Principles of Digital Communications

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Lecture – 54
M-ary PSK

We have learnt that M-ary ask gives a bandwidth saving, but it comes at the expense of either error performance or an increased transmission power, if the same error performance is required. Its signal constellation is 1 dimensional; we have also studied that QPSK and its variants and MSK have 2 dimensional signal constellations and provided bandwidth saving without increase in power or degradation in error performance.

Now, we will extend our study to M-ary phase shift keying that is M-ary PSK, where m is greater than 4. We will continue our study using the same earlier basic principle for the analysis of the digital modulation scheme. We will find a signal space representation of the modulator signal set, project the received signal onto the signal space basis to generate a set of sufficient statistics and process them to obtain the decision; so let us begin our study of M-ary phase shift key.

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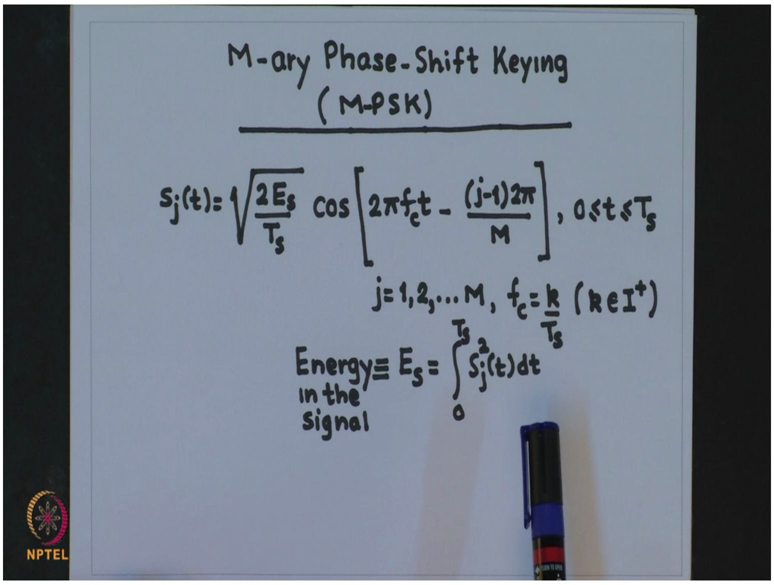

So, the transmitted signal is given by this expression we have the carrier at f c and that carrier frequency is equal to k by T S, where k is a positive integer and T is the symbol duration and E S denotes the energy in the signal S j t. In this case the energy in all the message signals from j equal to 1 to M is equal.

So, this expression for the transmitted signal can be rewritten using the trigonometric relation as follows.

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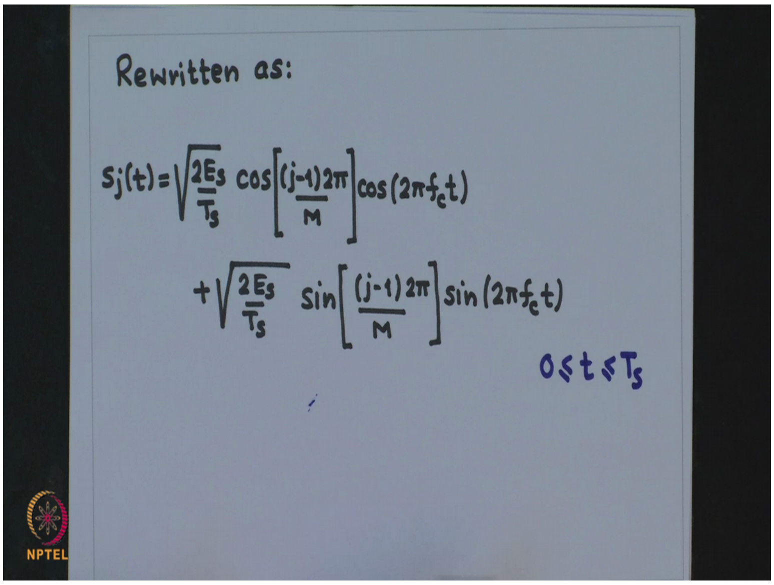

Now, examining this transmitted signal, this duration is over T S we can easily find out the basis signal for the message signal set.

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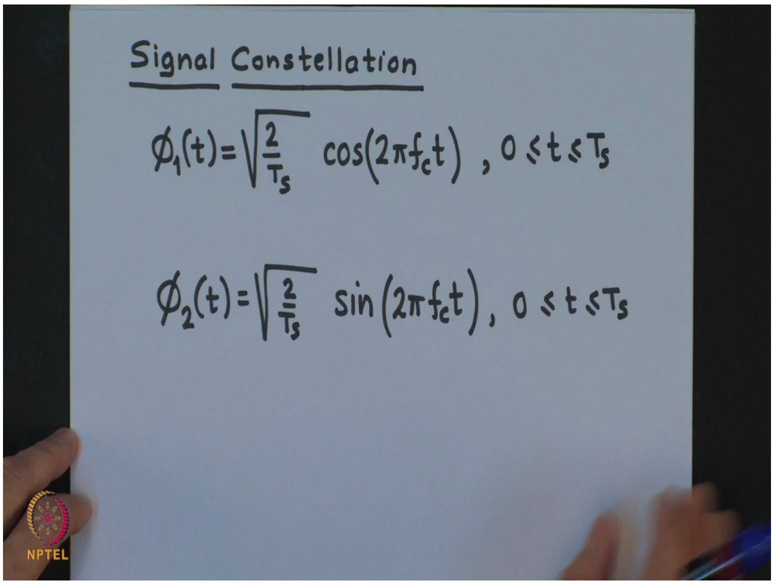

And that would be given by this 2 basis signals which is phi 1 t; root.of 2 by T S; cos 2 pi f c t and phi 2 t which is the root of 2 T S sin 2 pi f c t. Given this 2 basis signal, we can find the signal constellation for the given message signal set and that we will do as follows.

