Principles of Digital Communications
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Lecture – 56
M-ary FSK

We have studied M-ary ASK and M-ary PSK. Now, we will study M-ary FSK that is frequency shift keying. We will see that M-ary FSK takes a very different approach to modulation.

And we will also learn that the error performance of M-ary FSK, improves with the increase in M with the decrease in the required signal to noise ratio per bit, but at the expense of bandwidth. So, let us begin our study with M-ary FSK.

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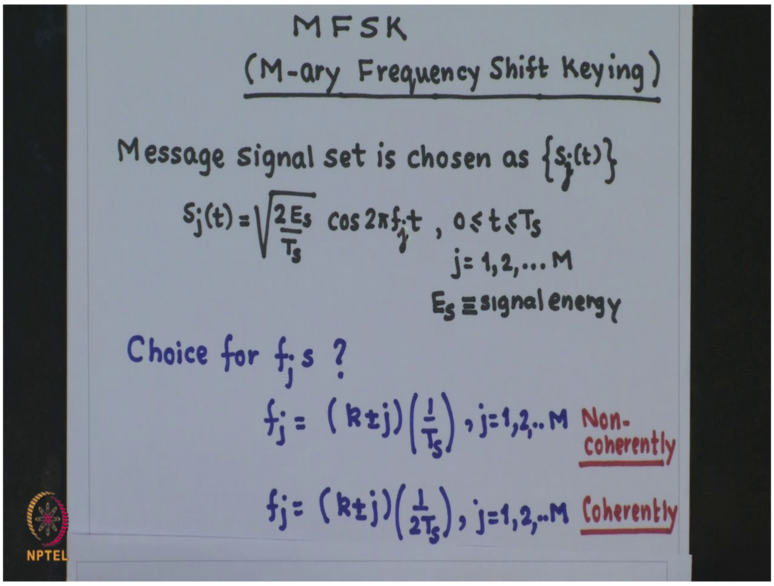

So, the signal set for M-ary FSK is denoted by M sinusoids of frequency each of the frequency f j and each of this sinusoid is over the duration, 0 to T S where T S denotes the symbol interval.

The choice of f j can be done in two ways; one we can choose the difference between the two adjacent f j as 1 by T S and this is known as non-coherently orthogonal. And, another choice to make this signal set orthogonal is to choose f j, where the difference between the adjacent frequencies is 1 by 2 S this implies, that it is coherently orthogonal.

Since, the signal set chosen for the M-ary FSK is orthogonal set; the natural choice for the basis signal to get the signal consolation would be to use the signals, in the message signal set directly after normalizing with the energy of each of the signal which is E S.

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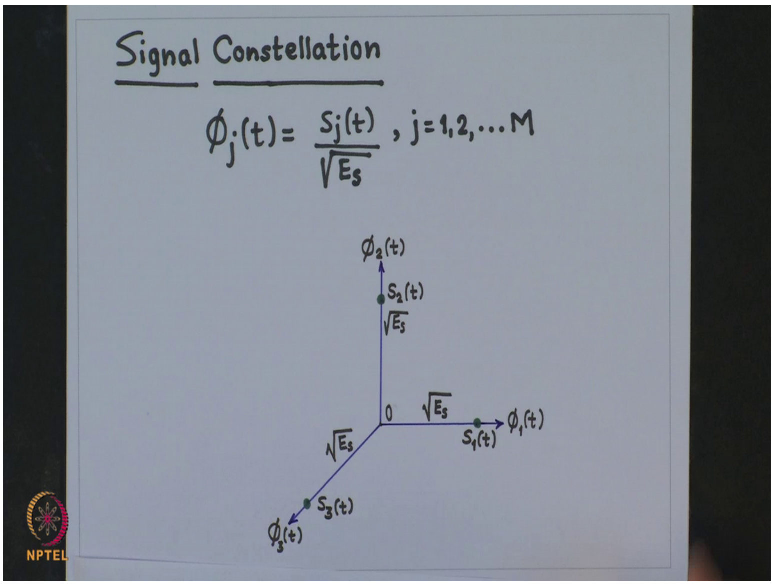

And, it is same for all the signals in this signal sets. Once, we have this orthonormal basis we can obtain the signal consolation for the M-ary FSK signal set. And here is an example for the case of M equal to 3. So, we have 3 signal S 1 t S 2 t and S 3 t and the orthonormal basis signals are shown here. So, the projections of S 1 t on phi 1 t, S 2 t on phi 2 t, and S 3 t on phi 3 t also shown here they are all equal and it is equal to root E S.

Now, let us determine the optimum receiver for the same we will assume the additive white Gaussian noise channel model.

