Principles of Digital Communications
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Lecture - 27
Nonuniform Quantizer (Lloyd-Max Quantizer)

We have studied uniform quantizer for a uniform pdf, and we have also studied how to do uniform quantization for the case when the pdf is non-uniform. The uniform quantizer for a non-uniform pdf still restricts the step size to be uniform and equal to a constant except for the last n intervals of your input range. So, next question is, is it possible for us to make it little more generic and have a variable step size. So, if the motivation for this is as follows. If the input distribution has more mass near the origin; that means, the probability of occurrence of the input is high about the origin then the input is more likely to fall in the inner levels of the quantizer.

Now, we have seen that in the lossless representation of the source that is lossless source coding in order to minimize the average number of binary digits which we require to represent a symbol from the information source. We assigned shorter code words to those symbols that occur with higher probability, and longer code words to those symbols that occur with lower probability. So, in analog expression in order to decrease the average distortion due to quantization we can attempt to approximate the input better in regions with high probability perhaps at the cost of worse approximation in regions of lower property. Now, we can do this by making the quantization intervals smaller in those regions that have more probability mass then those intervals which have low probability mass.

So, what we will get by this is that we will get smaller quantization intervals near the origin. So, if we have quantization levels say L to be constant then what will happen is that we will get larger quantization interval away from the origin. So, when we have a quantizer with non-uniform intervals then it is known as non-uniform quantizer.

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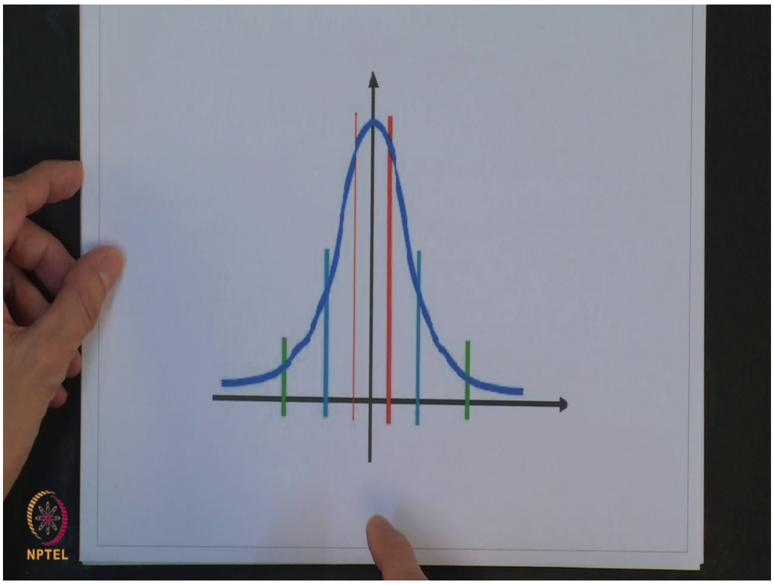

Now, we will study this and so the problem is as shown in this slide suppose the example I have a pdf of this form, then what I warned that since the probability mass is high here near the origin I will start alerting my contention intervals such that near the origin the step size a small compared to as we move away from the origin correct fine. And in this regard we will study this non-uniform quantizer and it is also popularly known as Lloyd-Max quantizer.