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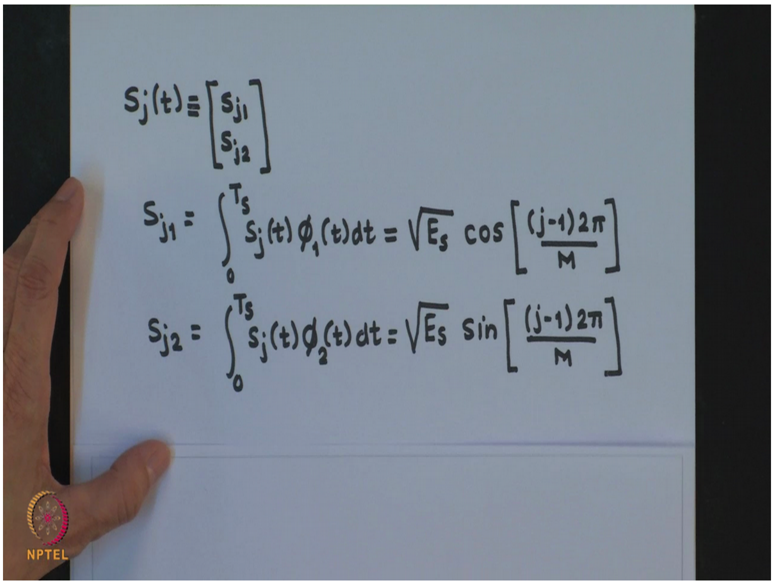

What we are required is to find the components S j 1, S j 2 of the vector which represents the message signal S j t. And S j 1 and S j 2 are nothing, but the projection of S j t on phi 1 t and phi 2 t respectively. Then we take the projection of these 2 signals, remember phi 1 t and phi 2 t are orthogonal and phi 1 t and phi 2 t forms orthonormal basis signal set. So, the energy of phi 1 t and phi 2 t both are equal to 1.

So, plugging in the expression for S j t and evaluating these projections, we get this 2 values for S j 1 and S j 2. Once we have this, we can easily get the signal constellation for the given message signal set and that would be as shown in this figures.

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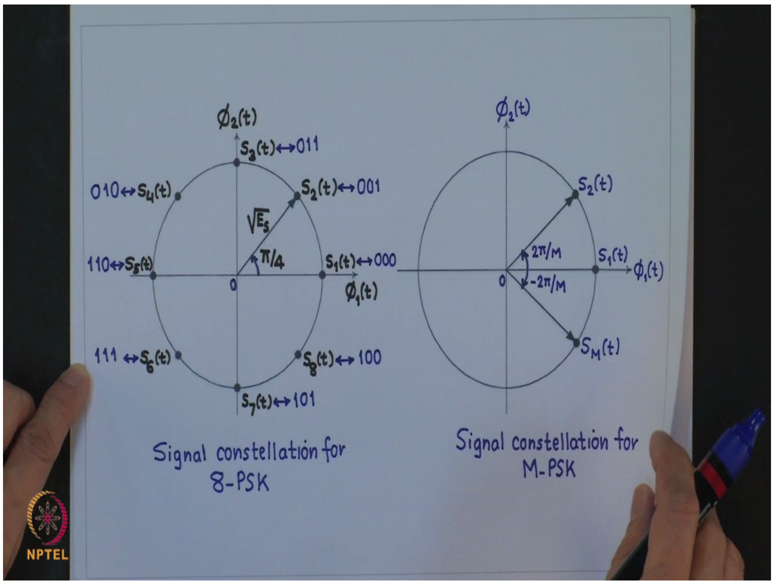

So, we see that the signals lie on a circle of radius given by root of E S and they are spaced every 2 pi by m radians around the circle. So, this figure illustrates for m equal to 8; so, 2 pi by 8 we will get the spacing between the 2 message signal points to be pi by 4 radians. So, we start with this S 1 t the other one will be place at pi by 4 radians that is S 2 t and similarly from here to here it will be pi by 4 correct.

And this is a signal constellation for any general M and in this figure we have just shown the 3 message points S 1; on this side it will be S 2 and on this side of S 1, it will be S M; that is the last message signal in the message signal set.

And for the case of 8 PSK, we have also shown the great mapping from 3 bit patterns to the signal point. And it is important to note that from this message point to this message point; the bit pattern differ only by 1 bit. And similarly from here to here it differs only by 1 bit and this is true of any other message point in this signal constellation.

Now, let us try to find the optimum receiver for this.

