Principles of Digital Communications
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Lecture – 58
Non-coherent BFSK

When a signal is transmitted on a communication channel, then due to noise and other effects there could be phase uncertainty in the received signal. We have learnt coherent binary FSK modulation schemes and also seen how to demodulate binary FSK. But in the demodulation scheme which we studied earlier, we assume that the receiver was synchronized to the transmitter in the sense that there is no phase uncertainty.

The question is that if there is a phase uncertainty, then how do you take care of it? If you still want to do the demodulation using coherent modulation scheme, then we need to estimate the phase of the received signal. But if you want to avoid the complexity of the phase estimation, then we can deploy or adopt what is known as non coherent demodulation technique; in which the phase uncertainty is taken care of.

So, today we will study non coherent optimum receiver for the binary FSK modulation technique.

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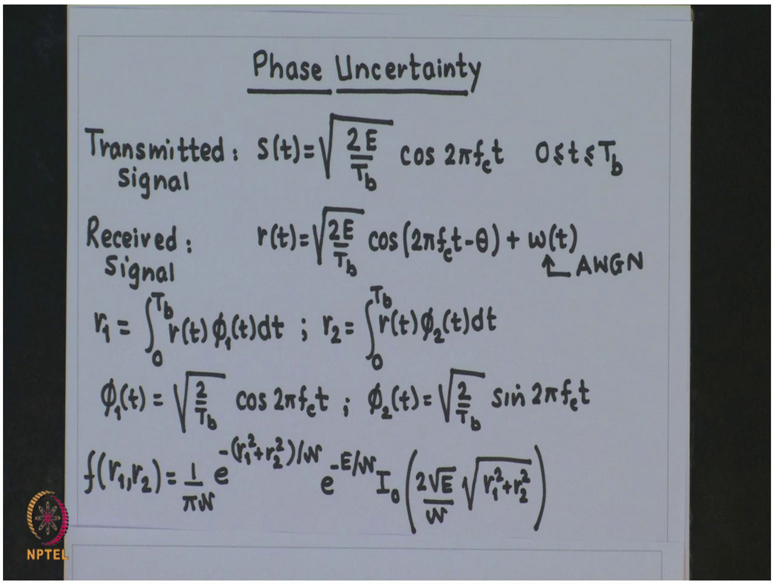

In the previous class, we learnt that given a transmitted signal which is a sinusoid of a frequency f c, then if the received signal has the phase uncertainty given by theta with the additive white Gaussian noise.

And if you are interested in the sufficient statistics r 1 and r 2 which are the projections of the received signal onto the orthonormal quadrature carriers; phi 1 t and phi 2 t shown here, then we can find out the joint pdf of r 1 and r 2 and it was shown that the expression involves the modified Bessel function of the first kind.

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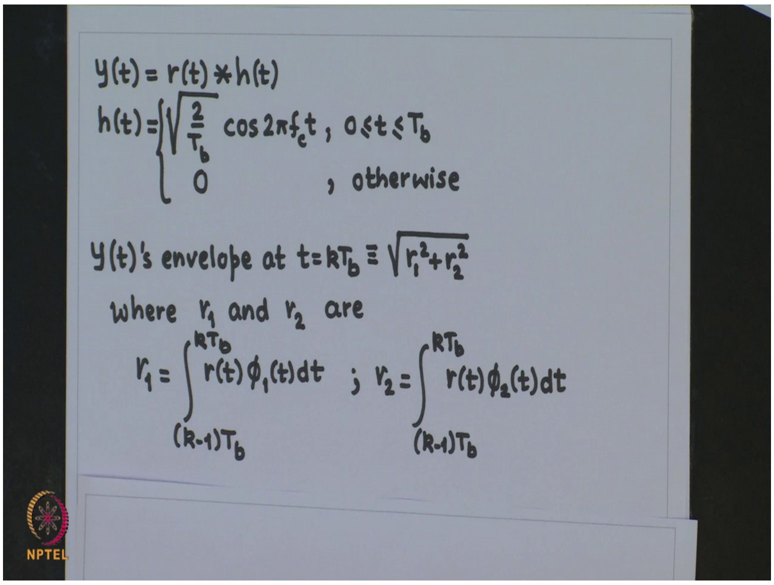

We also studied that in order to obtain the square root of the squares and summation of the sufficient statistic, we can take the received signal and pass it through a filter h t, which has an impulse response which is a sinusoid over a duration from 0 to T b. And if we sample the output envelope at t equal to k T b; then we obtain the square root of r 1 squared plus r 2 squared. Now, we will use this results for our study on non coherent binary FSK.