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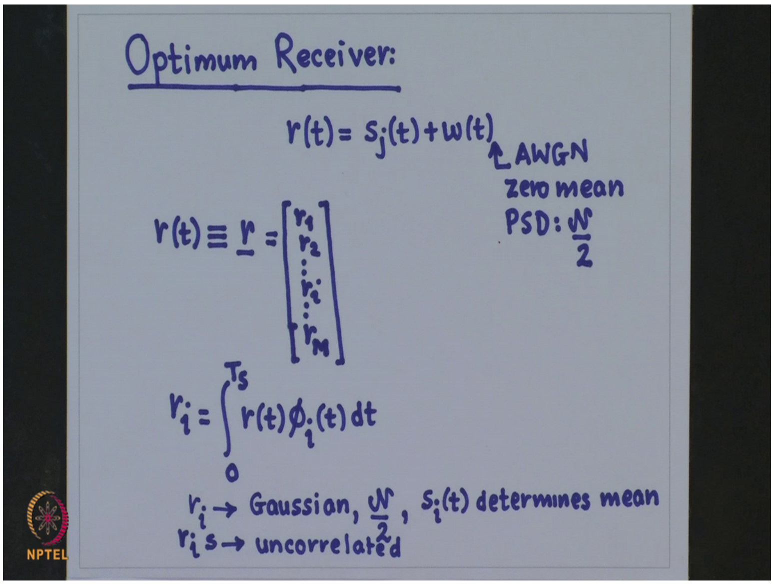

So, your received signal would be r t is equal to S j t plus w t S j t will be decided by the message signal, which we are transmitting. So, the first thing we have to do is basically try to project r t onto the basis signal. So, let us denote the projection vector as r and it is component from r 1 to r M and any r i the component of this projected vector r, would be given by this expression.

So, r i each of this would be Gaussian, since we have assumed the noise to be additive white Gaussian noise with 0 mean and power spectral density italic n by 2. This r i s will be Gaussian with the power spectral density equal to italic N by 2, and the mean of this r i would be determined by the sit, that is the signal which we transmit. And, please remember that we have shown earlier this r i s are uncorrelated and being Gaussian they are also independent.

So, we have to get the message vector for each of the message signals, by projecting the message signal onto the orthonormal basis signal.

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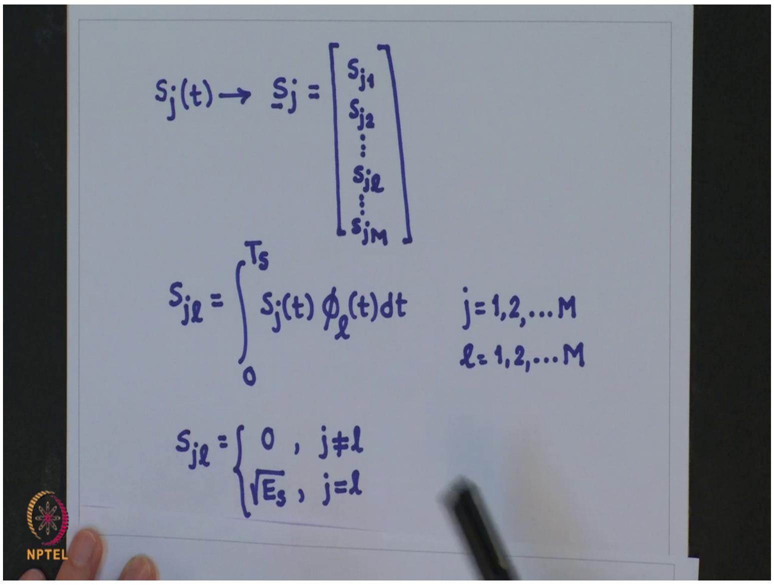

So, we will get M components for the signal S j t and we denote the message vector corresponding to this signal S j t as S j and the components are denoted as S j 1 S j 2 up to S j M. And important to observe that we will get only 1 component in this vector and rest of the other components would be 0. So, S j l would be equal to 0 for j not equal to l and it would be equal to root E S for j equal to l and this is true for all the j from 1 to M and l from 1 to M this is straightforward.

And, we will also assume that our message signals are equiprobable in that case our map detector will reduce to the maximum likelihood detector, which is in this case would turn out to be the minimum distance receiver. And the decision rule would be given by trying to decide in the favor of that i for which this term turns out to be minimum.

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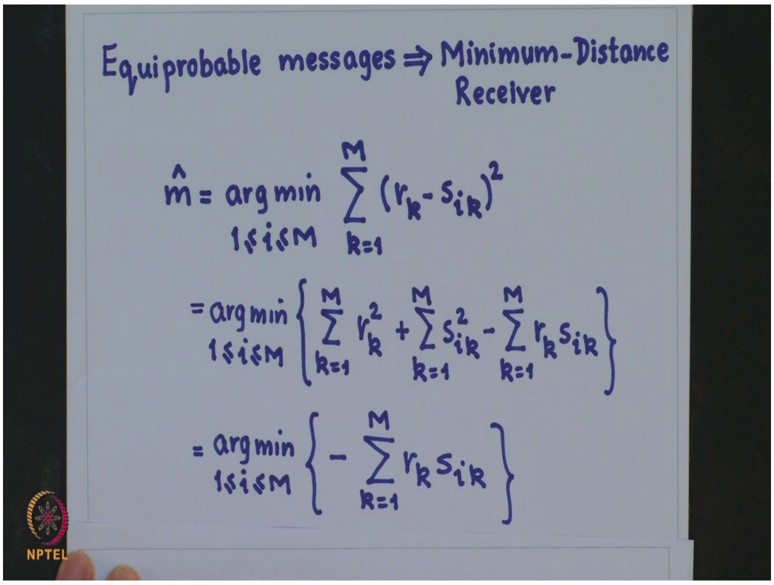

So, this is nothing, but taking the Euclidean distance between the received vector and the message vectors correct.