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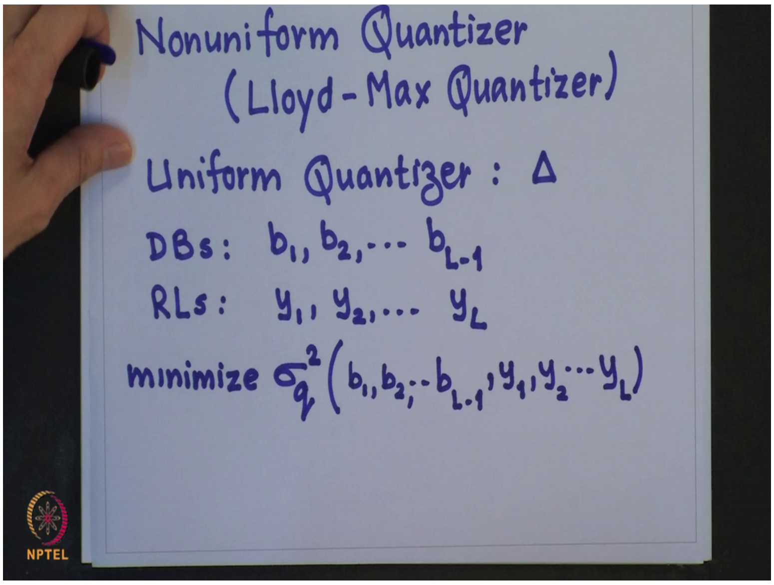

The first algorithm to obtain the values of the quantization boundary decision boundaries and the reconstruction levels for this non-uniform quantizer was provided by Lloyd and Max, and that is why it is known as Lloyd-Max quantizer also, ok. So, recall that for uniform quantizer our minimization problem had only one variable which was in the form of the step size which was constant, ok.

Now, our problem a little more complex we need to optimize with respect to decision boundaries and the reconstruction levels. So, our decision boundary DB denotes the decision boundaries these are b 1 b 2 up to b L minus 1, correct. Remember this is for unbounded pdf. So, b 0 is going to be minus infinity for generic and b L is going to be plus infinity and your reconstruction level which is RL is equal to y 1, y 2 up to y L.

So, now, we have to minimize our quantization noise which is a function of these decision boundaries and also of the reconstruction levels.

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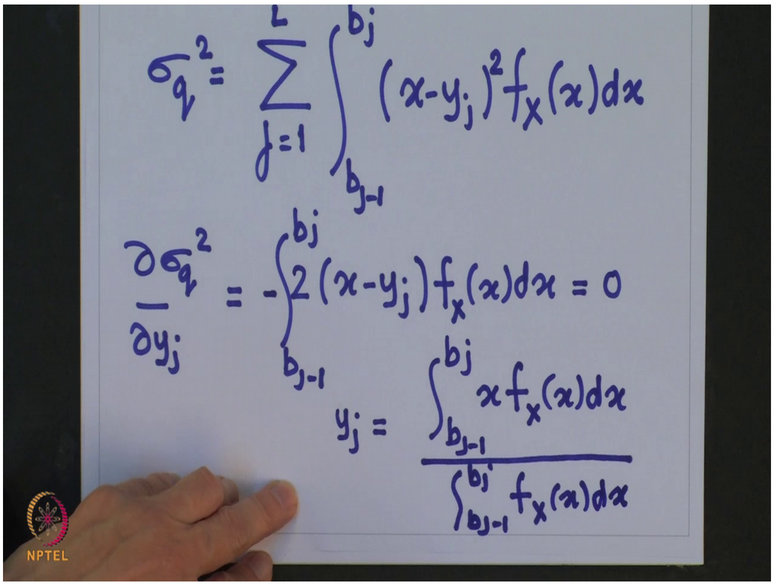

And this we know that can be written as follows, ok.

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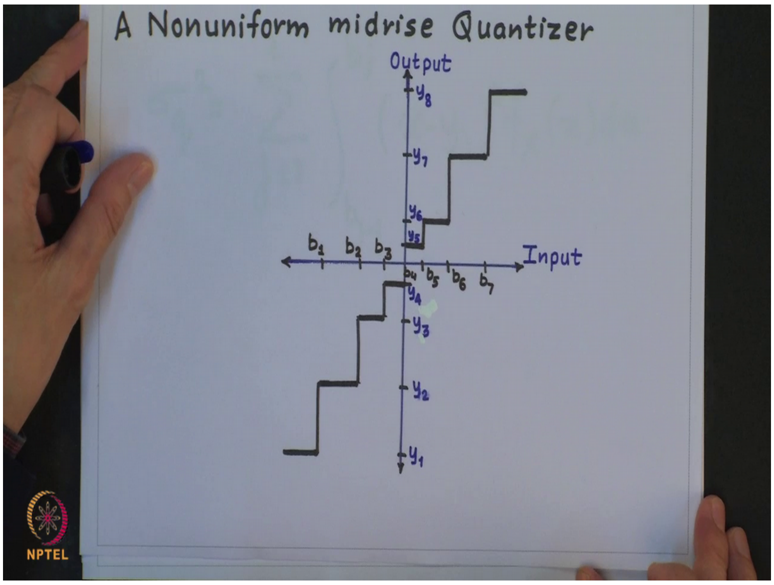