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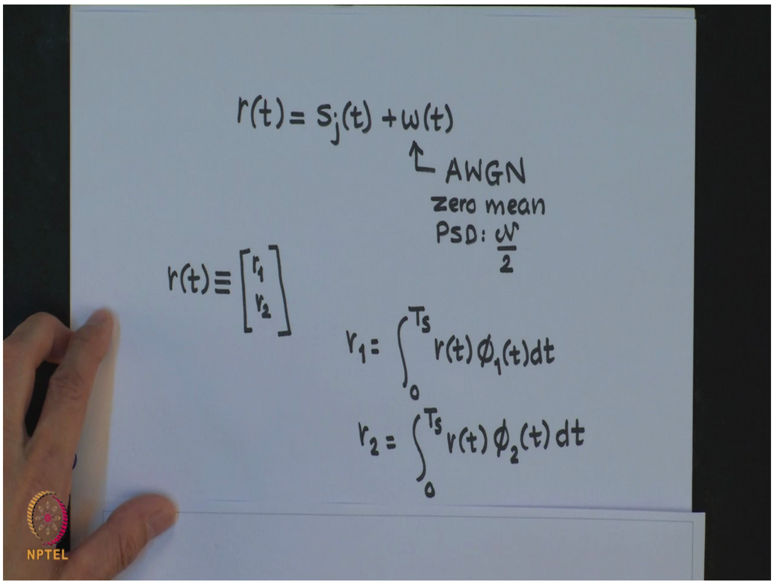

Now the received signal is r t which is S j t; the transmitted signal particular transmitted signal plus the additive white Gaussian noise which we will assume as usual it is a zero mean and power spectral density is italic N by 2 and it is a white Gaussian noise.

So, the received signal has to be projected onto the basis signal phi 1 t and phi 2 t to get the components r 1 and r 2 and this components are obtained using the expressions shown here. So, r t is the projection on phi 1 t and r t here projection on phi 2 t to give r 1 and r 2 respectively.

So, now the optimum detector for this we have learnt map detector.

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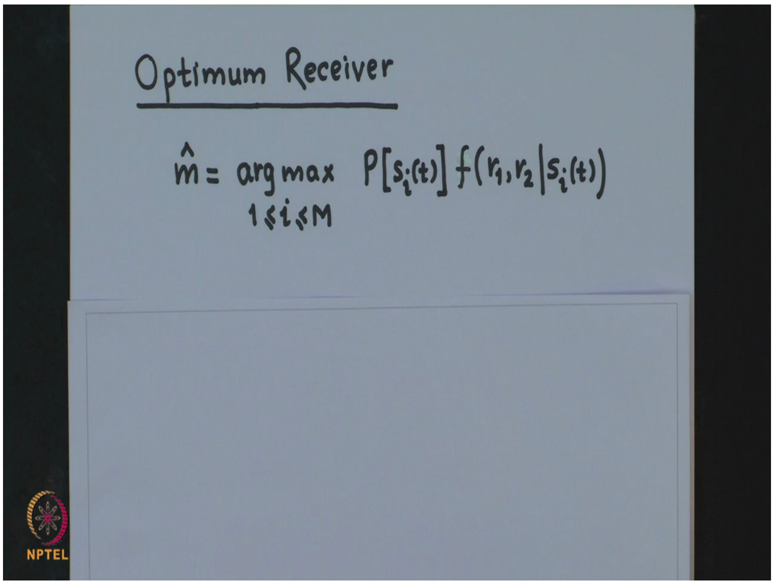

So, we have the decision will be in the favor of that argument of i for which this quantity is the maximum. So, this is the probability of transmission of a particular message signal S i t and this is the conditional joint pdf of r 1 and r 2 given S i t.

Now, when the messages are equiprobable; in that case this term will be constant and the optimum receiver will reduce to a minimum distance receiver. And we know the block diagram for such a receiver we have studied this on earlier occasions and that block diagram would be as shown in this figure.

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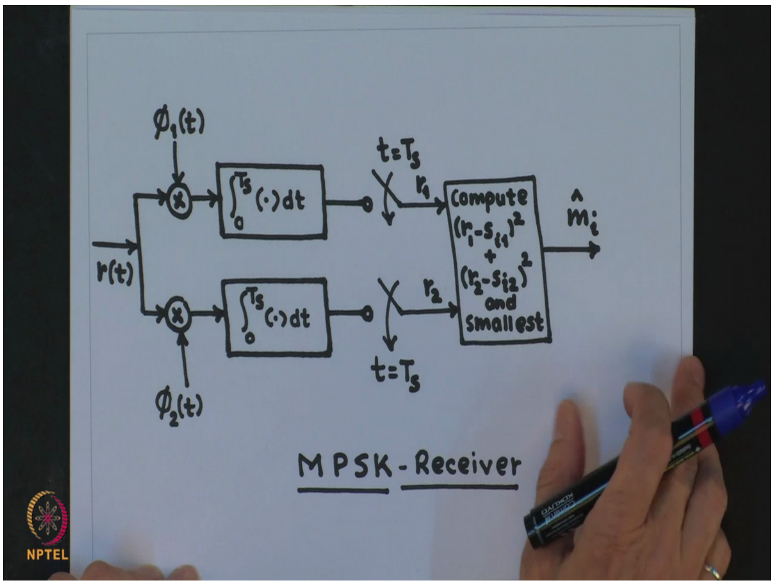

We have r t; take its projection, pass it through a correlation receiver and then sample at appropriate time to get the 2 values r 1 and r 2 and then compute the distance with the templates stored for the S i t signals and choose the smallest one; so, this is the minimum distance receiver.

