Principles of Digital Communications
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Lecture - 43
Principle of Invariance of Probability of Error

One of the important measure for the evaluation of the performance of a digital communication system is Probability of Error. Repeat one of the important parameter used to evaluate a digital communication systems performance is probability of error.

In today's class, we will seek the answer to the following question does the probability of error get affected by the rotation or translation of the coordinate axis which we use to represent the signal vectors? And we had also learn how to calculate the probability of error for any general binary signals.

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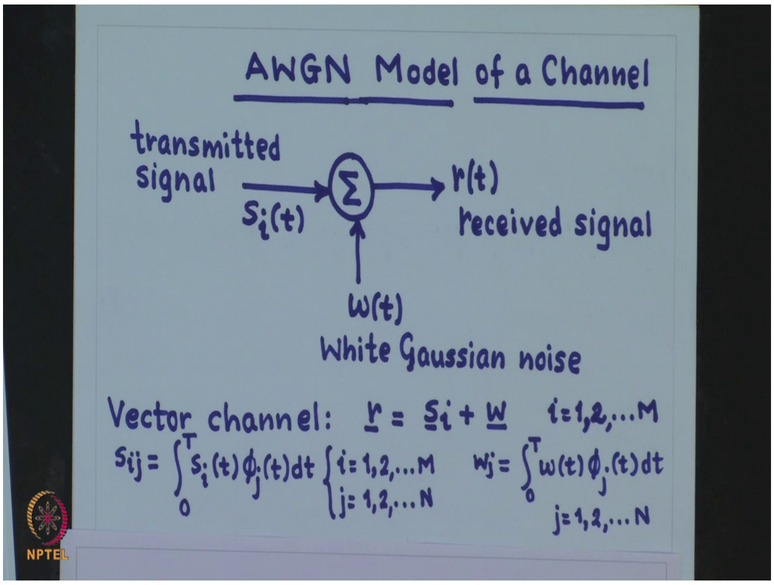

We have studied how to convert continuous additive white Gaussian noise model of a channel to a vector additive white Gaussian noise model. So, let us quickly recollect it this is the model which we have used for continuous AWGN channel, we have the transmitted signal on the channel white Gaussian noise gets added and we have the received signal r t.

Then using the Gram-Schmidt procedure, we know how to find out the orthonormal basis signal set for the given message signal set. And using that orthonormal basis signal set we can convert the continuous signals in terms of vectors as shown by this expression. And these vectors are obtained by taking the projection of the signal onto the basis signals used to represent the message signal set. So, we are assuming here that we have M message signals and the maximum number of basis signals which you could have would be equal to N, where n is less than equal to capital M.

Now, signal constellation is nothing but this signal vector representing the particular message signal.

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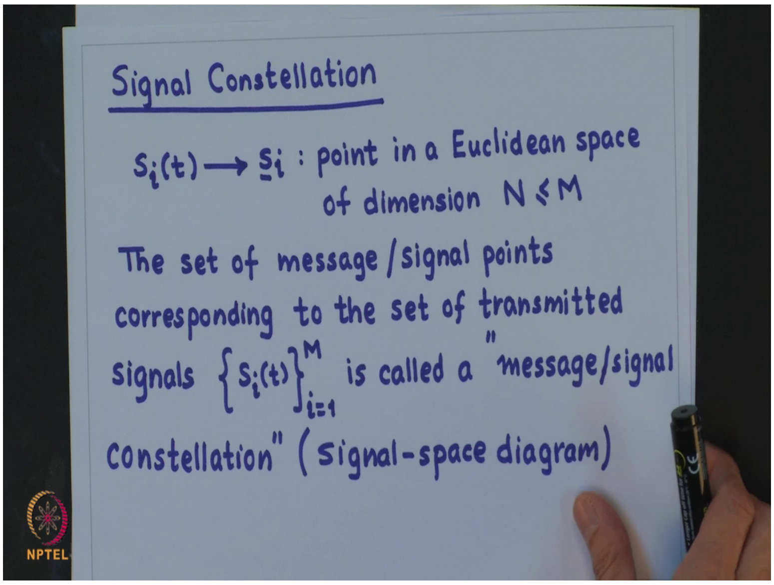

So, a particular message signal S i t gets converted to a message vector S i which is a point in a Euclidean space of dimension N less than or equal to M. Now the set of message signal points corresponding to the set of all the transmitted signals is called a message or signal constellation and sometime in the literature this also known as signal space diagram, this we have studied earlier.