So, we can expand this term if we expand this term we will get these three terms. Now, remember this is constant for all is because this energy in the projected received vector, this is the energy in the signal S i in our case this is same for all the signals in the message signal set. So, the only variable is this quantity, when we vary i from 1 to M. So, this rule reduces to this rule I have to compute the minimum of this value.

Now, again recollect that from this result this will be non 0 only for k equal to i, and that value S i i would be equal to root E S and that root E S would be same for all I. So, I can change my decision rule to the following decision rule you will decide in the favor of that i for which r i is maximum.

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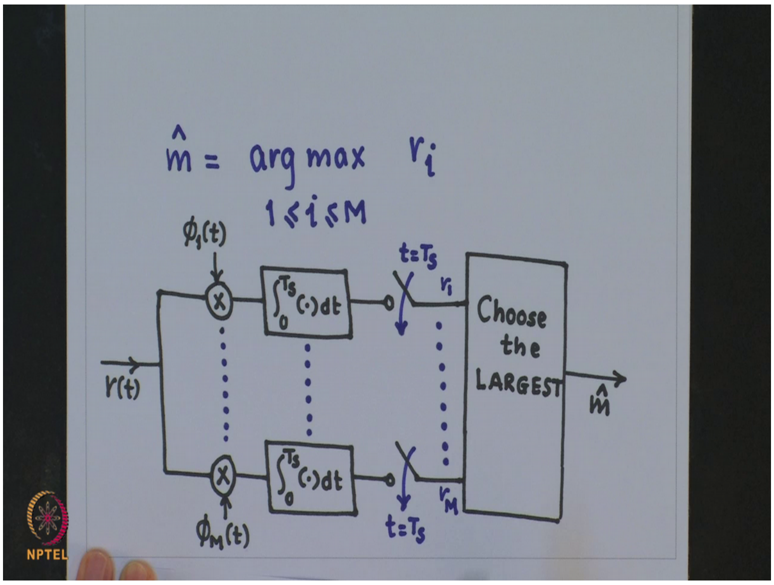

And based on this decision rule the optimum receiver or demodulator would be as shown in this figure, we have r t we can take it is projection correlated sample it appropriate time t equal to T S correct. So, this receiver is shown for the first symbol interval, this can be extended to any other interval, only the limits for these integrals will change.

So, the sampling instances would be equal to k T S. And appropriately these limits would change, this would become k minus 1 T S and this would become k T S and then we choose the largest of this to get the decision.

Now, having done this let us try to calculate the probability of the symbol error, now the signal constellation is symmetric, because the signal constellation is symmetric probability of error would be the conditional probability of error for given any S i t without loss of generality we can assume this sit to be equal to S 1 t.

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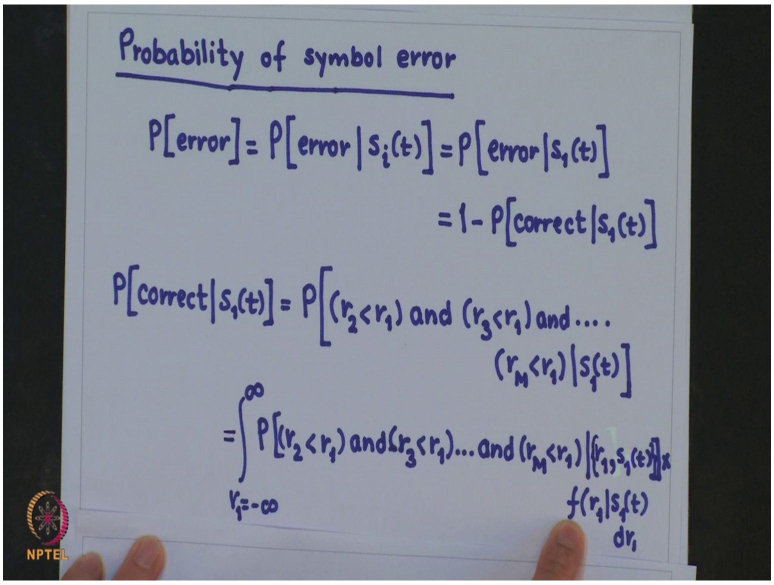

So, the probability of error would be the conditional probability of error, which I would get when I transmit S 1 t. And this probability of error is same as 1 minus conditional probability of correct detection given S 1 t.

So, we will try to compute this value now. So, what we want is that if the probability of correct detection given S 1 t, then this condition should be satisfied remember r 2 r 1 or these are all random variables. So, we want that r 2 should be less than r 1 and r 3 should be less than r 1 and so on. For, all r j S where j is not equal to 1 it should be r j should be less than r 1 and finally, you will get r M less than r 1 for given S 1 t. We, have to evaluate this probability now we can evaluate this probability very easily.