So, for whichever i this quantity is small we will decide that the message transmitted is m i or S i t the decision regions for the M-ary PSK is as shown in this figure.

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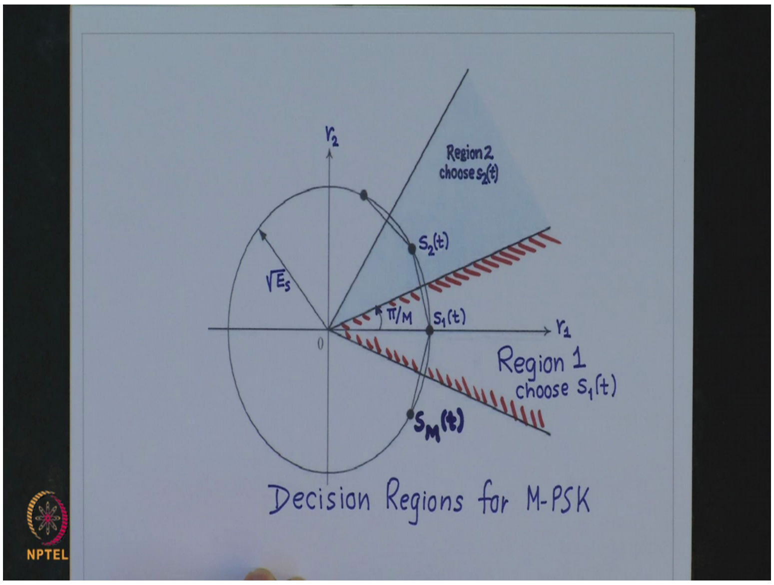

We have this points located on the circle of radius root E S and this region corresponds to the S 1 t, this region corresponds to S 2 t; remember that the decision regions are obtained by taking the perpendicular bisector of the line joining S 1 t and S 2 t, you will get this.

And then similarly the decision region between S 1 t and this is say S M t would be the perpendicular bisector of the line joining S 1 t and S M t. So, this will be the region corresponding to S 1. So, whenever r 1 and r 2 lie in this region we will decide in favor of message m 1 or signal S 1 t; given this now let us try to evaluate the probability of error.

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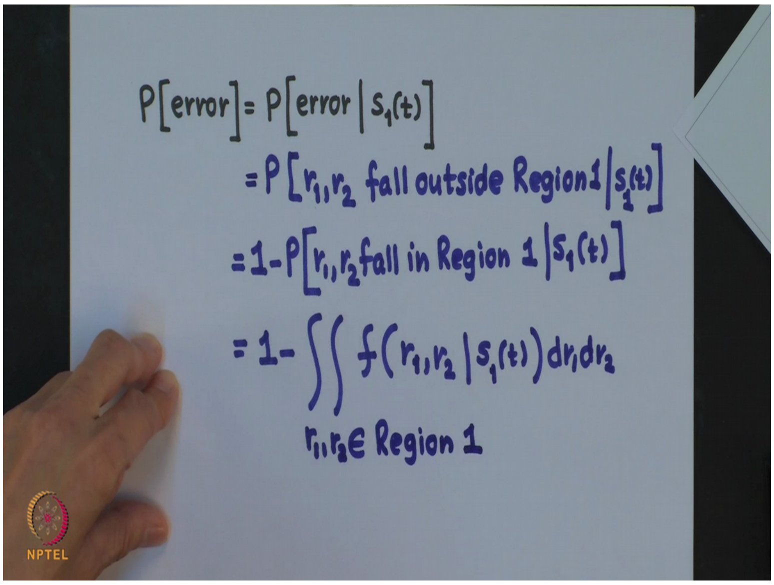

Now, since our message signals are equiprobable probability of error would be equal to the conditional probability of error. Since, the our message signals are equiprobable; probability of error is equal to probability of error given any S i t for our discussion without loss of generality, we have assumed S i t to be equal to S 1 t. So, what this implies is that my r 1 and r 2 should fall outside the region one given that I have transmitted S 1 t.

So, this would be equal to probability of r 1, r 2 falling outside region 1; given I have transmitted S 1 t. And this is equal to same as 1 minus probability of r 1, r 2 fall in region 1 given S 1 t. So, this we can evaluate as follows these are the conditional joint pdf of r 1 and r 2 r 1, r 2 belong to region 1 where this conditional joint pdf would be given by this expression.

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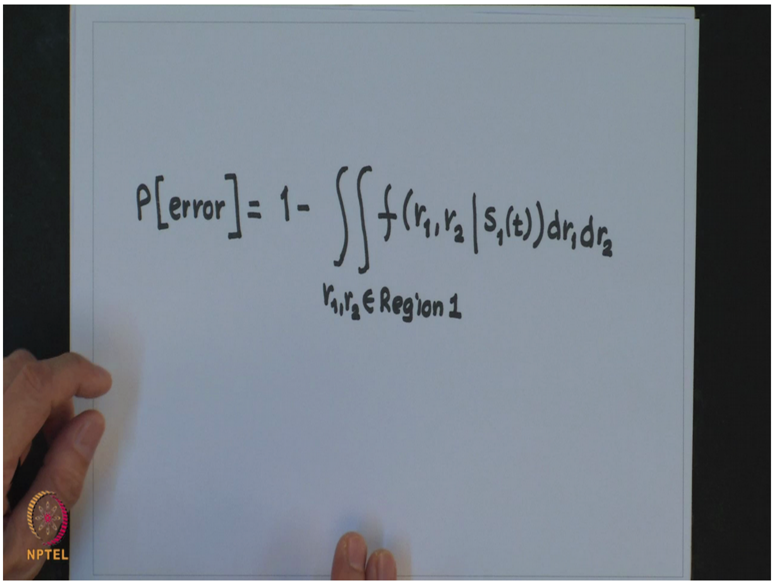

Because remember at this point the mean value for r 1 is going to be decided by the transmitted signal S 1 t and that would be equal to root E S. And for r 2 the mean value would be equal to 0 because its projection on r 2 is 0.