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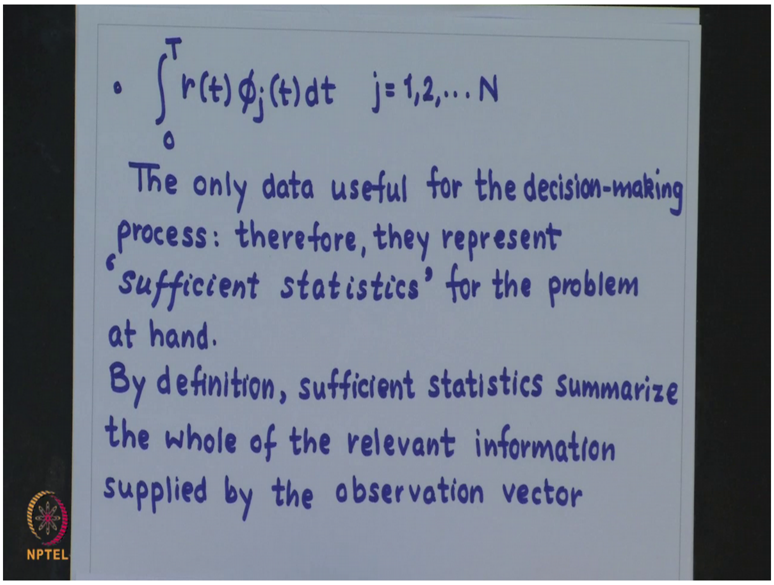

Now, we know that when we get the received signal r t; then we take the projection of r t onto the phi j t where phi j t represent the orthonormal basis set. Now we also seen that the only data useful for making the decision is obtained by taking this projections. And this therefore, represents sufficient statistics for the problem and at hand ok. So, by definition sufficient statistics summarizes the whole of the relevant information supplied by the observation vector.

So, when we do this kind of a projection the noise components which fall outside the space represented by the orthonormal basis set, do not affect our decision making process. So, it becomes an irrelevant information; we have also studied that the optimum receiver for an additive white Gaussian noise channel and for the case when the transmitted signals are equally likely is a correlation receiver which can also be implemented using a match filter. So, in such a case this correlation receiver or match filter implementation gives us what is known as maximum likelihood detector.

Now, in order to evaluate the performance of AWGN channel; we have to calculate the average probability of symbol error.

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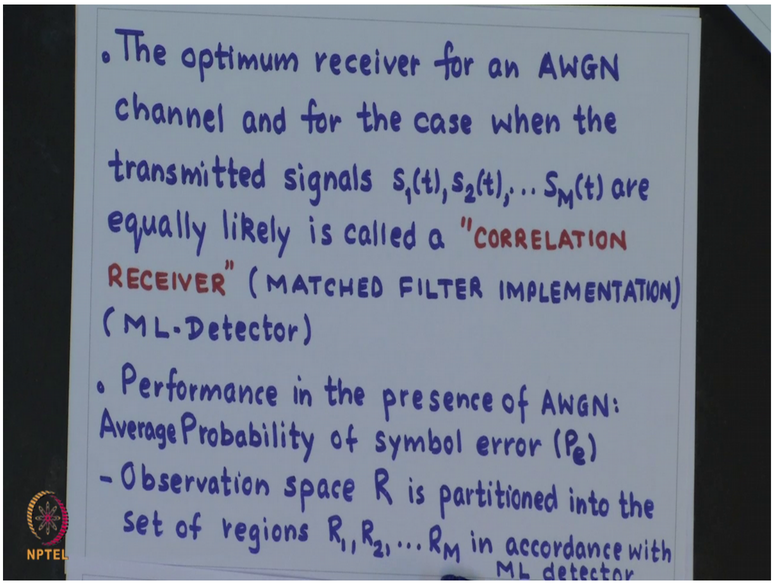

And this we have seen earlier that this is done by partitioning the observation space which we call R in sets of regions; R 1 to R M in accordance with the maximum likelihood detector, all this we have studied earlier this was to just recollect quickly.