So, what we want that we want this to be true for a given r 1 remember this these are random variables. So, this r 1 out here is a specific value of r 1, this condition is holding good for S 1 t, we have transmitted S 1 is important to remember that correct. So, for this condition, when I have transmitted S 1 tree and if I take a particular value r 1 I have to compute this.

And, then r 1 will vary from minus infinity to infinity. So, in that case what I have to do is that are to multiply this quantity by the conditional PDF of r 1 given S 1 t.

Now, if we have to find this now expression let us do that.

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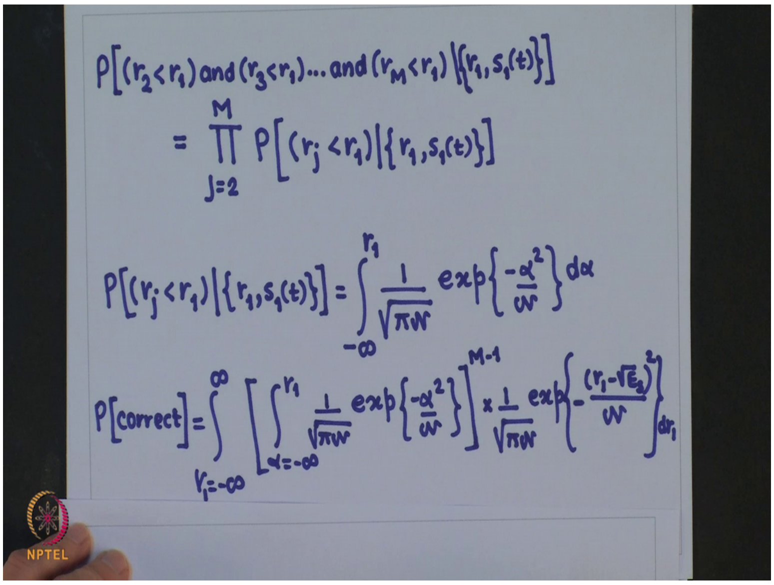

Now, this joint probability distribution is very simple to evaluate remember r 2 r 3 r M all these are projection, will be only due to the noise. The signal contribution is only to the component r 1. So, and we have said that r i s are uncorrelated being Gaussian they are also independent. So, the probability distribution for this would be the product of this conditional probability distribution correct.

So, I evaluate this for j equal to 2 to M. Now, we have to evaluate this term again this term is very simple to evaluate remember we have done this earlier. So, probability of r j less than r 1 for given r 1 and under the condition that, we have transmitted S 1 t would be the integration of the Gaussian distribution or Gaussian PDF, with 0 mean and the variance equal to italic N by 2.

And, this limit would be minus infinity to r 1 because here we have said r j is less than r 1. So, I evaluate this quantity. So, now, this quantity would be this quantity raised to M minus 1 it will be product of this. So, probability of correct detection would be given by this quantity. So, this is this quantity out here is equal to this quantity out here.

And, now we need the conditional PDF of r 1 given S 1 t we know that is also again a Gaussian distribution with the variance italic N by 2 and the mean in this case would be root E S. So, we write this expression for the conditional PDF of r 1 given that, we have transmitted S 1 t.

Now, once we know this probability of correct detection unfortunately you cannot get the closed form solution for this, but we can make this expression little more explicit in terms of the number of messages that is M and the signal to noise ratio per bit and let us do that. So, that is our goal in the next slide.

So, what we do is basically here in this expression out here, we use change of variable we substitute x is equal to root of 2 by italic and alpha and if I do that, I get this expression.

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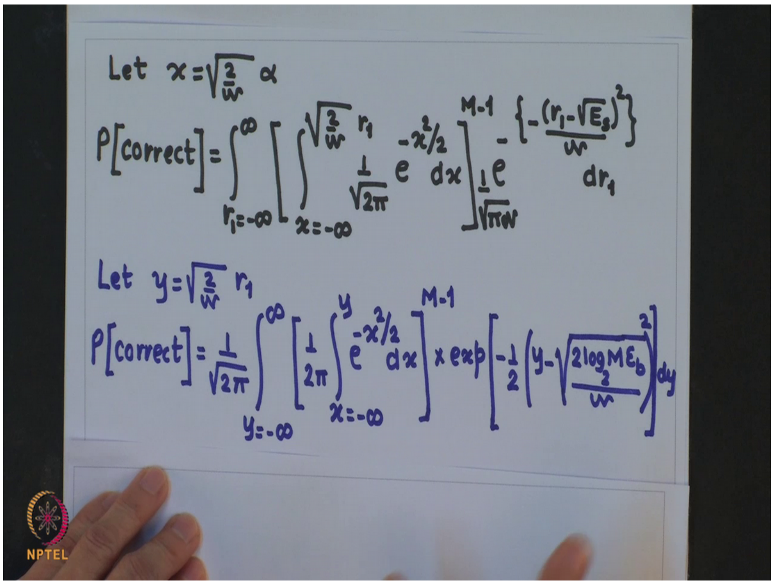

This, expression is the same as this expression out here, after change of variables very simple to see that correct and this has been left in that.

