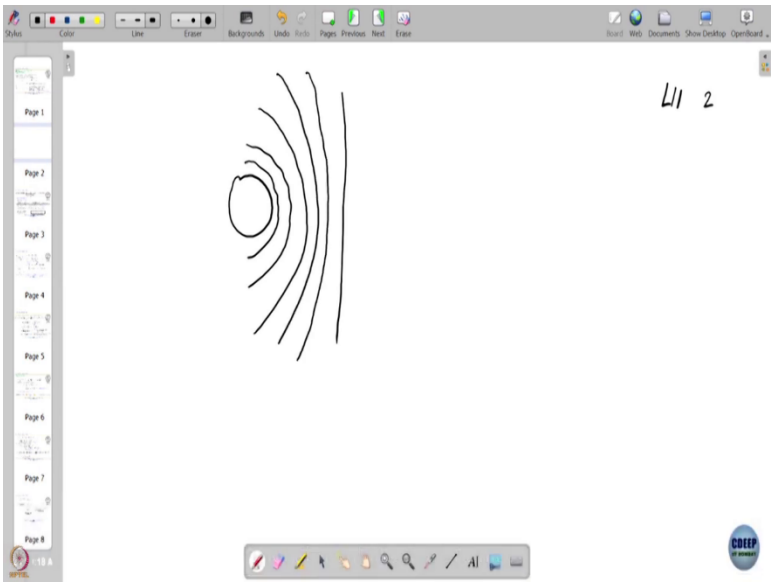

So, you consider the other case (lead to ground problem) which we have been seeing previously. In this case, the equipotential lines are going to be like the way shown in the above slide. We have seen this distribution in the previous lecture. Now, the equi-potential lines in the above figure are not equi-spaced. So, the exact solution of this problem would correspond to equipotential lines, which we are not equi-spaced. Because lines which are closer to the conductor are more closely spaced as compared to equipotential lines which are towards the ground electrode, which are sparsely spaced.

In this case, the exact solution corresponds to this non equi-spaced, equipotential lines and if you assume equi-spaced, equipotential lines then you will calculate higher energy than the exact solution which is non-uniform case and not equi-spaced lines.

In this case, equi-potential lines can be obtained from FEM solution and that is not very straightforward to calculate analytically. In the previous example of the parallel plate capacitor, it was very simple to prove the concept of energy minimization. But here it would be difficult, but I hope intuitively you understood why energy minimum condition is the exact solution from the previous capacitor problem.

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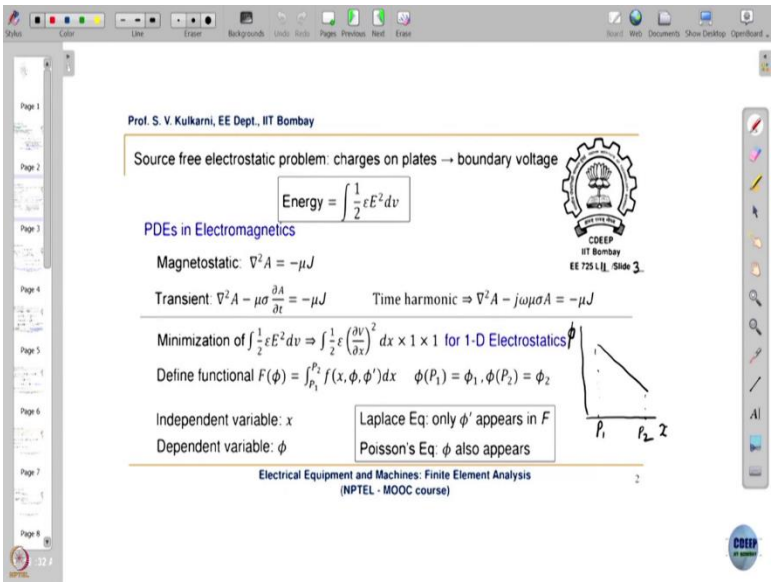


Now, let us go further. We are again considering a source free electrostatic problem because we are not considering charges on the capacitor plates. But the source of charges is being represented by corresponding boundary conditions on the top and bottom plates and the corresponding energy can be calculated by using the following equation.

$$\text{Energy} = \int \frac{1}{2}\varepsilon E^2 dv$$

Now before going into the details of how to solve this electrostatic problem, let us see the two most standard partial differential equations. PDE stands for partial differential equations. So we will be considering three typical cases (defined by the following equations) for low-frequency electromagnetics which are used when we are dealing with electrical machines and equipment.