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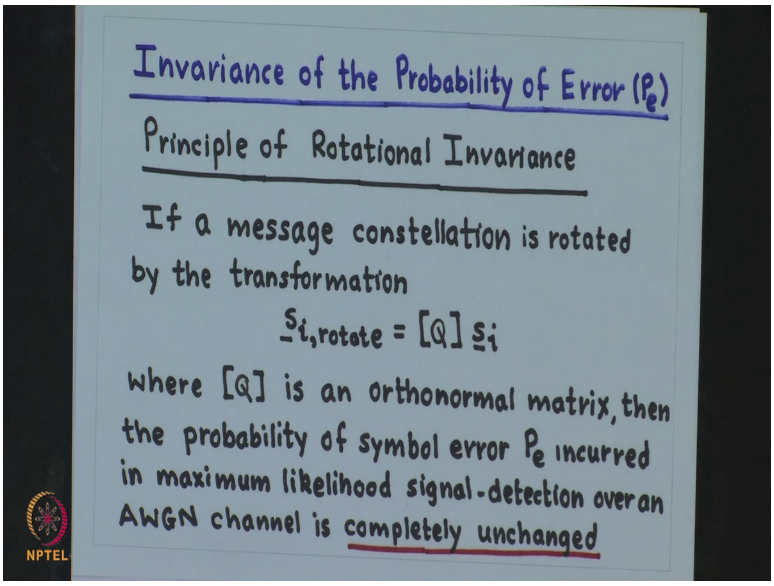

Now, the next question is that what happens to the probability of error if this coordinate axis is rotated or translated. So, let us take the first the case of rotational; now we will study what is known as principle of rotational invariance. What it says is as follows if a message constellation is rotated by the transformation S i rotate. So, S i is your vector corresponding to message signal S i and to that we have the message vector or signal vector S i.

And now this signal vector is rotated using this transformation matrix; this transformation matrix Q is an orthonormal matrix correct. Then we will show that the probability of symbol error which we denote as P e incurred in maximum likelihood signal detection over an AWGN channel is completely unchanged. So, P e is rotational invariant; so, let us take a simple example to understand what we are studying. So, let us assume that I have a signal space diagram or signal constellation as shown here there are four points in this.

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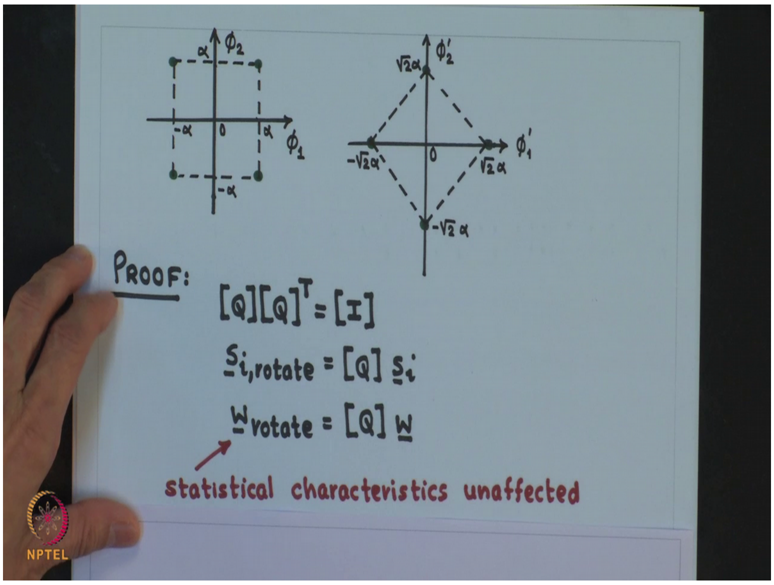

Now, these four points are being plotted with reference to this 2 signal phi 1 and phi 2 which are orthonormal. Now if this coordinate axis is rotated say by 45 degrees if I rotate this by 45 degrees I will get this signal space diagram and the rotated axis, I am just calling it phi 1 dash and phi 2 dash. I do not change the relative positions of this signal vector except for rotation of the coordinate axis.

Now, if such a thing happens and if you were to calculate the probability of error for this signal constellation, then we will see that the probability of error for this signal constellation is exactly the same as the probability of error which we will get using this signal constellation or the signal space diagram.

Let us look at the proof for this; so, we have been given that the transformation matrix which we use is orthonormal what it implies that this condition will be valid where I is identity metric, all your diagonal elements are 1 and off diagonal elements are 0 I carry out the this transformation. So, your each of these points get transformed to four different points using this transformation matrix Q. When I do this I know my noise vector; the earlier noise vector also will go through the same transformation correct; so, I put up this transformation here.