So, our problem is like this input output mapping for the non-uniform we will assume midrise quantizer. I have my decision boundaries b 1, b 2, b 3, b 4, b 5, b 6 its going to be symmetric about 0 right midrise quantizer I repeat b 0 is assumed to be minus infinity, b L is assumed to be plus infinity. Your reconstruction levels have been shown here y 1, y 2, y 3, y 4, y 5, y 6. So, anything lying between b 5 and b 6 would be given reconstruction value of y 6 similarly between b 6 and b 7 would be given the reconstruction value of y 7. So, our problem is basically to determine this decision boundary points and to determine this reconstruction levels, ok.

So, we solve this problem. Let us first take the derivative of this with respect to boundary is b j and also with respect to y j. So, first we will take the derivative of this quantity with respect to y j. Now, when I do this basically in this remember that only one term in the summation will remain after the derivative, and that is easy to see this what I will get is from here ok, and this will be a minus sign here ok, fine and this is equal to 0 I have to equate. So, if I do this basically from here is easy to see that I will get my value of y j to be equal to ok. So, this is nothing, but the centroid of the pdf in the interval between b j minus 1 and b j. So, y j is a centroid of that interval fine.

Next we have to take the derivative of this with respect to the boundary points b j and to do that basically it is important for us to understand that this quantization noise which is equal to this expression consists of following terms.

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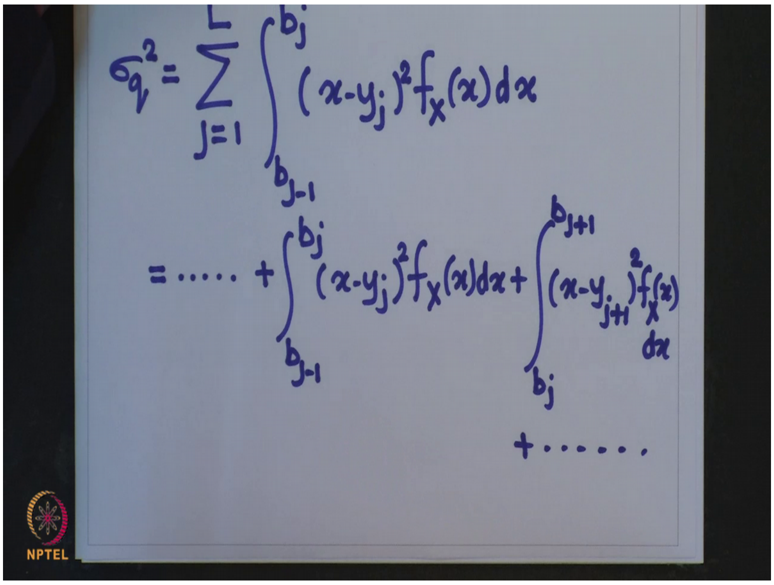

There are some terms out here plus you will have b j b j minus 1 x minus y j square plus and there will be some other terms. So, when you take the derivative of this respect to b j, these two terms are going to be involved rest other terms will go to 0.

And now, to take the differentiation of these two terms will again use the Leibniz rule.