PDEs in Electromagnetics

Magnetostatic: $\nabla^2\mathbf{A} = -\mu\mathbf{J}$

Transient: $\nabla^2\mathbf{A} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} = -\mu\mathbf{J}$ Time harmonic $\Rightarrow \nabla^2\mathbf{A} - j\omega\mu\sigma\mathbf{A} = -\mu\mathbf{J}$

The first equation ($\nabla^2\mathbf{A} = -\mu\mathbf{J}$) is used for magnetostatic analysis. Now again you are very comfortable with this equation and it is Poisson's equation for magnetostatics, where **A** is the magnetic vector potential and **J** is the current density. The second equation $\left(\nabla^2\mathbf{A} - \mu\sigma\frac{\partial\mathbf{A}}{\partial t} =\right.$

$-\mu\mathbf{J}\big)$ also we have seen and had some discussion about this equation. I have also explained about the term $\mu\sigma\frac{\partial \mathbf{A}}{\partial t}$. This is the induced eddy current term. Because $\frac{\partial \mathbf{A}}{\partial t}$ is the induced electric field intensity. So, $\sigma\frac{\partial \mathbf{A}}{\partial t}$ is **J** induced.

So, that is why the units are matching. The units of all terms in the second equation have same unit. $\nabla^2\mathbf{A} = -\mu\mathbf{J}$ is used if it is a static case and $\nabla^2\mathbf{A} - \mu\sigma\frac{\partial \mathbf{A}}{\partial t} = -\mu\mathbf{J}$ is for a general transient case. In the two cases, $\mu\mathbf{J}$ is the source current term and $\mu\sigma\frac{\partial \mathbf{A}}{\partial t}$ is representing the eddy current. And as I mentioned in that previous lecture when you bring $-\mu\sigma\frac{\partial \mathbf{A}}{\partial t}$ to the right hand side, the sign of the eddy current term becomes plus and as it should be plus because the signs of source current and induced eddy current should be opposite.

But their phase difference will not be exactly 180^o because all the quantities in the equation will be phasors and angle will get decided by many other quantities also. But maybe a little later we will discuss this. But in general, the sign should be opposite. And now if you are dealing with time-harmonic case that means all the quantities are varying sinusoidally with time then you can convert $\frac{d}{dt}$ as $j\omega$ and then you get the following equation where now **A** and **J** are phasors.

$$\nabla^2\mathbf{A} - j\omega\mu\sigma\mathbf{A} = -\mu\mathbf{J}$$

I have not represented phasor quantities by another symbol for sake of brevity and simplicity. But remember in this equation, **A** and **J** they will be phasors. Now, let us go further.

Now we are going back to the electrostatic problem. We have to minimize the following.

$$\text{Minimization of } \int \frac{1}{2}\varepsilon E^2 dv \Rightarrow \int \frac{1}{2}\varepsilon\left(\frac{\partial V}{\partial x}\right)^2 dx \times 1 \times 1 \quad \text{for 1-D Electrostatics}$$

So, in the remaining course, what we will do is, we will start with electrostatics. Then we will go to magnetostatic fields, time-harmonic, axi-symmetric, permanent magnets, transients, forces, coupled circuit field and in that way, we are going to proceed in this course by increasing

complexity. Electrostatics is probably the simplest thing to analyse. So, we are starting with electrostatics. So, you are minimizing the energy in electrostatics. Also note that for 1D electrostatics $dv = dx \times 1 \times 1$.