Now, let us see what happens to the statistical characteristic of this noise vector and we will show that this noise vector statistical characteristic remains the same as the earlier noise vector; let us try to prove this. So, the first thing is if we assume that here this noise vector was Gaussian, then it is not very difficult for us to see that this will still remain Gaussian, because this Q matrix is nothing but the linear combination of the Gaussian random variables. And we know that linear combination of the Gaussian random variable will also give us the Gaussian random variable, so this still remains Gaussian.

Then let us look at the mean value of that.

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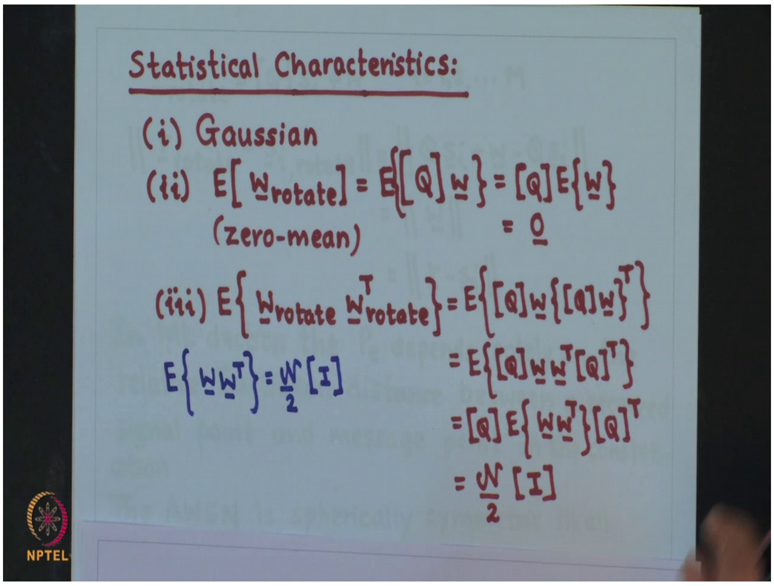

We can calculate the mean value as follows expectation of this noise vector, I carry out the expectation of this I interchange the expectation and the Q operation this is given to be 0, so, I get this to equal to 0. So, this noise vector after rotation also remains 0 mean.

Let us look at the covariance matrix of the rotated noise vector, I have this expectation of this I am supposed to evaluate this I can rewrite it as this expression easy to see that. I take the transpose, once I take the transpose I will be able to like this term out here as this term. Now we know that W W transpose is equal to given is italic N by 2 multiplied by the identity matrix. So, if I substitute this out here what I will get from here is this quantity because remember this is I matrix identity matrix and then this also multiplied by this again identity matrix; so, you get this. So, what this shows that our noise rotated noise vector statistical characteristic remains unaffected.

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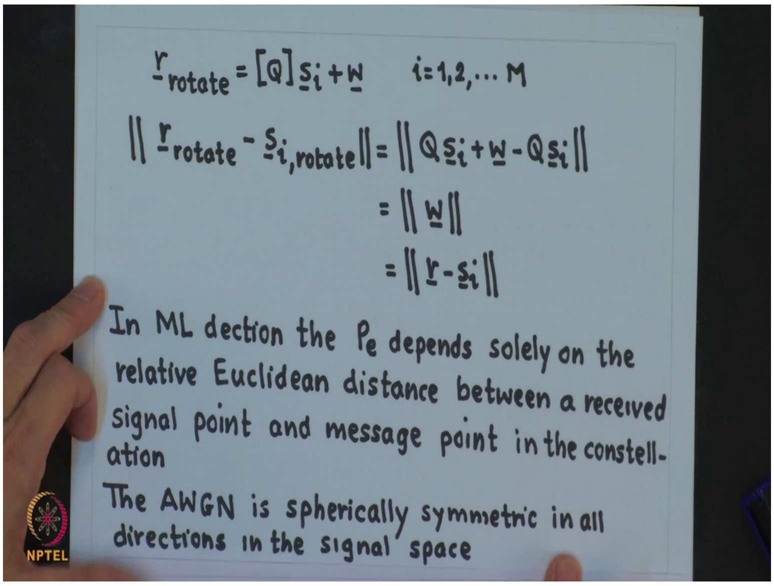

So, in that case now let us try to evaluate r rotate which is nothing but Q S i plus this now I will write this as W; some vector noise vector we have just shown that this rotated noise vector statistical characteristic does not change. So, I have just used this noise vector with the same characteristic as the earlier one.