And now here we change once more the variable y is equal to root N by italic N r 1 and if I do that it is not very difficult to see that this expression will reduce to this expression as shown here. As I have said that this is little complicated to evaluate, but we can find out the numerical solution for this.

Now, due to the symmetry of the M-ary MFSK constellation all mappings from sequence of N bits to signal points yield the same bit error probability. So, now, we will try to evaluate the bit error probability, we will get the exact relationship between the probability of bit error and the probability of symbol error for M-ary FSK.

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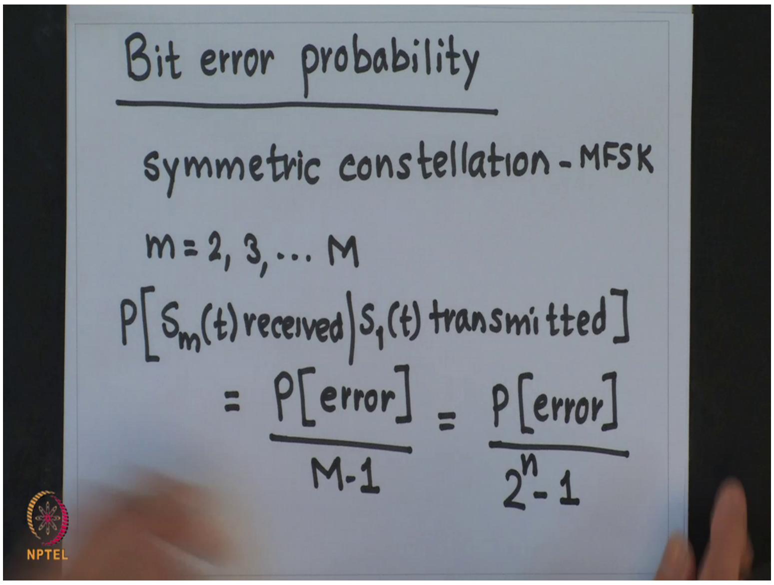

Now, note that M-ary FSK has a consolation, which is symmetric. So, what this implies that the probabilities of receiving any of the message signals. If a from M equal to 2 3 up to M, when S 1 is transmitted I am assuming S 1 t without loss of generality would be equal.

So, probability will be given S 1 t transmitted would be equal to first of all probability of the symbol error there has been error correct and that would be divided by M minus 1. So, this I can write as probability of when I write probability of error, it means probability of symbol error and remember M is equal to 2 raise to n. So, we get this quantity ha.

Now, assuming, that message signal S 1 t.

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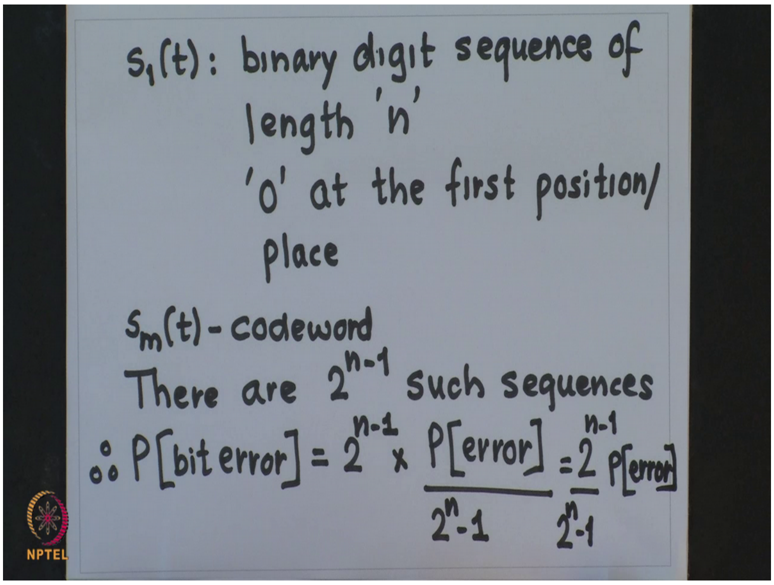

Corresponds to a binary digit sequence of length 'n' with a binary digit of bit '0' at the first position or place this argument can be for any bit in this binary digit sequence, but without loss of generality we are assuming at the first position or place. Now, the probability of an error at this position is the probability of detecting another signal S m t S, which will have the code word corresponding to a sequence with a binary digit 1 at the first position.

Now, there are 2 raised to n minus 1 such sequences, which will differ from the code word sequence corresponding to S 1 t and which has 1 at the first position of place. So, what this implies therefore, that probability of bit error would be equal to 2 raise to n minus 1 because there are so, many sequence multiplied by the this term which is the probability of the symbol error.

So, this is equal to 2 raise to n minus 1 upon 2 raised to n minus 1 probability of error. And note that this ratio probability of bit error by probability of symbol error is equal to 2 raise to n minus 1 upon 2 raised to n minus 1. And, this is precisely the ratio between the number of ways that a bit error can be made and the number of ways that a symbol error can be made. And furthermore this ratio approaches half as n tends to infinity.