So, it is a both are independent statistically; so I can take the joint pdf to be the product of individual conditional pdfs. Unfortunately no closed form solution for the above integral is feasible; therefore, we try to determine the error probability bounds as follows. So, let us first calculate the lower bound; to do that consider the message signal point S 1 t and its adjacent message signal point S 2 t.

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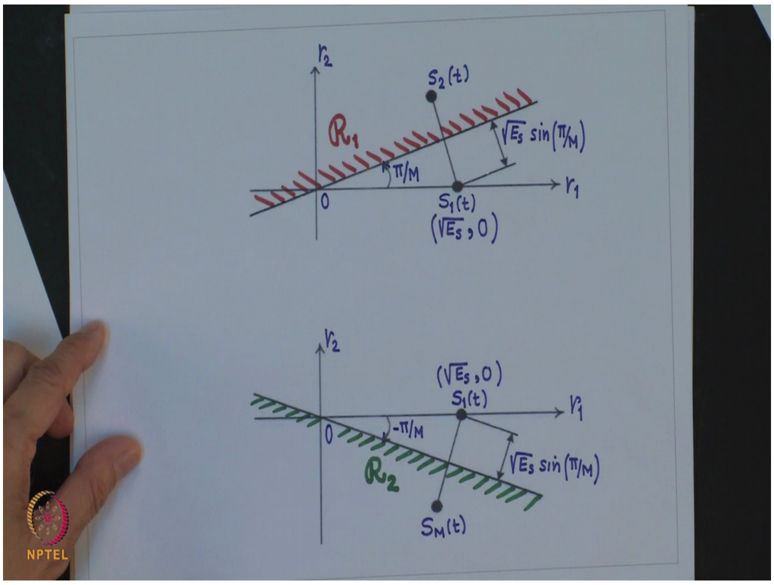

So, the region between this 2 is given by this line and anything on this side indicated by this red hashes is denoted as r 1. So, probability of let us we will first find out the lower bound for this.

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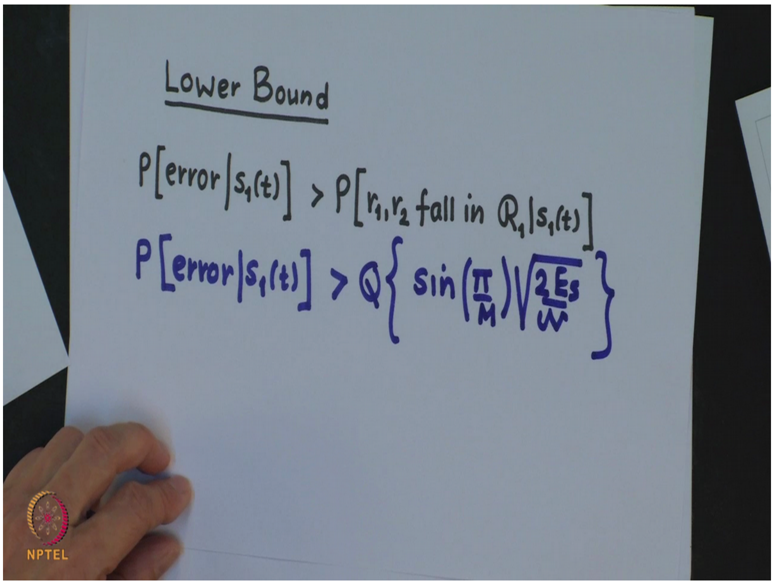

So, probability of error given S 1 t would be larger than the probability of r 1, r 2 falling in r 1 given S 1 t correct; it is very obvious from this. So, what this implies is that probability of error given S 1 t is greater than; we want to find out what is the probability that it lies in this region.

Now, remember that our noise is circularly symmetric correct. We can use the principle of translation invariance and rotational invariance by taking this point coordinate axis to this point and rotating it. In that case what will happen? You will get a Gaussian distribution like this about this point and what we are interested is to find out what is the probability of that Gaussian distribution area lying here.

So, this distance is very easy to see that this is equal to root E S multiplied by sin of pi by M. This is root E S this is pi by M for generic M M-ary PSK, so you will get this equal to this value here. So, from this we can write the probability of error given S 1 t greater than Q times sin pi by M root of 2 E S correct.

Now, let us calculate the upper bound now to calculate the upper bound, we will use r 1 and r 2 which is given here. Now, we choose another adjacent point of S 1 t on this side which is S M t and the decision boundary between the 2 is given by this green hash line and let us call this portion as r 2. And similarly here this distance is root E S sin pi by M.