For a standard capacitor problem, we know that the energy is $\int \frac{1}{2}\epsilon E^2 \, dv$ because $\frac{1}{2}\epsilon E^2$ is energy density. And that is why we started directly by assuming the expression of energy (given in the above equation) that has to be minimized. But in general, for a given partial differential equation how do we derive the expression of energy and how the corresponding energy is minimized?

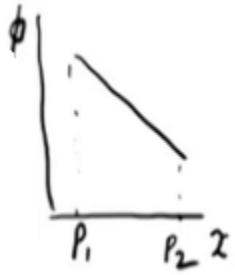
Later on, we will see how do we find the corresponding energy functionals that need to be minimized for each of the above discussed partial differential equations. So, then we are coming to an important concept what is known as functional. Now, functional is the expression for energy. In variational calculus, we term energy as functional $F(\phi)$ where ϕ is potential.

That is the reason in the previous lectures on the basics of electromagnetics, we called flux by ψ otherwise it will interfere with this symbol of potential (ϕ). So, here ϕ is the potential. It could be electric potential, magnetic potential or any other potential. In this case, if you are dealing with electrostatics, ϕ will be electric potential *V*.

So, in this case, the functional in a one-dimensional problem is defined in general form, from P_1 to P_2, as given in the following equation and the potential is varying with *x*

$$F(\phi) = \int_{P_1}^{P_2} f(x, \phi, \phi') dx \quad \phi(P_1) = \phi_1 , \phi(P_2) = \phi_2$$

Potential may vary in any way, I have just shown by a straight line in the following figure,



But it can vary depending upon the boundary and source conditions. Also, the functional energy can be in general represented by $f(x, \phi, \phi')$ and ϕ' is derivative. For example, if you consider this functional for electrostatic case, only V' is in the functional (or energy) because E is nothing but $\frac{dV}{dx}$.

For this case in energy functional only ϕ' or V' term will appear. But in general, it will not be so. It will be a function of x, ϕ, ϕ'. For example, in case of a Poisson's equation, you will get it as a function of ϕ, that will see later. So, there will be some term in the functional expression which will be a function of ϕ.

In this equation, you have one independent variable (x) and one dependent variable (ϕ). Because, you have Laplace's equation, only ϕ' appears in the functional. We will go further.

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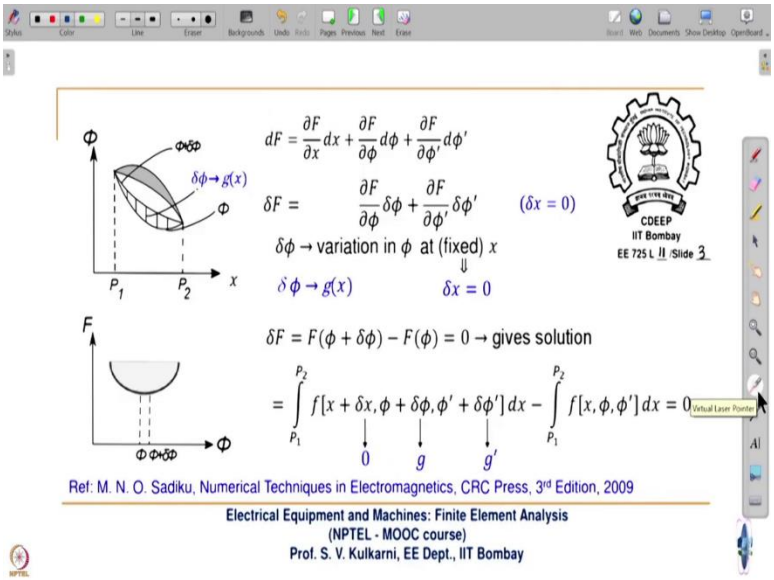
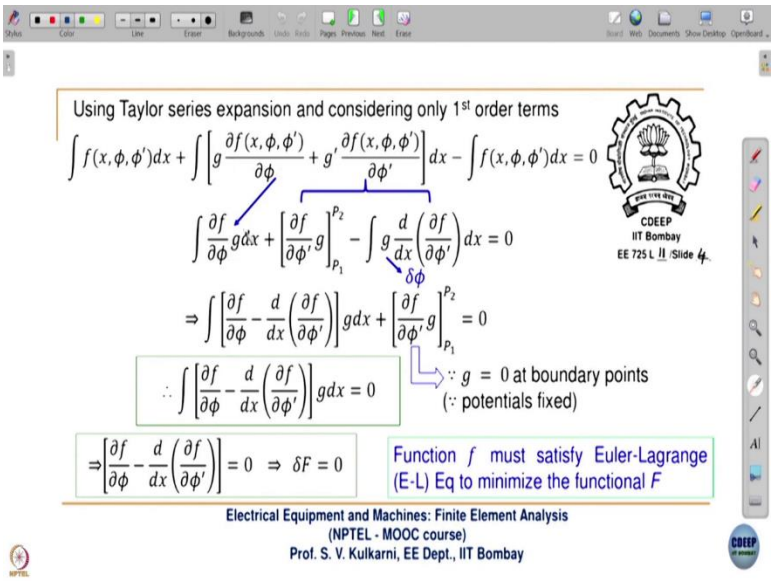