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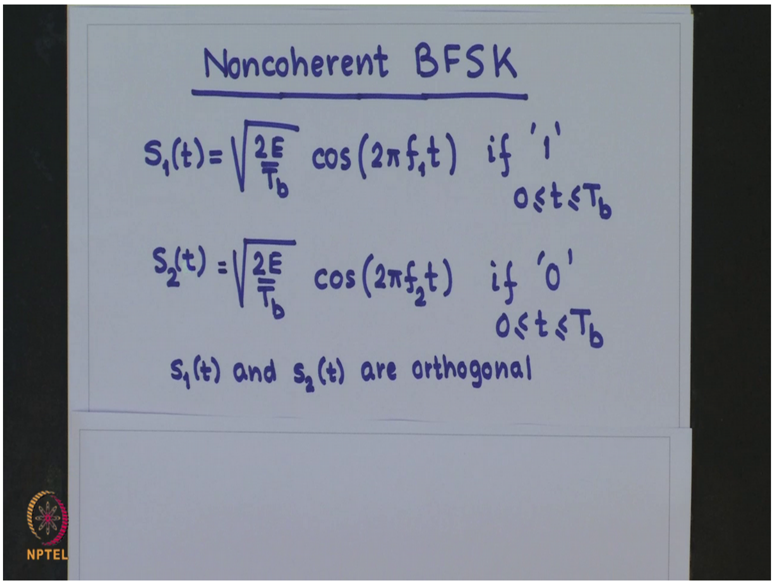

So, non coherent binary FSK has 2 signals in the say message signal set given by S 1 t with the frequency f 1 and another signal S 2 t with the frequency f 2. For symbol 1, we transmit the frequency f 1 sinusoid and for symbol 0, we transmit the sinusoid with frequency f 2; both are over the duration T b.

And we will assume that S 1 t and S 2 t are orthogonal, so we have studied this earlier that for non coherent orthogonality; the difference between f 1 and f 2 should be integral multiple of 1 by T b. So, now, let us try to find out the optimum receiver for transmission of this signals on a communication channel, where there is phase uncertainty.

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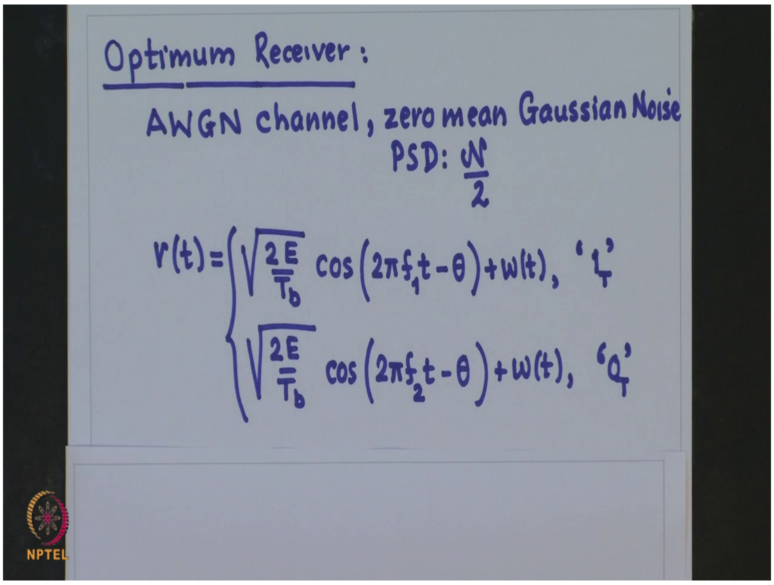

So, our model would be again additive white Gaussian noise channel with zero mean Gaussian noise and power spectral density given by italic N by 2. Our received signal would be given by the sinusoid with phase uncertainty in the term theta plus additive white noise, this is for if we have transmitted 1. And if we have transmitted 0; the received signal would be of frequency f 2 with additive white Gaussian noise.

Now, given this received signal it is very easy to see that the orthonormal basis signals which can be used to obtain the sufficient statistic for our decision making process would consist of the following 4 orthonormal basis signals.

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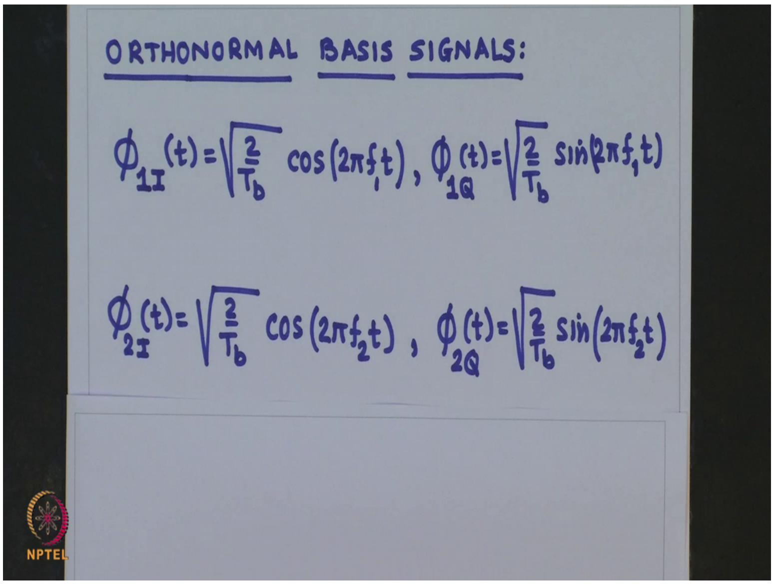

The this set phi 1 I and phi 1 Q will take care of the representation of the sinusoid at the frequency f 1 with the phase uncertainty theta. And this orthonormal signals will take care of the frequency f 2 with the phase uncertainty. So, the received signal would be projected on all this 4 orthonormal basis signals and the vector would be as follows.