Now, let us try to calculate the distance between the received vector and each of the message vector in the signal constellation. So, here I have shown the distance calculation for a particular message vector S i after rotation. So, this will be the Euclidean distance which I will have this I can rewrite it as this expression because this have been rotated and it is easy to see that this turns out to be W correct. So, and this W is nothing but r minus S i that is the your earlier Euclidean distance which you had between the received vector and the message vector I in the old signal space diagram ok.

Now, we know that in maximum likelihood detection this should be detection corrected, in maximum likelihood detection the probability of error depends solely on the relative Euclidean distance between a received signal point and the message point in the constellation; this we have studied earlier. So, what this result shows that this relative Euclidean distance between the received signal point and a particular message point in the constellation remains unaffected. And remember that the additive white Gaussian noise is spherically symmetrical in all directions in the signal space.

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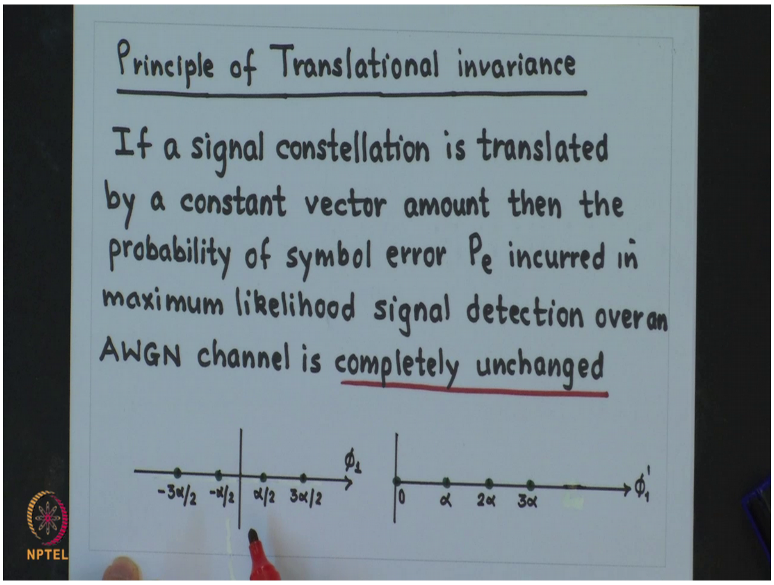

Now, let us take another property of this signal constellation and that is principle of Translation invariance. What this says is that if a signal constellation is translated by a constant vector amount, then the probability of symbol error P e incurred in maximum likelihood signal detection over an additive white Gaussian noise channel is completely unchanged correct.

So, this figure out here depicts what is to be done; I have signal constellation, I have just shown 1 dimensional signal space. And now this point is origin out here this has been shifted to some other point and so, maybe this point comes here correct. And rest of the thing the distance between this message vectors, they do not chang; so, they remain the same.

Now, in such a case also the probability of error will remain unchanged very simple to prove this.

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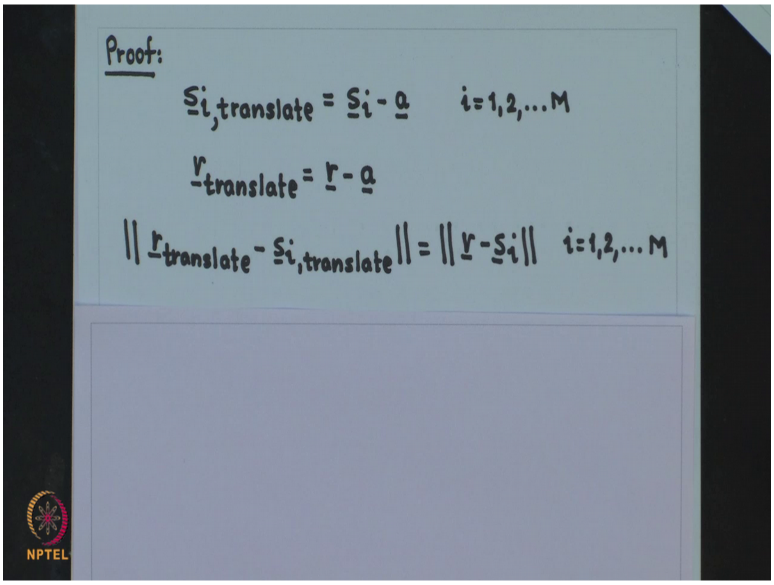

The proof follows for my signal vector, it has got translated a is some arbitrary shift vector, my the received vector will also get translated. Now if I take the distance between the 2 this is the new one received, this is my signal vector after translation I take the difference Euclidean distance refer it remains the same correct.