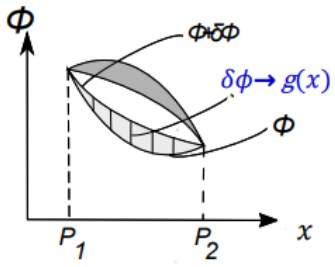
Now, we will go further and discuss more about the functional *F* and the corresponding variational calculus. We know the total differential of a function F is defined as

$$dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial \phi}d\phi + \frac{\partial F}{\partial \phi'}d\phi'$$

But when it comes to variation in *F* (which is defined in the following equation), we are not varying *F* with *x*. That is why we are saying $\delta x = 0$.

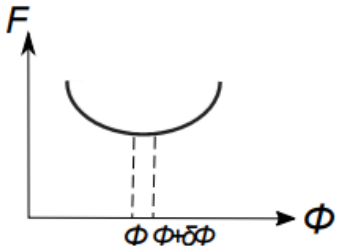
$$\delta F = \frac{\partial F}{\partial \phi}\delta\phi + \frac{\partial F}{\partial \phi'}\delta\phi' \qquad (\delta x = 0)$$

What it means? The energy *F* is minimized by varying the potential at each point in the domain and for one combination of potentials at various points we will get minimum energy. Now in the following figure of a one dimensional domain, there are so many points in between the points P1 and P2.



At each of these points, we will vary the potential values. For example, we are varying potential from some initial guess value (ϕ) to a higher value $(\phi + \delta\phi)$. So, at every point, we are varying the potential, but we are not varying *x*. We are not varying potential as a function of *x*. At every *x*, we vary potential. That is why, here we are varying potential which is given by $\delta\phi$. We are doing variation in ϕ. Similarly, as ϕ is varied, ϕ' also will vary, because ϕ is a function of *x*. So, again there will be variation in ϕ'. So, as you vary potential, there will be variation in potential derivative also. But we are not varying *x* and that is why here $\delta x = 0$. That is why $\frac{\partial F}{\partial x}\delta x = 0$ because $\delta x = 0$.

So this is the main concept of variational calculus. Again remember, $\delta\phi$ is variation in ϕ at a fixed *x*. In the above figure, various ϕ distributions that are attempting to minimize the functional *F* (which is the energy) are given. Now, here let us see the graph in the following figure which represents variation of *F* with ϕ.



Now, remember in above figure, ϕ and $\phi + \delta\phi$ are a set of values of potentials at various points in the problem domain.

Suppose if there are say hundred points in the domain and we calculate the energy associated with hundred points, so that corresponding total energy is given in the vertical axis of the figure, and the ϕ will correspond to a vector of the hundred values. So, all those hundred values will decide

the energy of that one-dimensional region. So, now if you change the potentials at those hundred points, you will get a new set of values of potentials that will give another curve of potential versus *x*.