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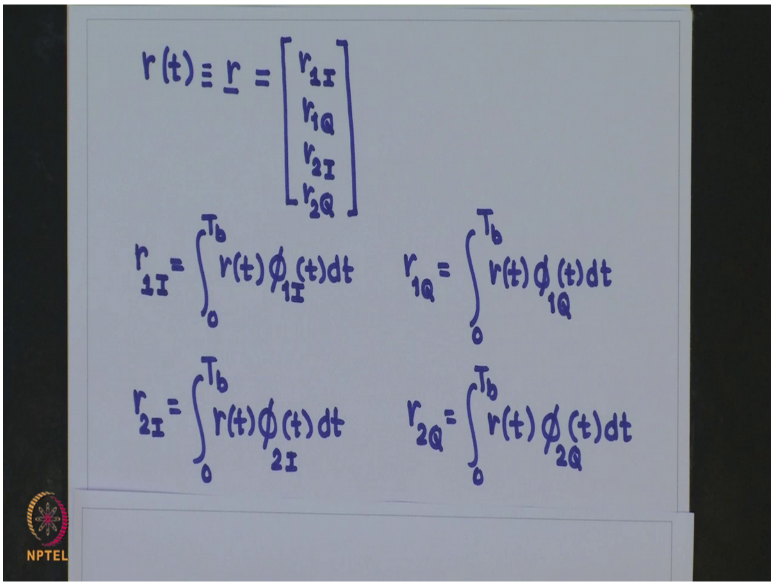

We have the rt signal and the projected vector is r with the components r 1 I, r 1 Q, r 2 I, r 2 Q, r 1 I will be the projection of r t on phi 1 it. And similarly r 1 Q would be the projection of r t on phi 1 Q t and similarly for the other components. And these are the expressions for obtaining this 4 components of the vector r.

Now let us look at; so, this 4 components now, they are sufficient statistics for our decision making process. Now let us see what are this outputs when we transmit a particular signal at the transmitter.

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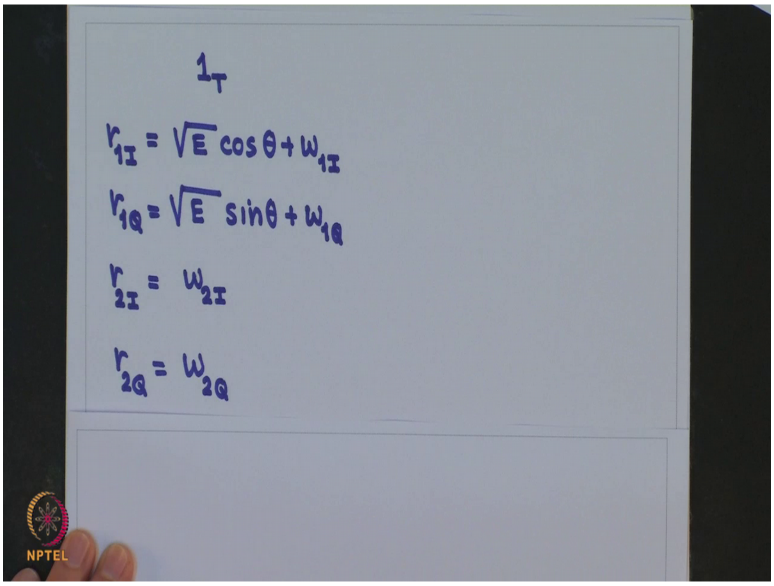

So, let us take first the case of transmission of symbol 1; in which case we will transmit the sinusoid with frequency f 1. Now, when we take the received signal, it will have this 4 projections; since the signal transmitted is 1, the first two components of the vector r will have the contribution from the signal and cos theta and sin theta is because of the phase uncertainty at the receiver.

And other two components will be consisting of only the noise because these two components will get the contribution from the signal only when there is a transmission of 0. So, similarly we will find out what are these components when we transmit the symbol 0.

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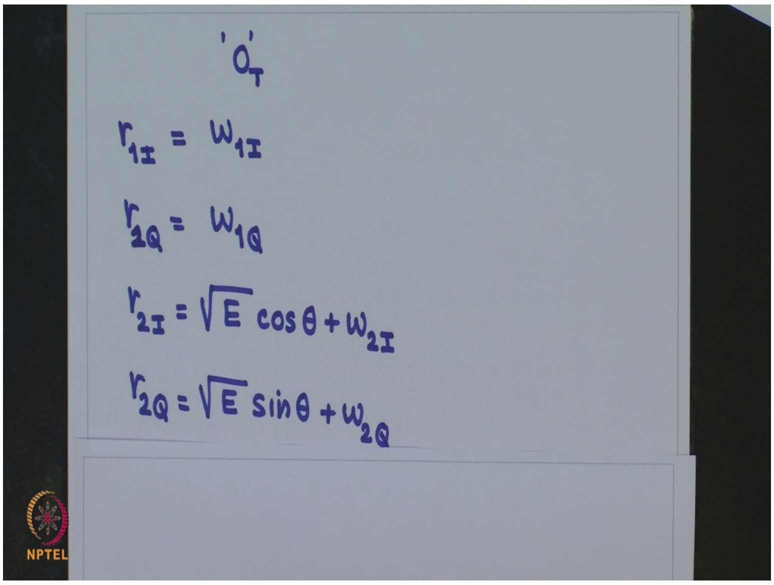

In that case, those 4 components would be as follows. So, we see that the first two components of the vector r have only the contribution from the noise; whereas, the third and the fourth components of the vector r have the contribution from the transmission due to the symbol 0 ok.