So, we have shown that basically that if I rotate or translate my coordinate axis; the probability of error does not change. It is an important concept which we should remember when we are trying to evaluate the probability of errors. Now, quickly we will also try to calculate the error probability for equiprobable binary signaling scheme.

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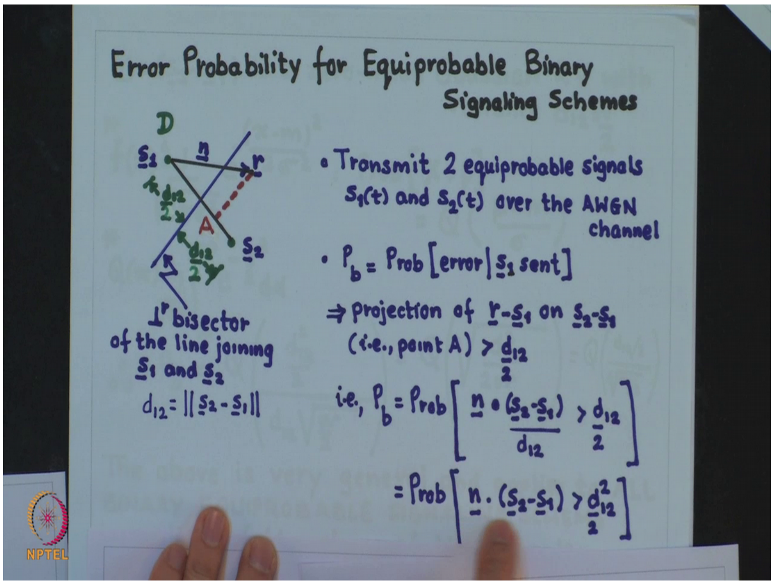

. So, let us say that the transmitter transmits 2 equiprobable signals S 1 t and S 2 t over the additive white Gaussian noise channel.

Now, let me denote let me denote those 2 points in the signal space diagram as shown here, this is S 1 and S 2. So, these are the 2 vectors corresponding to the 2 signals S 1 t and S 2 t. Now we know that signals are equiprobable; so, what this imply that the 2 decision regions R 1 and R 2 will be separated by the perpendicular bisector of the line joining S 1 and S 2 correct. So, I take the perpendicular bisector; this region out here would be R 1 and this region on this side of the line would correspond to S 2; the distance between S 1 and S 2 let it be denoted as d suffix 1 2.

Now, so this is the perpendicular bisector which decides your decision boundary between R 1 and R 2 presuming that both the signals are equiprobable. Now since the signals are equiprobable by symmetry error probabilities when S 1 or S 2 is transmitted are equal. And therefore, it would be equal to this probability; probability of error given S 1 is sent. So, let us try to calculate this probability and this probability will be the same as the probability of error for this equiprobable binary signaling scheme.

Now, to do that basically since we are assuming that S 1 is sent an error occurs if the received vector r is in the region R 2, which implies that the distance between the projection of r minus S 1. So, this is a vector correct on S 2 minus S 1 that is this is a point I get the projection that should be larger than the midpoint of this which is d 1 2 by

2 correct. So, this if this condition is satisfied then your vector r would be lying in the region R 2 and in that case the error will occur ok.

So, this implies that this condition should be the probability of error now is nothing but the probability that r minus S 1 now remember r minus S 1 is nothing but your noise vector n. So, what it implies? That the probability of the noise vector n projected onto this, this is the normalized by d 1 2; the unit vector in this direction, I am taking the projection that should satisfy this condition for the error to occur.

Now, if this condition is satisfied; it means that probability this condition should be satisfied correct.