So, again we want to find out what is the probability that it will r 1, r 2 falls in this region given by r 2. And that is again very simple it will be Q times what we saw earlier; same argument will be there. So, the upper bound will be obtained by taking the probability of r 1, r 2 falling in r 1 plus probability of r 1, r 2 falling in this r 2 when I say. So, we can write this probability as follows.

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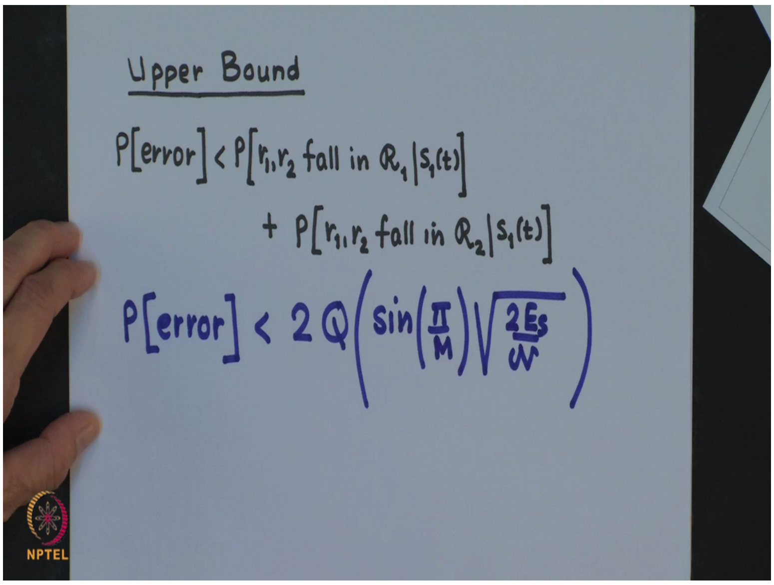

This is equal to this will be the upper bound; it will be twice Q of sin pi by M root 2 S italic N.

So, the common area under the joint conditional pdf that is f of r 1, r 2 given S 1 t where regions r 1 and r 2 intersect has been counted twice our probability of error is bounded between these 2 quantities.

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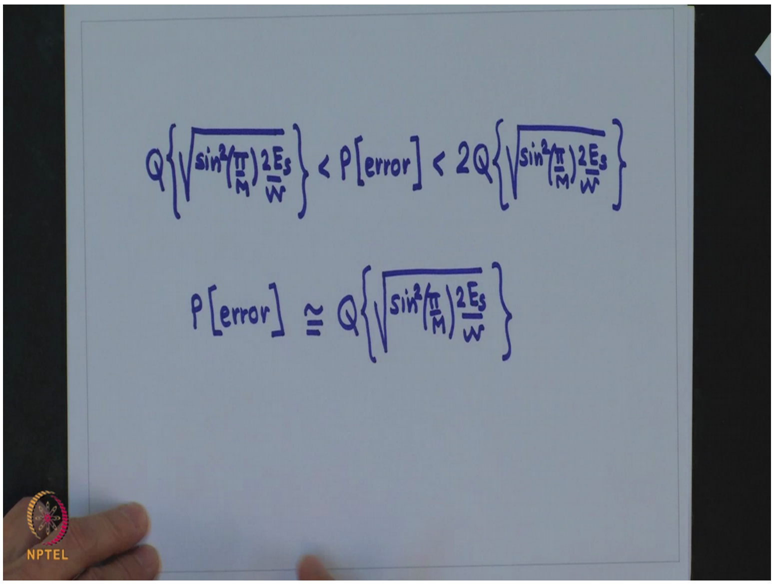

And for high signal to noise ratio we can approximate this probability of symbol error to be equal to this expression and we will see very soon that this is really true ok. Now, let us try to evaluate the probability of symbol error in terms of the average bit energy.

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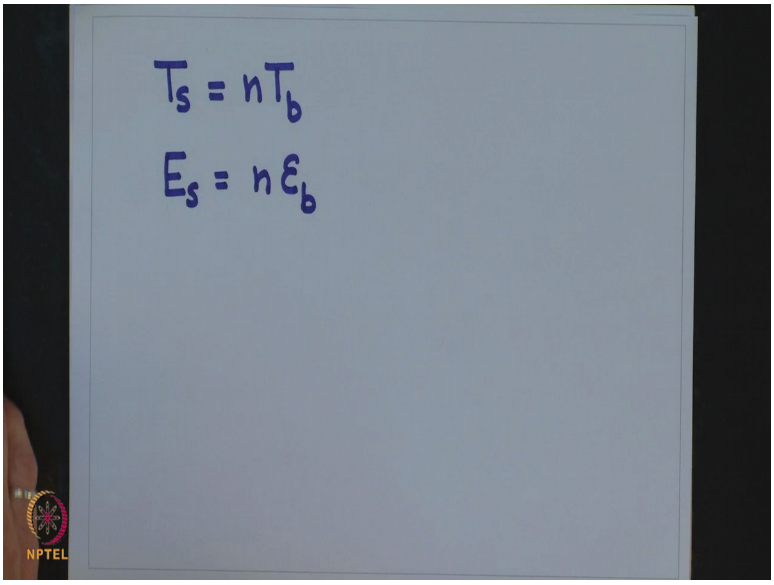

Now, we know that the message occupies the duration T S which is equal to n times T b where T b is the binary digit bit rate. So, this also shows that the bandwidth requirement for the M-ary scheme is one by n that of the binary case.