Now, note that all these noise components W 1 I, W 1 Q, W 2 I and W 2 Q are uncorrelated Gaussian random variables with zero mean and variance given by italic N by 2. And since they are uncorrelated Gaussian random variables, it also implies that they are statistically independent.

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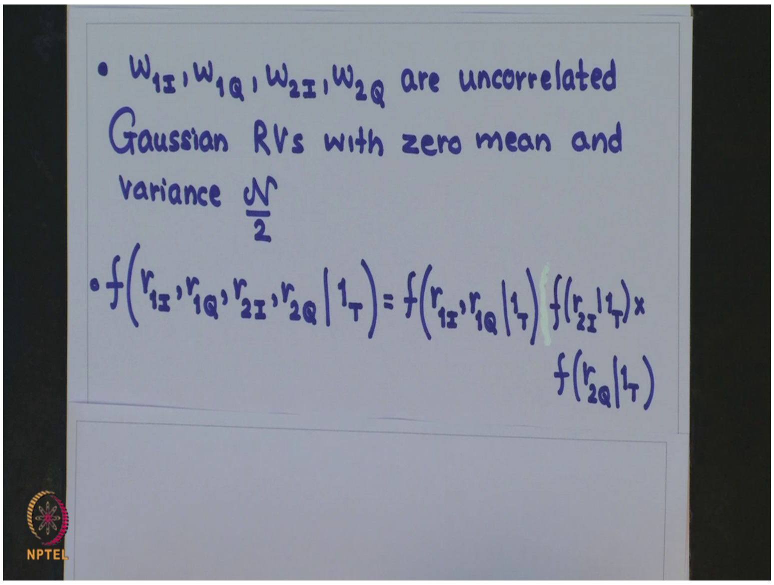

So, now, we need to find out the joint pdf of the components of the received vector under the condition that one is has been transmitted. And also calculate the joint pdf of these 4 components under the condition when 0 has been transmitted.

Once we get this joint pdf, then we can find out the maximum likelihood ratio and to do that we will proceed as follows. The first thing is that we compute the joint pdf of the 4 components of the vector r given that 1 was transmitting. Now, as mentioned earlier all this are uncorrelated Gaussian random variables. So, we can write this as follows the joint pdf would be equal to the product of the joint pdf of r 1 I, r 1 Q given one was transmitted.

And the joint pdf of r 2 I, r 2 Q given 1 was transmitted, but joint pdf of r 2 I, r 2 Q given 1 was transmitted is the product of the conditional pdf of r 2 I given 1 was transmitted multiplied by the conditional pdf of r 2 cube given 1 was transmitted.

Now, to find out this conditional pdfs it is very simple; now to find out the joint pdf of r 1 I, r 1 Q given 1 was transmitted, we will use the result from our earlier class. So, joint pdf of r 1 I, r 1 Q given 1 was transmitted is evaluated using the same approach as studied earlier. And based on that result; this result which we have derived earlier, we will use this result.

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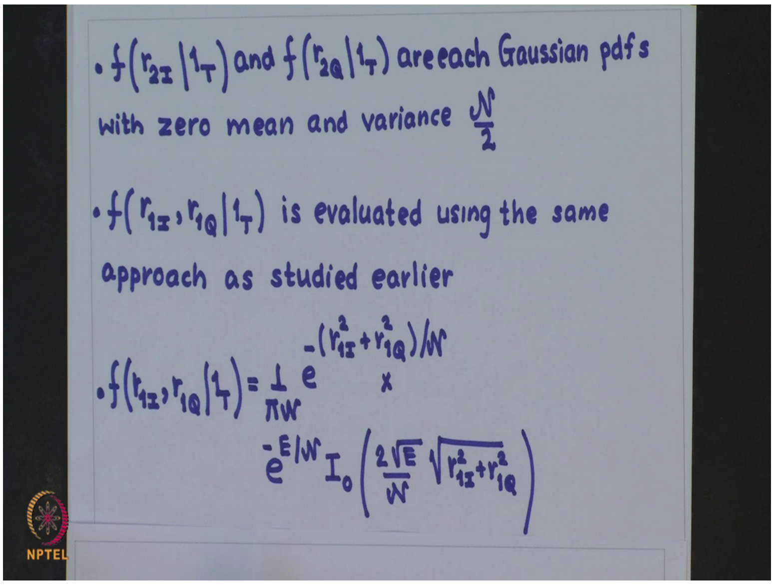

And we can write the joint pdf of r 1 I, r 1 Q given 1 was transmitted as this expression here.