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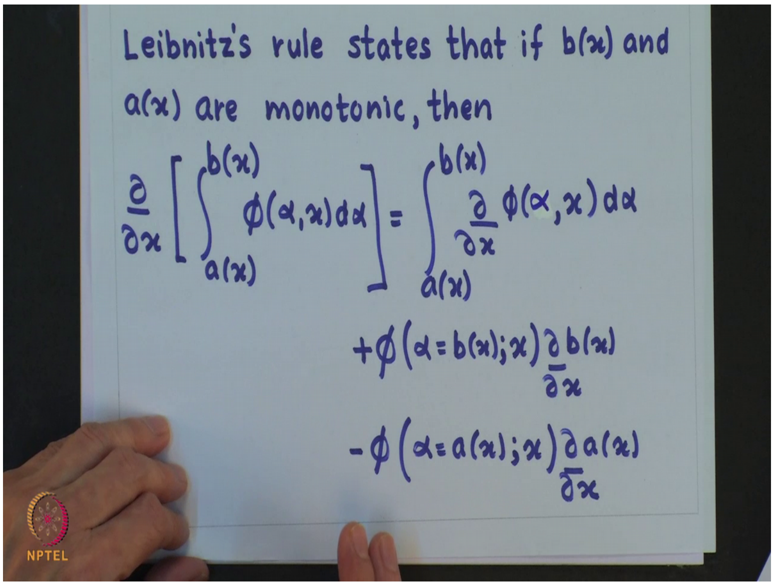

Given this function take the derivative with respect to x parameter. This is the first term and these are the other two terms which we get correct. Using this relationship very simple in this case unlike what we did for the uniform quantizer and the non-uniform pdf simpler than that here. So, if we do this that apply this rule to this case here we will get the following thing, rest other terms will go to remember when I take a derivative here with respect to b j rest of the other terms will go to 0 only these two terms will be involved, and the output from there, when I take the derivative equal to 0 implies I will get, ok. So, this implies that.

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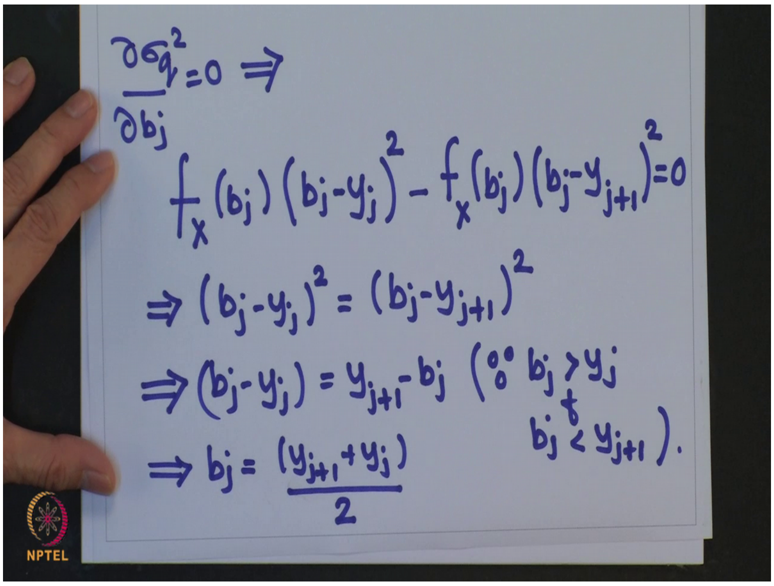

Now, this will imply that b j minus y j is equal to y j plus 1 minus b j because your b j is greater than y j and b j is less than y j plus 1 look at this convention which we have followed for naming the boundaries and the reconstruction level. So, from this it is easy to see that this condition is satisfied and from this we get this implies that b j is equal to y j plus 1 plus y j by 2. Here we see basically that your boundary points b j is the midpoints between the two reconstruction level. Recall for the uniform quantizer it was the other way round reconstruction levels were midpoints of the decision levels or decision boundaries.