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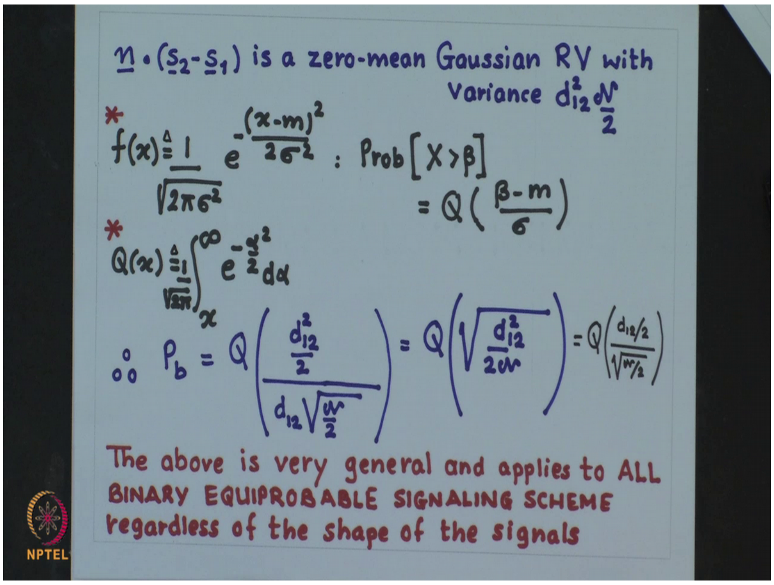

Remember that this noise vector projected on S 2 minus S 1; the line joining S 2 minus S 1, this is a 0 mean Gaussian random variable with variance given by this quantity we have done this earlier. So, I do not want to repeat it.

Now, quickly we know that if I give you a Gaussian pdf; a general Gaussian pdf with mean value to be m and the variance equal to sigma squared, then probability of the random variable X being larger than certain value beta would be given by this function Q function and the Q function itself is defined out here. So, using this we want this condition to happen. Remember this is the noise is additive white Gaussian noise, so we want this to happen.

So, this is your random variable Gaussian random variable with the 0 mean. So, m is equal to 0 in this case and your variance sigma squared is equal to this quantity. And if we use this relationship out here; it is easy to see that the probability of error in this case would be equal to this quantity divided by the standard deviation of the noise which I can rewrite as this.

So, what this shows is that this is nothing but the distance d 1 2 is a distance between the signal points and half of that. And then which is divided by the projection of the noise onto the unit vector correct. And we know that if the input power spectral density or the variance is italic N by 2, then here also the variance will remain same as italic N by 1; so, this is standard deviation. So, this is a standard a very general formula to calculate the probability of error for equiprobable binary signaling schemes correct.

So, this is valid regardless of the shape of the signals; now remember that this Q is a decreasing function. So, what is required to minimize the probability of error is to maximize the distance between the 2 signal point. And quickly basically that there is another way also to calculate this distance d 1 2, sometimes this is useful and specifically when you have very generic type of signals S 1 and S 2 t and we can do directly in the time domain and we can use this expression for calculating d 1 2 squared.

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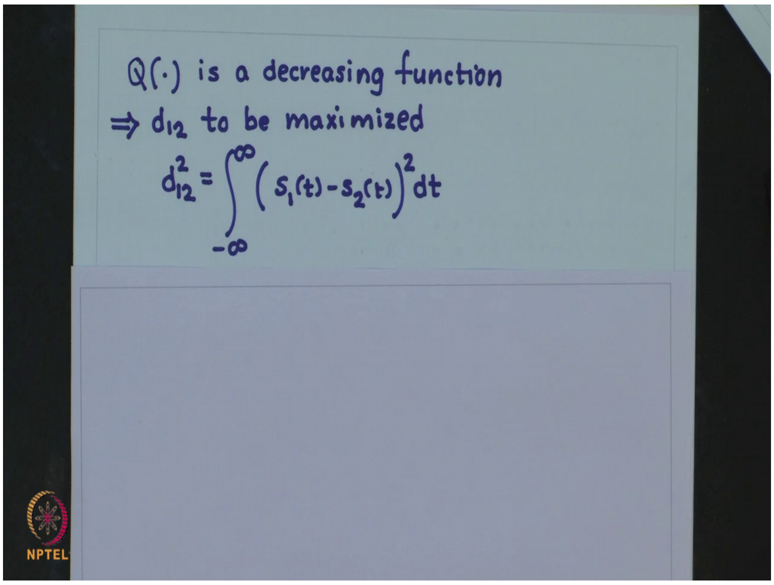

This is a very generic way for calculation of probability of error. Let me summarize, so today we have seen that the probability of error remains unaffected by the rotation and the translation of the coordinate axis, which is used to represent the signal vectors based on particular orthonormal basis signal set.

And we have also learnt how to calculate the probability of error for equiprobable binary signaling schemes where S 1 t and S 2 t could be of any shape. Now this will be useful for us when we are evaluating the performance of digital carrier modulation schemes which we will study next time.

Thank you.