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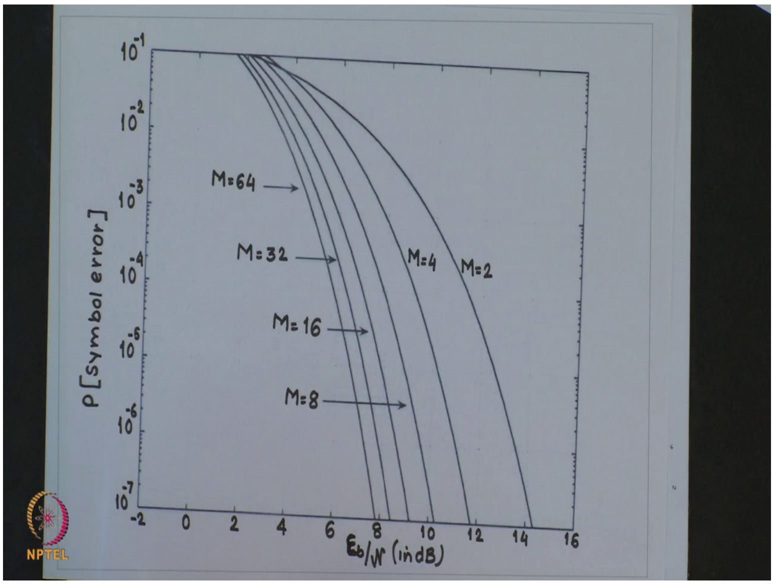

Now, this figure out here shows the exact symbol error probability of M-ary FSK as a function of SNR per bit, for different values of M. So, notice from here that in a completely opposite behavior as compared to M-ary SKA, M-ary FSK, and M-ary QAM, the required signal to noise ratio per bit to achieve a given error probability decreases as M increases in M-ary FSK signal. So, for given SNR per bit as you keep on increasing M the symbol error keeps on decreasing right.

So, but this important to note that this happens at the expense of a larger transmission bed I repeat please, it should be noted that this happens at the expense of larger transmission bandwidth in order to accommodate a higher number of orthogonal carriers.

Now, we have calculated the exact symbol error probability and the bit error probability, but from this expression, it is difficult to see how the error probability is behaved, which signal to noise ratio per bit or with the consolation size M. So, to overcome this difficulty we will consider upper bounding the error probability by using the union bound and let us derive the basic expression for this.

So, we will try to derive the error probability for M-ary FSK, which is an upper bound.

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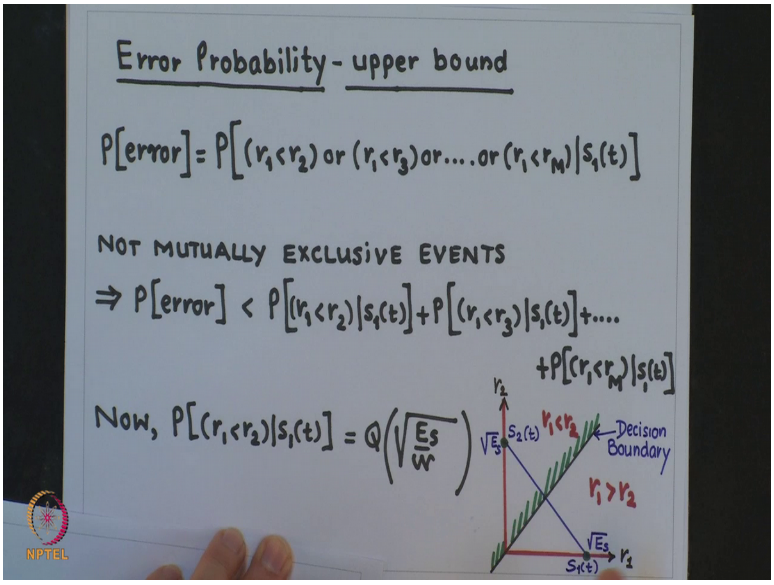

Now, we said that probability of error given that I have transmitted S 1 t remember I repeat that probability of error is the same as the conditional probability of error. So, probability of error is equal to probability of r 1 being less than r 2 or r 1 less than r on this correct.

Now, this are not mutually exclusive events. So, what this implies, that probability of error will be less than the summation of each of this conditional errors correct, because this events are not mutually exclusive. So, I can rewrite this probability of error as less than this quantity. Now, let us look at 1 of this term out here. So, let us look at the first term in this expression.

Now, first term in this expression can be represented as shown here. So, what we are saying? That, what we want to evaluate this probability? This probability if you look in the signal constellation, I have shown here only the 2 axis corresponding to signal S 1 t and S 2 t that is same as phi 1 t and phi 2 t.

Now, we want that this condition should be satisfied r 1 should be less than r 2. So, for these 2 signals then this will be the decision boundary, because equi probable bisector of the line joining these 2 points. So, this region would be r 1 less than r 2 and this region would be r 1 greater than r 2. Now, we know this very simple to evaluate the probability of error for such case and that would be equal to Q times this again, the distance divided by 2 and that 1 is divided by the root of italic N by 2 and you will get this term.

So, if you do this basically for each of this term they will be same, because they are all orthogonal. So, each of this term would be equal to this. So, that there will be M minus 1 term. So, I can write their probability of error in that case as less than M minus 1 and this I can write as less than M because upper bound ok.