So, now, we have to solve for this y j and b j using this equation. So, we have two equation for y j is the reconstruction level and this is the now, we have to do this in a iterative manner because both are interlinked. And one of the first algorithm to do this was basically what was known as Lloyd-Max algorithm. We will quickly have a look at that algorithm.

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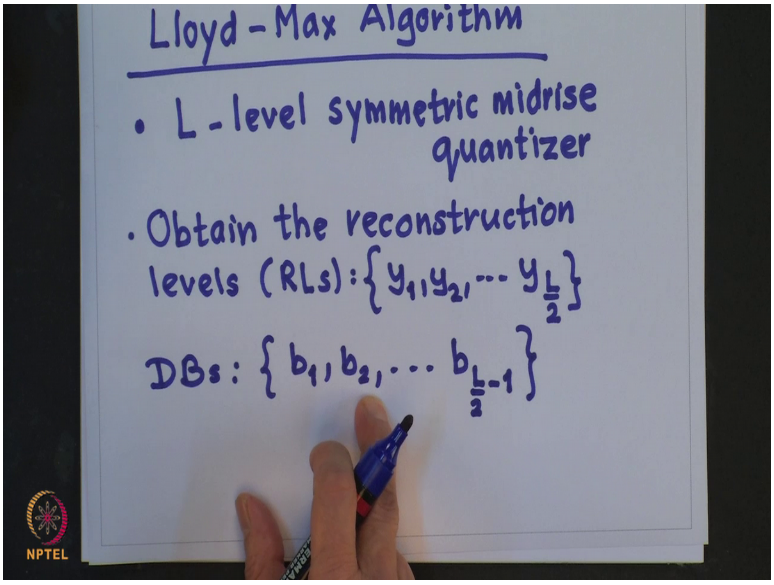

So, we have to design L level symmetric midrise quantizer, your given pdf is symmetric. So, now, a problem for finding out the boundary levels and the reconstruction levels reduced it what I have shown in this figure we will compute it only for the positive side and I am shown here basically this boundary. So, I have shown here basically the decision boundaries and their reconstruction levels what we would be interested is you know obtaining b 0, b 1, b 2, b 3, y 1, y 2, y 3, y 4. Once we get this then immediately we can get the other decision boundary points on the negative side and we can get the reconstruction levels on the negative sign. It is a just symmetric.

So, using this notation we can our problem is to obtain the reconstruction levels and the decision boundaries as follows. It is your RLs y 1, y 2 up to y L by 2, and decision boundaries b 1, b 2 up to b L by 2 minus 1 correct. And the reconstruction levels for the negative side here and the decision boundaries for the negative part will be obtained by a symmetry ok. Just going back to these equations this equation we can write the value for y 1, correct.

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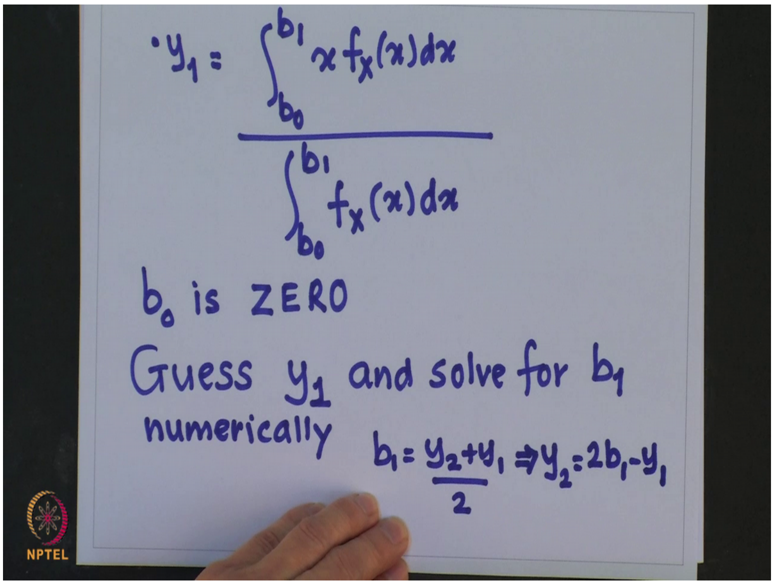

So, my first y 1 is equal to b 1 b 0 x divided by fine.