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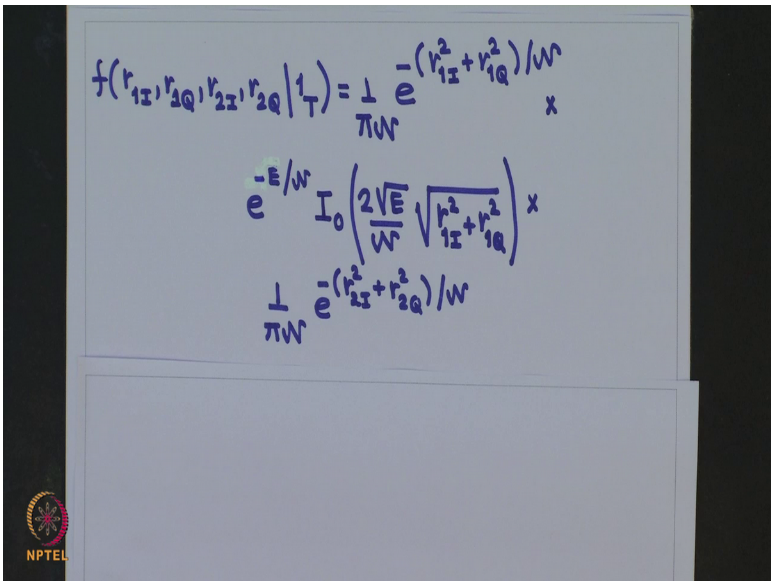

Now, given this expression; we can write the joint pdf for all the 4 components together given 1 was transmitted; this is a conditional joint pdf, it would be of this form. This has been obtained by just plugging the appropriate pdfs; now similarly we have to find out the conditional joint pdf of the 4 components of the vector r under the condition that 0 was transmitted. So, whatever we have studied so far we can extend to the case of obtaining the conditional joint pdf; given 0 was transmitted and that expression would be given as shown here on this slide.

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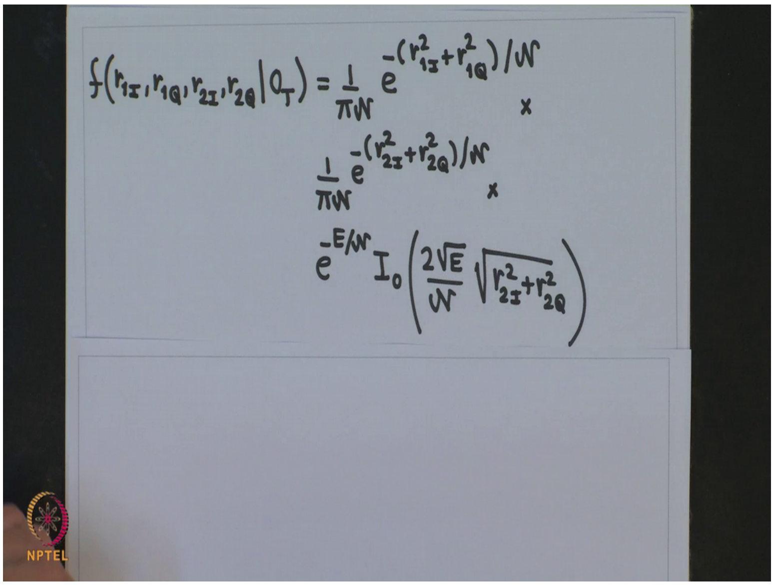

Now, if we assume that the symbols are equiprobable; then in this case the optimum detector or the receiver or demodulator would be to compute the likelihood ratios of the 2 conditional pdfs and if we do that we will get the following expression here.

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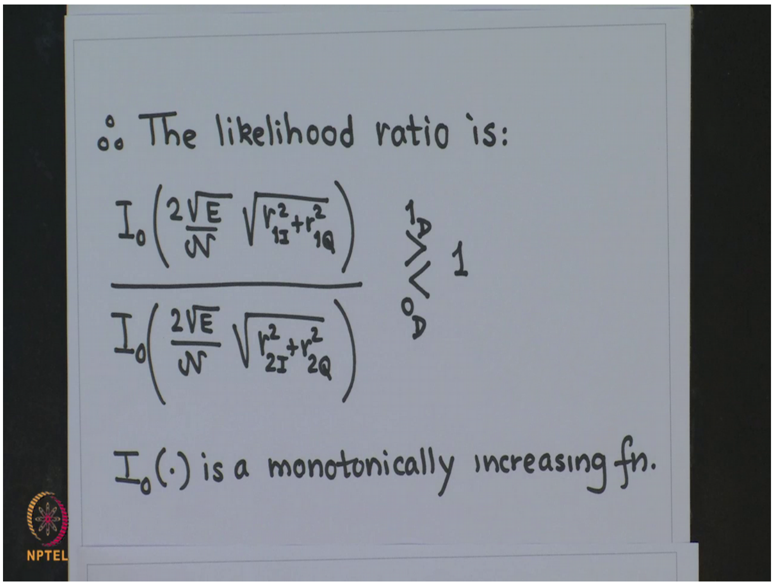

It is important to note that other factors which are there associated with the joint pdf will cancel out. So, if you look at this 2; this factor will cancel out with this similarly this factor will cancel out with this and this factor will cancel out with this. So, this is in a numerator and this is in a denominator.