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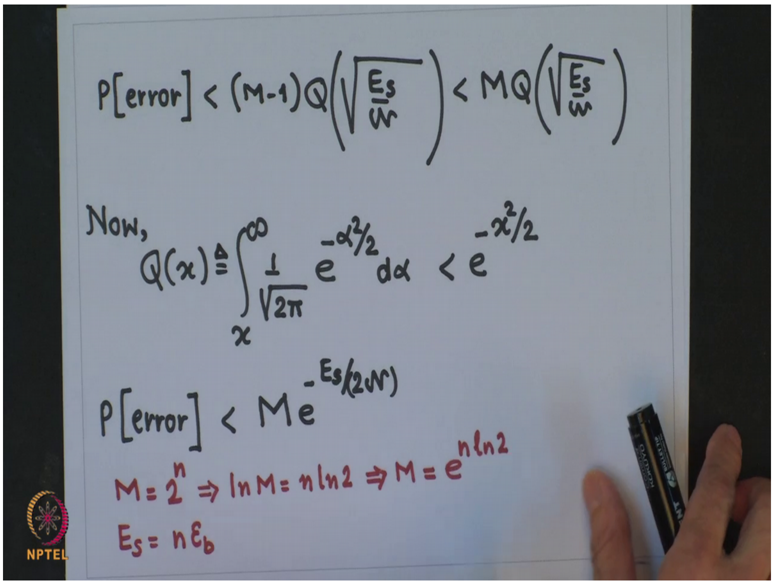

Now, we know that Q x is given by definition equal to this expression. Now, we can approximate this Q x correct.

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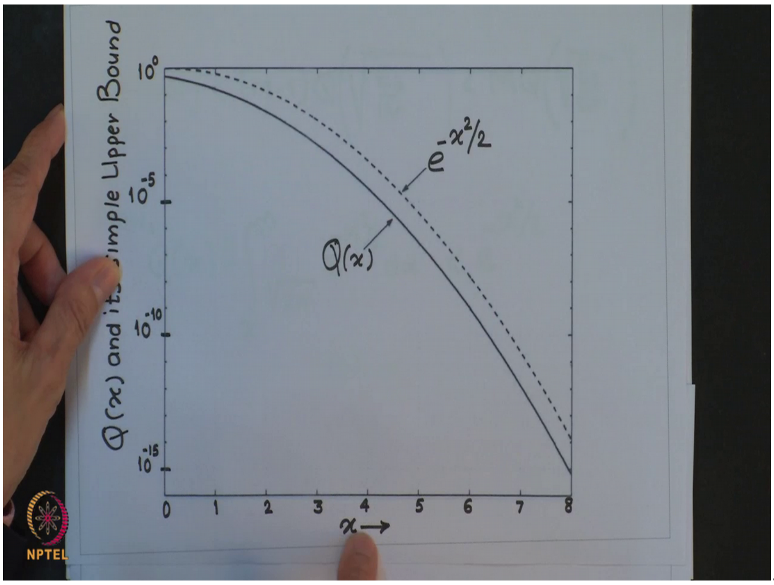

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So, the Q x as a function of x is given here and along with this curve we have also plotted here e raised to minus x square by 2 right as function of x. We see that the difference between Q x and this is there, but this quantity out here can serve as an upper bound for Q x. So, if we substitute this as an upper bound for Q x, in that case I can get a new bound for the probability of error in where I remove this Q. And, if I do that I get probability of symbol error to be equal to this using this relationship.

And, now quickly let us look at this M I can express M as 2 raised to n which is same as log M is equal to n log 2, which implies that M is equal to e raised to n log 2. And remember that, energy in the signal is n times the average energy per bit. So, if I use this relationship and if you look at this expression probability of error I just substitute for the value of M equal to this. And, this quantity out here E S substitute with n times average energy per bit I get this as my new upper bound for the probability of symbol error.

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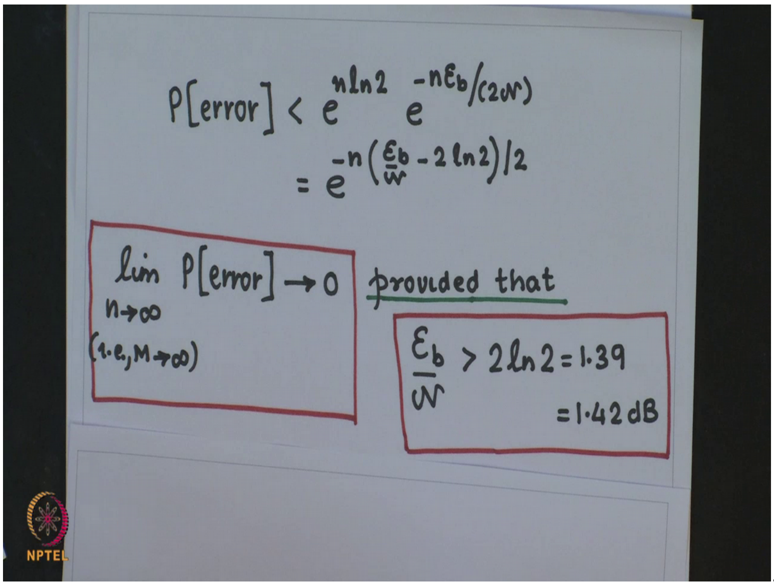

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And, this I can rewrite it as this quantity here just removing and writing this, now this is interesting out here.