So, the way we have assumed again I show you the slight b 0 is 0, correct. So, b 0 is 0. Then what we do is guess y 1 and solve for b 1 numerically from this equation, correct. So, I will get my b 1. Now, we know that b 1 is equal to y 2 plus y 1 by 2. So, this implies that my y 2 is equal to 2 b 1 minus y 1. So, I will get my y 2.

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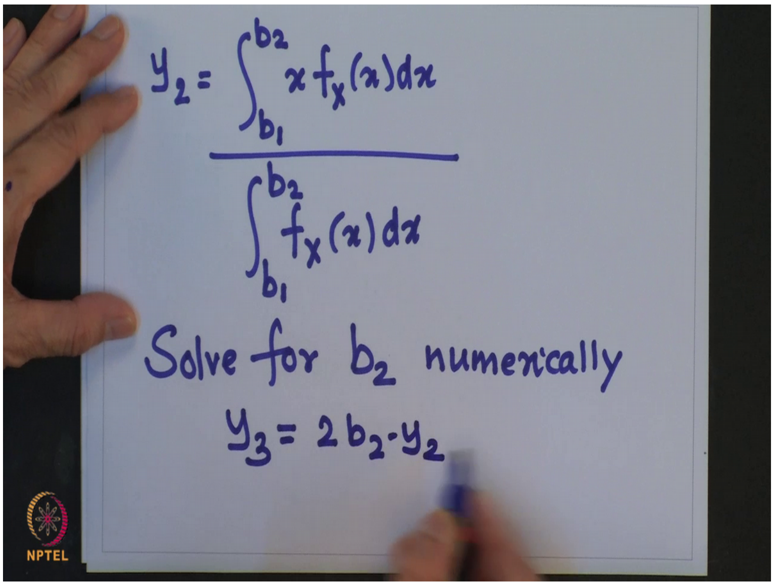

Once I do that I know my y 2 is related to b 2 as follows. So, what I do is now, solve for b 2 numerically using this equation. And we know that b 2 is basically the midpoint of y 3 and y 2, so I will get my y 3 from here to be equal to 2 b 2 minus y 2.

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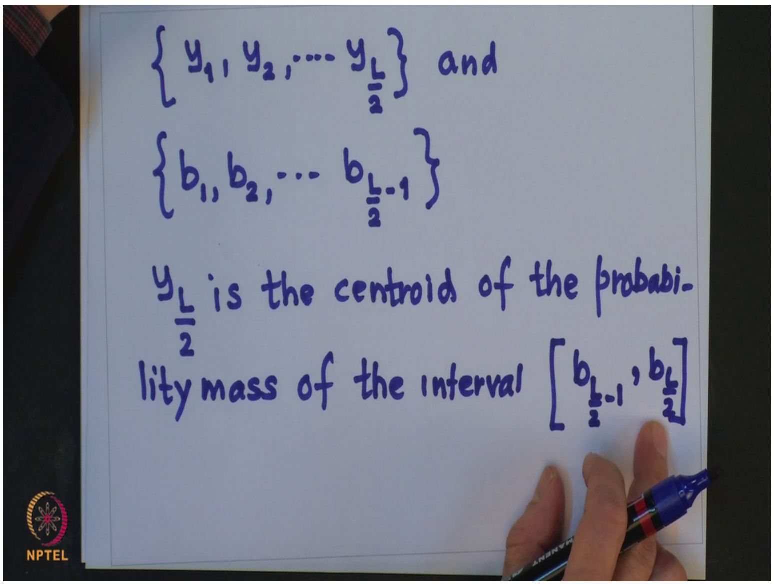

So, I continue like this process until we obtain a value for the following reconstruction levels and the decision boundaries. Now, the accuracy for all the values obtained at this point depends on the quality of the initial estimate of y 1. Now, we can check this by noting that y L by 2 is the centroid of the probability mass of the interval b L by 2 minus one b L by 2.