So, what is the left out in the numerator is only this term and then the denominator would be only this term fine. So, our decision rule would be now that if this ratio is larger than 1, then I decide in the favor of 1. So, I will take the decision that one has been transmitted and if this ratio is less than 1; then my decision would be in the favor of symbol 0. So, now, this basal function is a monotonically increasing function of its argument. So, what this implies that we can simplify our decision rule to the following rule.

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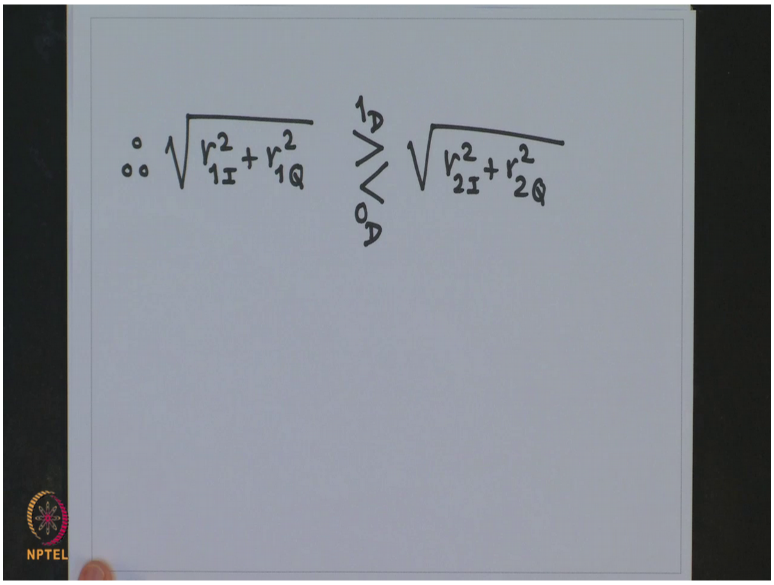

So, this becomes our test statistics now; so, using this test statistic we can take the decision whether 1 was transmitted or 0 was transmitted. Now to obtain this test statistic, we could use the projections of r t on phi 1 I t and phi 1 Q t and then take the output and square them and sum them up and then take the square root.

But we have learned that this same thing could be obtained by passing the signal r t through a linear time in variant system which has an impulse response; which is a sinusoid over a duration of 0 to T b. And then sample the envelope of the output at the correct instances say t equal to k T b. And at that instance the envelope will be same as the quantity given here either on the left hand side or the right hand side correct.

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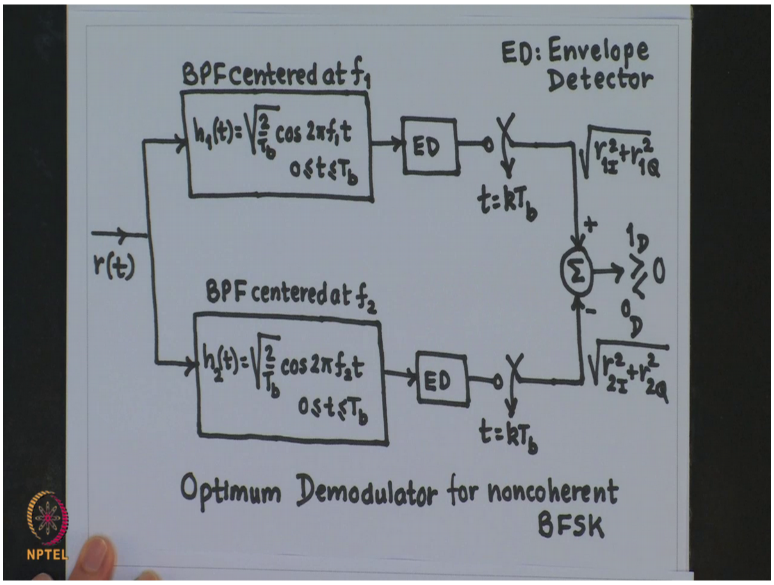

So, we will use this idea for and that case we will get the optimum demodulator which is as given on this slide. So, we have a r t and it passes through a band pass filter which is centered at f 1 and the impulse response of that band pass filter centered at f 1 would be nothing, but a cosine sinusoid with a frequency f 1 over the duration 0 2 T b. And similarly to r t is passed through another band pass filter centered at f 2 with this impulse response. And then both these outputs are passed through the envelop detector and these are sampled at appropriate time.

So, t equal to k T b the sampling instance, the output of this would be the square root of the summation of the squares of the sufficient statistics. And then you compare this with this; if it is a large this is larger than this I decide 1 has been transmitted, otherwise it is 0. Now, we will calculate the probability of symbol error for the non coherent binary FSK case.