So, now your energy in the symbol would be equal to n times the bit energy E b because there are n binary digits or bits in each symbol. So, using this relationship; we can write down the probability of symbol error in terms of the bit energy and that would be as follows.

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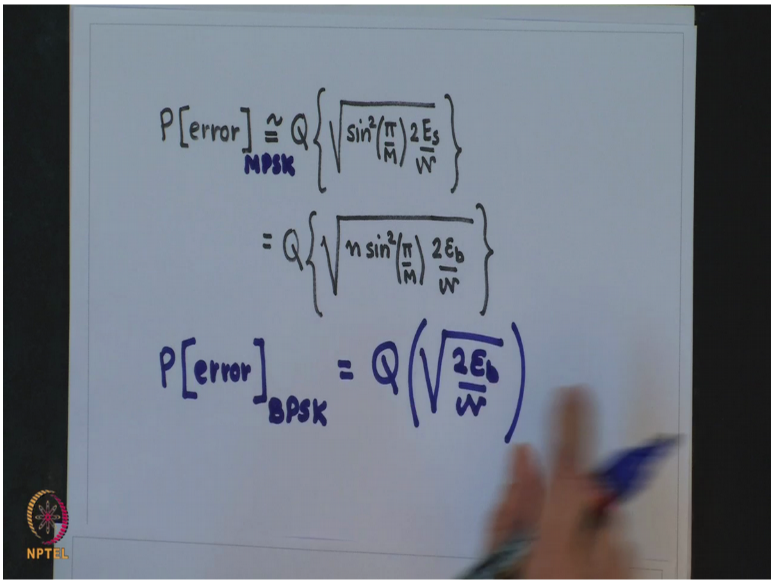

This was the expression which we had for the symbol error; now for E S; I substitute as n times bit energy I get this relationship.

Now, remember the probability of error for the BPSK case was equal to Q of root 2 E b by italic N. So, for the same error performance M-ary PSK; signal energy needs to be increased by the reciprocal of the factor n sin squared pi M. And the table out here summarizes the performance of M-ary PSK and BPSK; these are the different choice for n.

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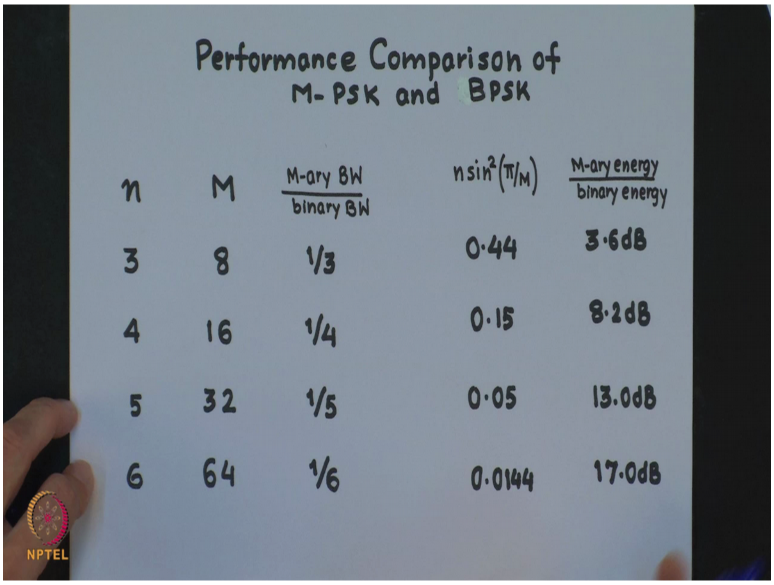

Given this choice M is equal to 2 raised to n. So, 2 raised to 3 is 8; 2 raised to 4 is 16 and so, on. And if we take the ratio of M-ary bandwidth required to the binary bandwidth requirement, this would reduce as shown in this column out here and this quantity n sin squared pi by M and if you evaluate, it turns out to with this.

So, the M-ary energy upon the binary energy; this is the bit energy required average. What it shows that M-ary PSK signal needs 3.6 dB more energy than the binary case for M equal to 8. And for M equal to 64, the M-ary bit energy requirement goes up by 17 dB. It should be noted that the values in the table are comparison for the symbol error of M-ary PSK to the bit error of binary PSK.

Now, using the gray approximation the bit error probability of M-ary PSK can be written as follows.

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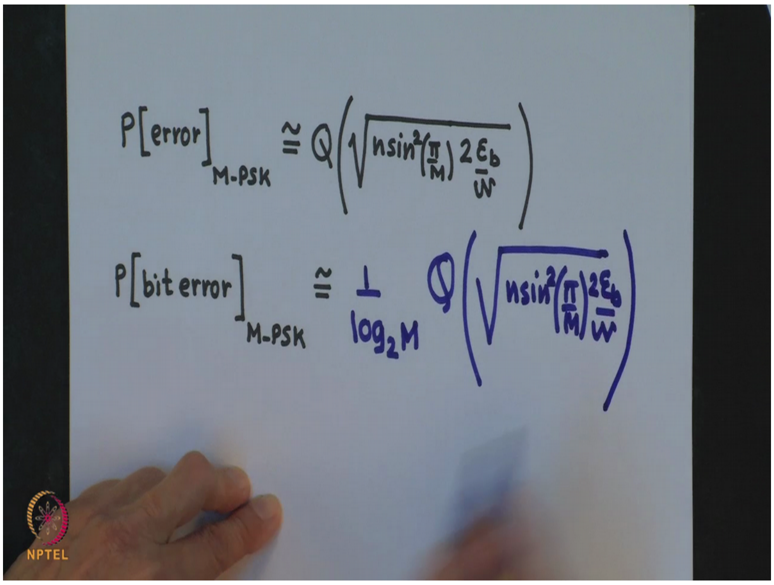

So, this is the symbol error; so, the bit error using the gray approximation would be equal to 1 by log to the base 2 of M multiplied by this quantity out here; figure out here shows the symbol error probability of M-ary PSK for different values of M.