Now, we know this b L by 2 from our knowledge of the data correct, so if is unbounded pdf it would be infinity, correct. So, it is thus it is simply the largest value that that input can take and we are aware of this. So, I know my b L by 2 minus 1, I know my b L by 2 from the knowledge of the input data. So, I could have calculated my y L by 2 using this relationship, but remember once I get this quantity out here I already got my y L by 2. So, y L by 2 I am getting from two places.

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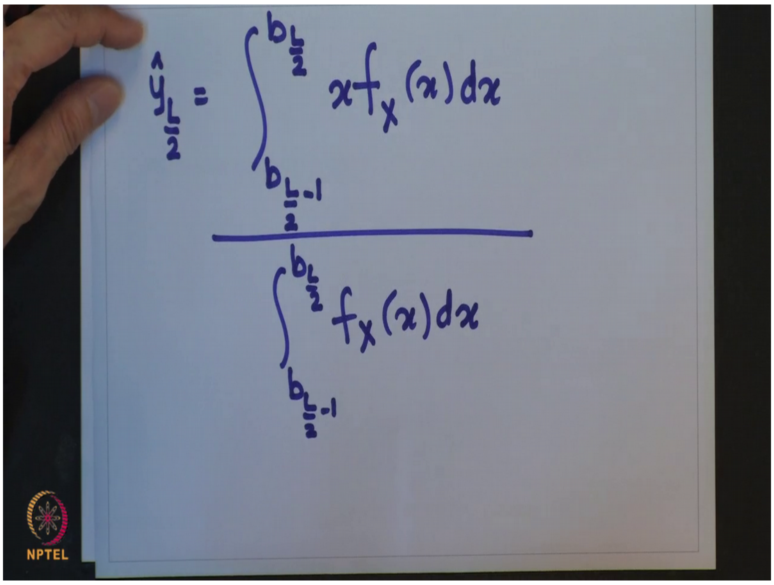

So, let me call the output which we get using this and my knowledge of b L by two as y hat L by 2, this I can compute as follows is a centroid. Now, you compare this point with what you have got from your calculation because once I get this point and I know my y L by 2 minus 1 I can get my y L by 2, correct. So now, compare this with the previously computed value of y L by 2. If the difference is less, then some tolerance threshold we can stop otherwise you will have to change the estimate, and the estimate of y 1 that is your initial estimate will be in the direction indicated by the sign of the difference between what you have got here and what you have obtained using this boundary point and the earlier reconstruction level and then you repeat the procedure, right. So, this is how you do it.

And. So, in the literature you will find that again the table has been provided for the Gaussian and Laplacian pdf again assuming that we have unit variance for different levels 4, 6 and 8.

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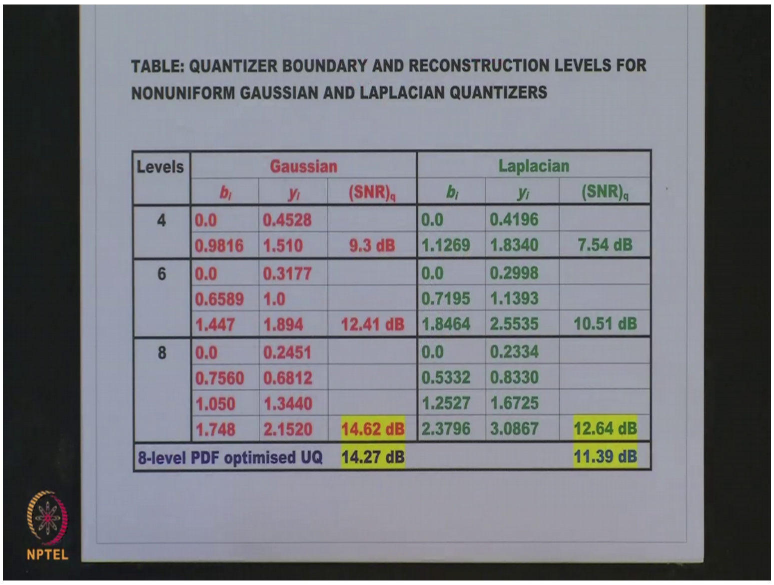