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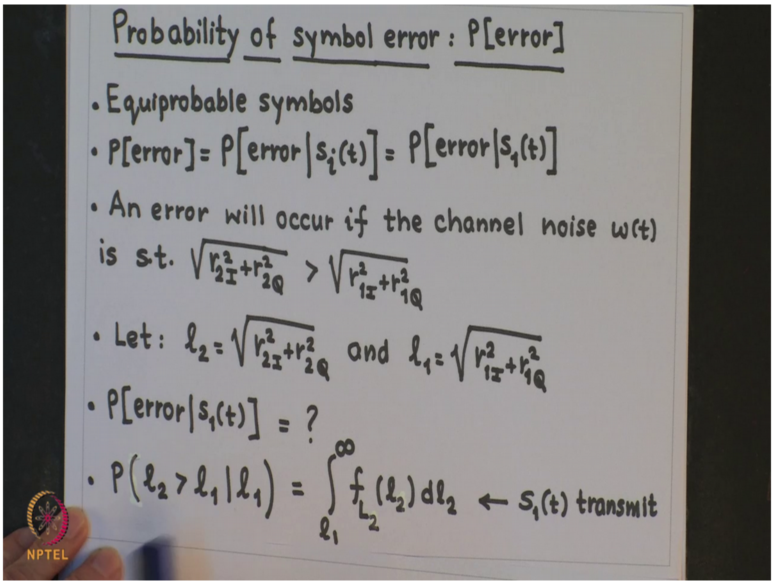

We will assume that we have transmission of equiprobable symbols; so, that implies that the probability of error can be computed as the conditional probability of error given transmission of any message signal S i t and without loss of generality we will assume that S i t to be S 1 t.

So, we will compute the conditional probability of error given S 1 t transmission has taken place. Now an error will occur if the channel noise W t is such that the quantity on the left hand side is larger than the quantity on the right hand side of this inequality. Now, let us define l 2 as the square root of this quantity and l 1 as the square root of this quantity; this new definitions will help us to simplify our calculation. Now, we are required to calculate this conditional probability of error and to do that first we will lead given that S 1 t has been transmitted; what is the probability of this random variable l 2 being larger than the random variable l 1 for some specific value of l 1.

So, we will need to integrate the pdf of the random variable l 2 over l 1 to infinity. Now l 1 is equal to square root of this quantity and an l 2 is the square root of this quantity. Now r 2 I and r 2 Q are Gaussian random variables with zero mean and variance italic N by 2.

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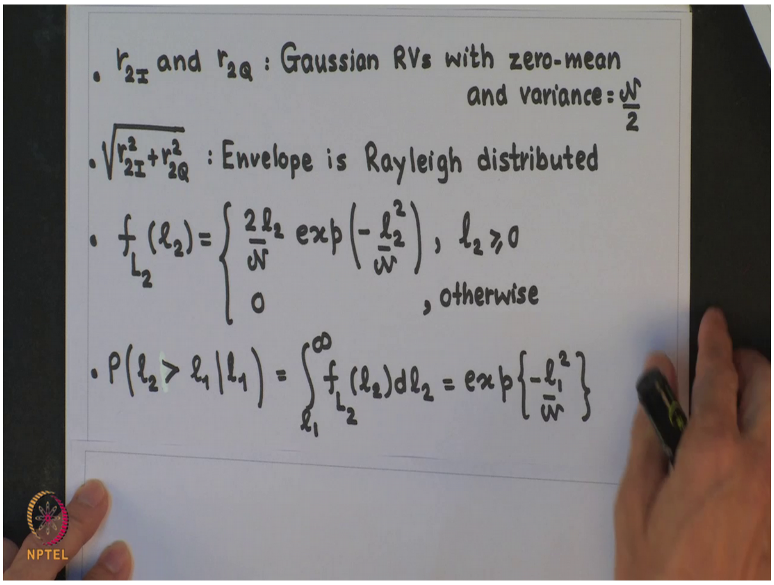

Now, from the theory of probability we know that the square root of the sum of the squares of two random variables with zero mean and variance italic N by 2 is nothing, but the envelope and this is Rayleigh distributed. So, the pdf of l 2 is given by this Rayleigh distribution; so, we have to compute this probability which is integrating this quantity over l 1 to infinity. And it is very straightforward if we integrate, we get exponential of minus l 1 squared by italic.

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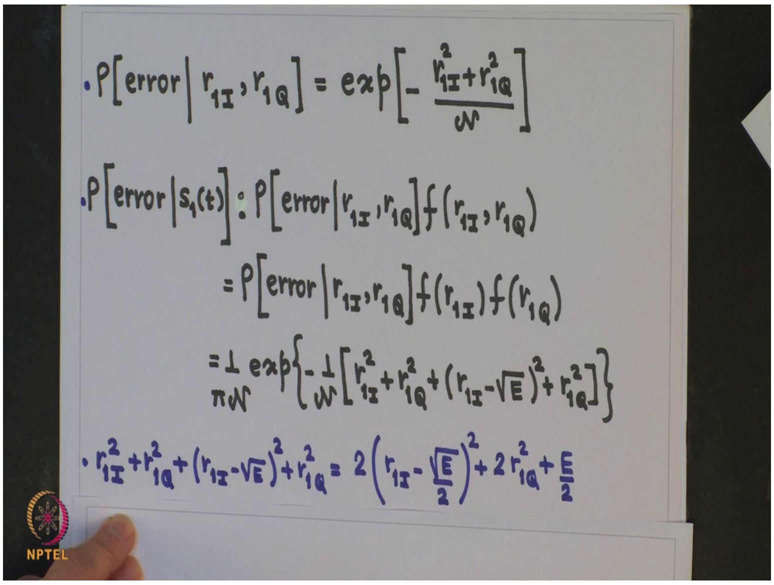

Now we have the probability of error given l 1; that is equivalent to probability of error given r 1 I and r 1 Q. So, in this expression instead of l 1 squared, I substitute l 1 squared to be r 1 I squared plus r 1 Q squared, this will simplify our integration process as we will see soon.