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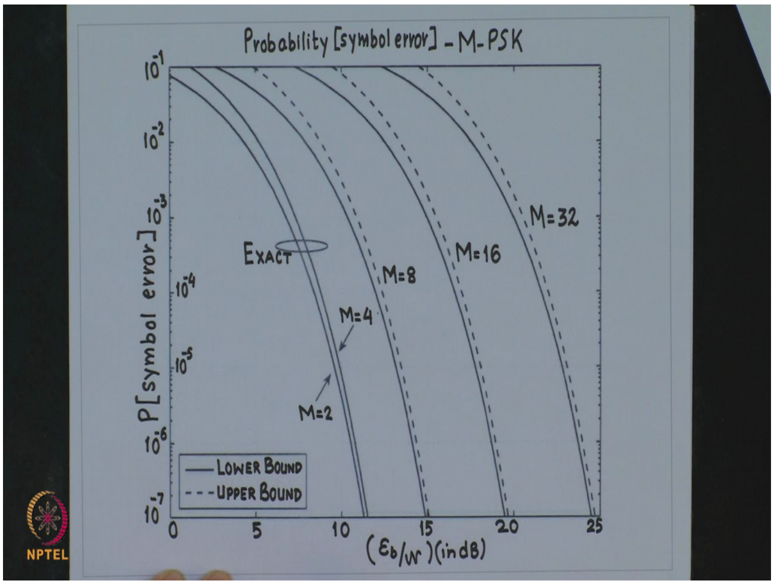

So, in particular upper and lower bounds on the symbol error probability are plotted for M equal to 8, 16 and 32. So, the dotted one out here is the upper bound and the solid line is for the lower bound whereas, for M equal to 2 and M equal to 4, these are the exact results.

Now, observe the tightness of the lower bound and upper bound in the high SNR regions. So, the performance curves clearly illustrate the penalty in signal to noise ratio per bit as M increases beyond M equal to 4.

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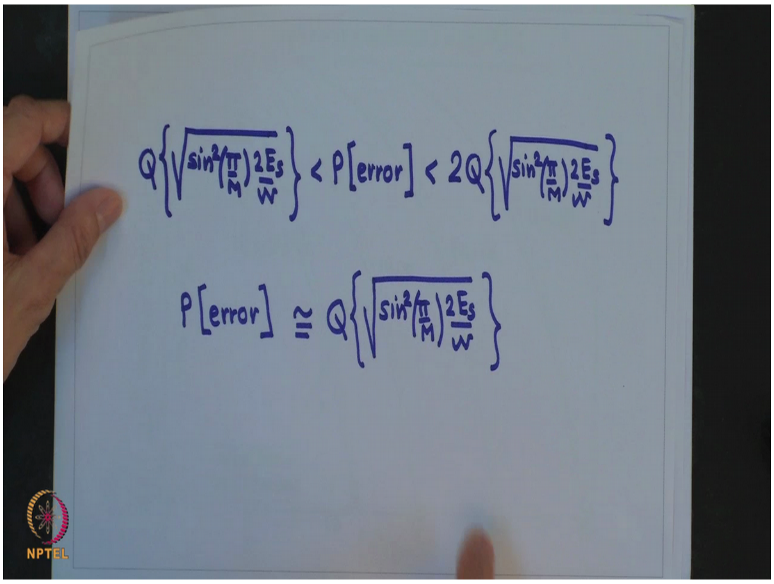

So, we had said that earlier that the probability of error can be approximated for M-ary PSK by this expression. And this is clearly shown in this figure for high signal to noise ratio these 2 curves will merge.

Now, let us take a probability of error 10 raised to minus 5; the difference between M equal to 4 and M equal to 8 is approximately about 4 dB. And the difference between M equal to 8 and M equal to 16 for the probability of symbol error equal to 10 raised to minus 5 turns out to be approximately 5 dB. So, this result is same as what we got in the case of M-ary ASK and for the large values of N; doubling the number of messages requires an additional 6 decimals of power to maintain the same error performance.

M-ary PSK therefore, behaves much the same as M-ary ASK in terms of bandwidth and error performance. In retrospect, this might have been expected since the energy E S grows linearly with the number of bits n. So, the radius of the circle that the signals are mapped onto grows as root of the signal energy that is E S and hence so, does the circumference, but the number of signals mapped onto the circumference grows exponentially with n. So, what this implies that a distance between the signals therefore, becomes smaller and smaller.

However, as the circle grows larger there is a lot of space inside it that should be able to house signal points. And this is what quadrature amplitude modulation does; we will study this modulation scheme next time.

Thank you.