Here, it shows that limit of this tends to 0 as n tends to infinity, which implies that M also tends to infinity provided that this quantity is has to be positive. So, what it implies basically that e b by root N has to be larger than this quantity which is 1.4 2 d B correct. So, so the upper bound on the probability of error implies that as long as the signal to noise ratio per bit is larger than 1.4 2 d B, we can achieve an arbitrary low probability of error.

Now, we can have a different interpretation of this and that we can do it as follows we will rewrite this expression, this expression out here has this quantity.

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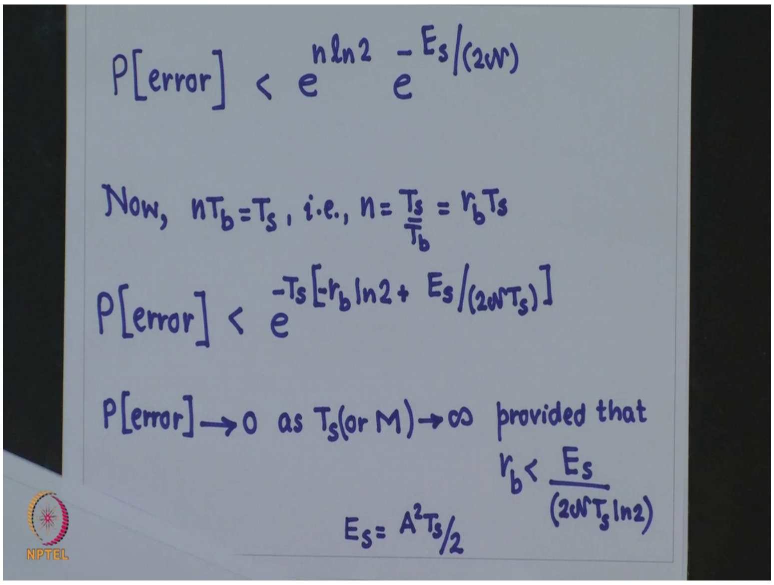

And, now what I do is that I again use this relationship n times the bit duration is equal to the symbol duration. So, I can write n is equal to r b into T S. And, if I use this relationship and just try to see what happens to this expression out here on the right hand side, I can rewrite this and log 2 as this quantity because n is equal to r b T S. So, I write it here and this I write it here.

Now, this gives us a different viewpoint. What it shows that, the probability of error will tend to 0 as T S which also implies that M tends to infinity. Provided that now this quantity has to be greater than equal to 0, which implies that r b should be less than this quantity.

Now, remember that this is the energy in the signal and for the sinusoid, the energy in the signal is given by a square T S by 2 where a is the amplitude of the sinusoid. So, if I write this quantity in terms of the amplitude of the signal the same relationship I can rewrite it as follows.

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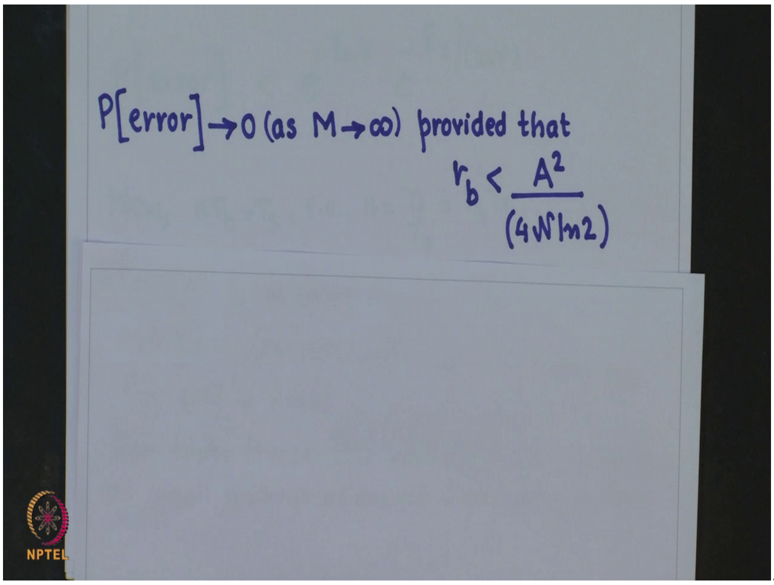

Probability of error tends to 0 as n tends to infinity provided that my rate satisfies this constraint.

This behavior of the error probability is surprising, since what it shows is that provided the bit rate r b is small enough the error probability can be made arbitrarily small even though the signal to noise ratio is finite. What it means that equivalently? The transmitter power can be finite and still 1 can achieve as small an error as desired when the bit stream is transmitted by means of M-ary FSK.

So, we are having studied all these M-ary schemes now we need to make some kind of a comparison between all these M-ary schemes, and we will do it in the next class and then we will conclude our study of M-ary coherent modulation techniques.

Thank you.