Now, to calculate this conditional probability of error given S 1 t; I will require to compute the this quantity which is the conditional probability of error given r 1 I, r 1 Q and the joint pdf of r 1 I, r 1 Q and then we will have to integrate this over the domain of r 1 I and r 1 Q. Now r 1 I and r 1 Q; they are uncorrelated Gaussian random variables. So, the joint pdf of this would be equal to the product of the two individual pdf because uncorrelated Gaussian random variables implies statistical independence.

Now, we know that probability of r 1 I and r 1 Q are Gaussian pdfs with the mean of root E and 0 respectively. So, using this expression and using the pdf for r 1 I and r 1 Q; we can write this expression as written here. And now we do a little manipulation with this term out here, we try to complete the square by rewriting this term as shown here on the right hand side of this equation. If we use this relationship for this term out here and substitute that and then integrate over the domain of r 1 and r 1 Q; we will get the conditional probability of error given S 1 t and that is what we will do next.

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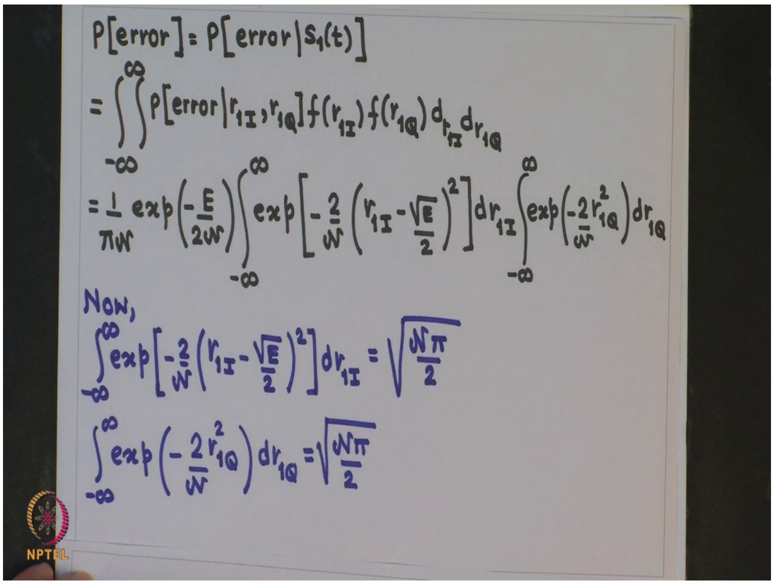

So, this conditional probability of error given S 1 t would be equal to the solution to this integral correct. And we just substitute the values out there and separate out the 2 integrals, here we have used the property of a Gaussian pdf of mean of root E by 2 and the variance italic N by 4. And similarly here this way we have used the property of the Gaussian random variable mean to be 0 and its variance to be italic N by 4.

So, we get this 2 quantities and simplifying this expression after plugging in this 2 values out here, you will get the probability of error for the non coherent binary FSK to be of this.

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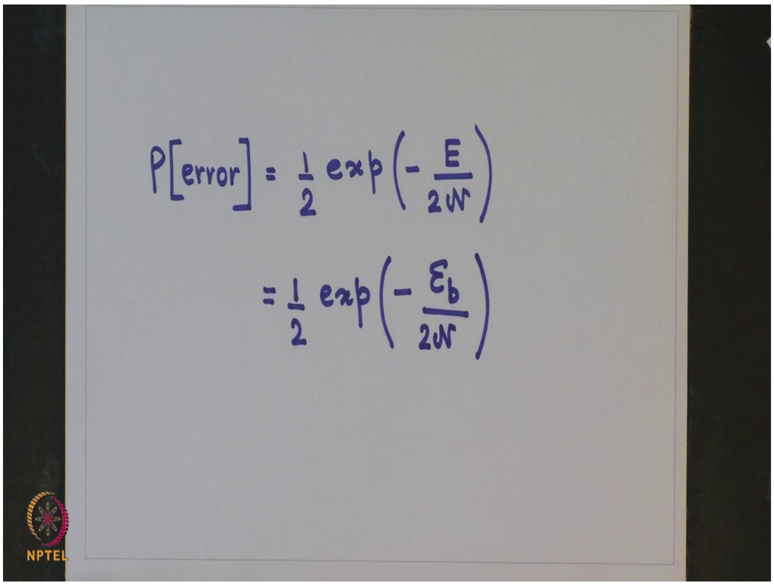

And now remember that in this case, this expression is equivalent to the following expression because in this case the signal energy is the same as bit energy. So, this is the final expression which we get for the probability of error, for the optimum receiver, for the non coherent binary FSK.

Now, the results which we have derived for the non coherent binary FSK can be extended to non coherent M-ary FSK. In principle, this can be extended to any non coherent M-ary orthogonal signaling scheme. Now, we will extend the concepts of non coherent binary FSK to non coherent version of phase shift key popularly known as DPSK and this we will do next time.

Thank you.