TABLE: QUANTIZER BOUNDARY AND RECONSTRUCTION LEVELS FOR NONUNIFORM GAUSSIAN AND LAPLACIAN QUANTIZERS

Levels	Gaussian			Laplacian		
	b_i	y_i	$(SNR)_q$	b_i	y_i	$(SNR)_q$
4	0.0	0.4528		0.0	0.4196	
	0.9816	1.510	9.3 dB	1.1269	1.8340	7.54 dB
6	0.0	0.3177		0.0	0.2998	
	0.6589	1.0		0.7195	1.1393	
	1.447	1.894	12.41 dB	1.8464	2.5535	10.51 dB
8	0.0	0.2451		0.0	0.2334	
	0.7560	0.6812		0.5332	0.8330	
	1.050	1.3440		1.2527	1.6725	
	1.748	2.1520	14.62 dB	2.3796	3.0867	12.64 dB
8-level PDF optimised UQ			14.27 dB			11.39 dB

So, for 4 levels you will have two boundary points, one is obviously 0 and the other boundary point is therefore, the same you will get the reconstruction levels. So, similarly for the Laplacian 1 6 8 level also I have been given. We notice one thing here also look for example, 8 level the Laplacian signal to quantization noise ratio comes out to be lower than this right because Laplacian has a heavier tail that is a reason, and one more thing is that this quantizers give you better performance compared to 8 level pdf optimized uniform quantizer, right. So, you get some improvement what for the gaussian case and for the Laplacian case, right, ok.

Now, again the question is how good are these quantizers these are quite good quantizers provided we have a good knowledge of the pdf of the input with all its parameter. So, the same comments which we made for the uniform quantizer with non-uniform pdf holds good, there could be mismatch the mismatches could be that we know the pdf of the input, but not sure about the variance of we choose the wrong pdf itself in both the cases basically the performance deteriorate.

Now, in a practical scenario sometime it becomes difficult to implement this kind of non-uniform quantizers. So, the idea is basically is it still possible to get good performance and exploit the ideas from what we have learned so far and design a uniform quantizer. For example, if you take a speech signal speech signal usually has heavy mass around the origin and, but we may not know the exact pdf or its variance. So, is it possible for such signals to design by using some kind of a preprocessor before we send it to a uniform quantizer. So, my job is basically take the signal input signal use some kind of a preprocessing and then apply a uniform quantizer on it and see that this preprocessing is a reversible transformation. So, at the receiver basically I can get the proper values by using the inverse reversible transformation.

So, in conclusion Lloyd-Max quantizer which is a non-uniform quantizer is also an optimum quantizer in the sense that it minimizes the mean squared quantization error for the given number of contention levels and the input pdf. Now, the performance of this quantizer could deteriorate severely due to a mismatch. The mismatch could be due to the wrong estimate or choice of variance for the particular pdf or the wrong choice of pdf itself.

In practice implementation of non-uniform quantization is always a difficult task. So, in practice this non-uniform quantization is achieved by the process known as companding. In companding the signal before quantization passes through a compressor characteristic which really stretches out the lower amplitude of the signal and compresses the higher amplitude of the signal, and then it goes through a uniform quantization process and this is followed by an expander characteristic which is the inverse of the compressor characteristic.

So, this process of Companding, we will study next time.

Thank you.