And that may give some other energy value. So, now when you are approaching the minimum energy point, if you change the potential values at all the nodes by some small value, it is not going to lead to any appreciable change in the total energy content. That means we have reached the minimum. So that is the meaning of the statement $\delta F = 0$. Now the δF can be represented by the following equation.

$$\delta F = F(\phi + \delta\phi) - F(\phi) = 0 \rightarrow \text{gives solution}$$

That means when we are near the minimum energy point, then the change in energy with variation in ϕ is 0. And that corresponding ϕ is the solution. Now let us further expand the above equation for a one-dimensional domain between P_1 and P_2 as given in the following equation.

$$\delta F = F(\phi + \delta\phi) - F(\phi) = 0 \rightarrow \text{gives solution}$$

$$= \int_{P_1}^{P_2} f[x + \underset{0}{\delta x}, \phi + \underset{g}{\delta\phi}, \phi' + \underset{g'}{\delta\phi'}]\, dx - \int_{P_1}^{P_2} f[x, \phi, \phi']\, dx = 0$$

Now again I just want to repeat *f* is just a function of x, ϕ, ϕ' and integration of *f* with *x* is *F*, so functional F is function of function *f*. So, again the difference between *F* and *f* is highlighted. Now, if we further simplify this expression by using the standard Taylor series expression $\left(f(x+h) = f(x) + hf'(x)\right)$ and remembering that in this case there are two variables ($\phi, \phi'$and $\delta x = 0$), the following expression is obtained.

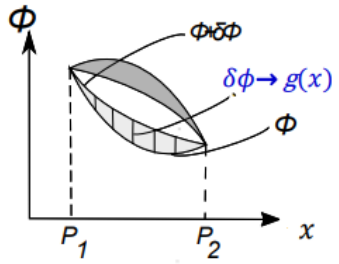
$$\int f(x,\phi,\phi')dx + \int \left[g\frac{\partial f(x,\phi,\phi')}{\partial \phi} + g'\frac{\partial f(x,\phi,\phi')}{\partial \phi'}\right]dx - \int f(x,\phi,\phi')dx = 0$$

$$\int \frac{\partial f}{\partial \phi} g\,dx + \left[\frac{\partial f}{\partial \phi'} g\right]_{P_1}^{P_2} - \int g\frac{d}{dx}\left(\frac{\partial f}{\partial \phi'}\right)dx = 0$$

($\delta\phi$)

Now, the second integral can be written as indicated in the above equation.

And now g is $\delta\phi$, so g is a function of x, because this variation of ϕ at every point is a function of x as indicated in the following figure.



Because, as shown in the above figure, in the region near to the boundary points the variation is small and in the middle region, it is higher. So, the variation of ϕ is a function of x. So that is why g is a function of x.

And then the second term of the second integral is split by using integration by parts as shown in the above equation. Now the above equation is simplified as given below.

$$\Rightarrow \int \left[\frac{\partial f}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial f}{\partial \phi'}\right)\right] g\,dx + \left[\frac{\partial f}{\partial \phi'} g\right]_{P_1}^{P_2} = 0$$

$\because g = 0$ at boundary points
($\because$ potentials fixed)

And the second term becomes 0. Because when that term is evaluated at boundaries, it will reduce to 0 because g is 0 at the boundaries as the potentials are already fixed at the two boundary points. So, the second term becomes 0 and then the expression $\delta F = 0$ is simplified as the following equation.

$$\int \left[\frac{\partial f}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial f}{\partial \phi'}\right)\right] g\,dx = 0$$

So, when the above integral is equated to 0, we are minimizing the corresponding energy. So, if this integral has to be 0 and if energy has to be minimum, effectively the integrand has to be 0 as given below, because for any arbitrary $g(x)$ the whole integral can be 0 only if the integrand is 0 which leads to Euler-Lagrange equation. And the bracketed term on the left hand side is called as Euler-Lagrange expression.

$$\Rightarrow \left[\frac{\partial f}{\partial \phi} - \frac{d}{dx}\left(\frac{\partial f}{\partial \phi'}\right)\right] = 0 \quad \Rightarrow \quad \delta F = 0$$

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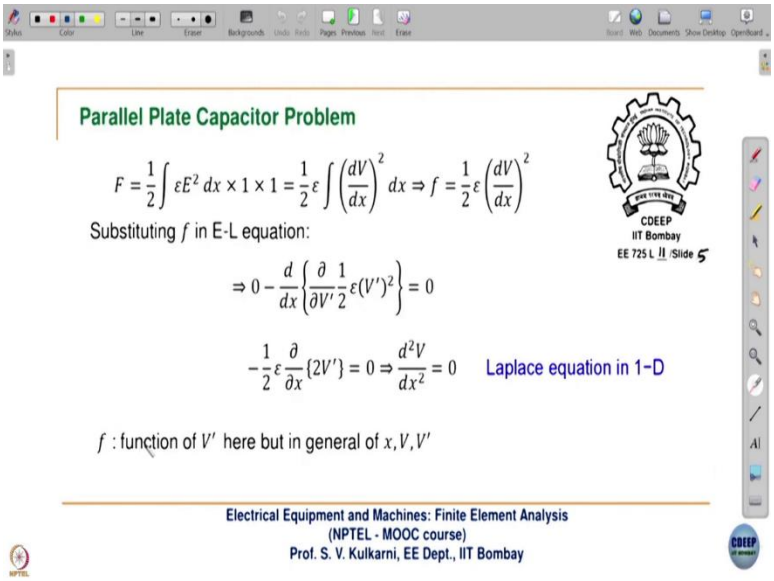

Now, let us see an application and an example to understand what we just now saw in a little bit more depth. Let us again take a parallel plate capacitor problem and we know the functional for that is as given below.

$$F = \frac{1}{2}\int \varepsilon E^2\, dx \times 1 \times 1 = \frac{1}{2}\varepsilon \int \left(\frac{dV}{dx}\right)^2 dx \Rightarrow f = \frac{1}{2}\varepsilon\left(\frac{dV}{dx}\right)^2$$

And it is one dimensional (fringing is neglected) that is why dv reduces to $dx \times 1 \times 1$. And E is nothing but $\frac{dV}{dx}$. And then $f = \frac{1}{2}\varepsilon\left(\frac{dV}{dx}\right)^2$.

Then $\int f dx$ is nothing but F (functional). Now, suppose if we want to solve this capacitor problem and find out the field distribution, we will assume some potential distribution (V). If that potential distribution V is the exact solution and then when it is substituted in this Euler-Lagrange equation, the energy due to the potential distribution will be minimized because the solution is known and it is exact.

So, when you substitute the exact solution V in Euler-Lagrange expression, you will get it equal to 0. And that is what we are assuming that this V is known and it is the exact solution. And when it is substituted in the Euler-Lagrange expression, we will get 0 on the right hand side. And then if that energy gets minimized in that way and then this V is the exact solution. Now, let us see when you substitute the above expression of f in Euler-Lagrange equation and then further simplify as given below.

Substituting f in E-L equation:

$$\Rightarrow 0 - \frac{d}{dx}\left\{\frac{\partial}{\partial V'}\frac{1}{2}\varepsilon(V')^2\right\} = 0$$

$$-\frac{1}{2}\varepsilon\frac{\partial}{\partial x}\{2V'\} = 0 \Rightarrow \frac{d^2V}{dx^2} = 0$$

You will get $\frac{d^2V}{dx^2} = 0$ which is Laplace's equation in 1D. So, when the exact solution was known to us and we substituted the corresponding f in Euler-Lagrange expression, we will get the minimum energy condition. And therefore, the right hand side of the equation will be zero satisfying Euler-Lagrange equation and this further leads to $\frac{d^2V}{dx^2} = 0$ which is Laplace equation.

So, again I want to highlight this. Here f was only function of $V' = \frac{dV}{dx}$. But in general, it can be a function of x, V and V'. We will see such examples in some other cases later. Thank you.

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