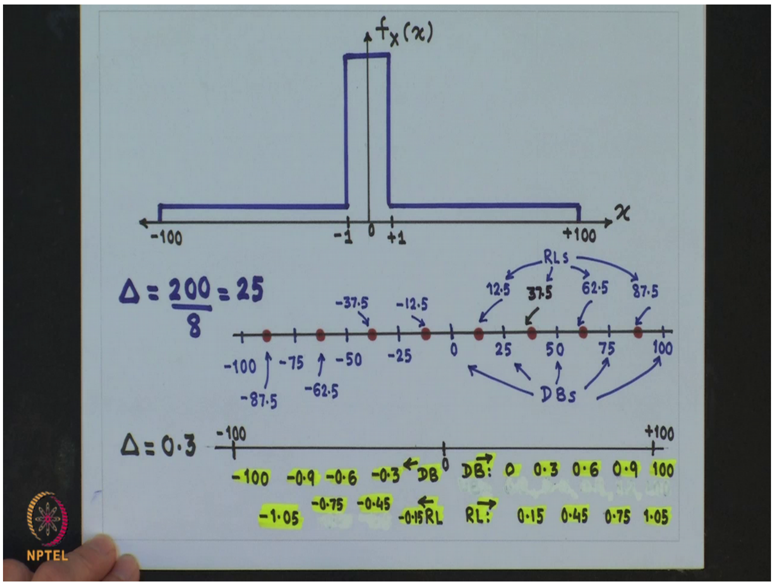
Principles of Digital Communications
Prof. Shabbir N. Merchant
Department of Electrical Engineering
Indian Institute of Technology, Bombay

Lecture – 26
Step Size & Quantization Noise

We have studied uniform quantization for a input PDF which is uniform, then an uniform quantizer is an optimum quantizer, in the sense that mean square quantization error will be minimum for the given number of quantization levels. And in this case the Quantization Interval or the Step Size is easily obtained by taking the input range and dividing it by the number of quantization levels, but if the PDF is not uniform, then this strategy is not a good one. Let us revisit the example which we had considered earlier.

(Refer Slide Time: 01:18)



So, I have PDF here. The input ranges from minus 100 to plus 100 and between minus 1 to plus 1 the probability of the input occurrence is 95 percent and in the rest of the interval is 5 percent. Now, if we deploy the strategy of designing a uniform quantizer which we use for the uniform PDF, then in this case my quantization interval would be the range which is 200 divided by 8. I am presuming that we are designing a 3 bit quantizer that is 8 levels. I will get the step size to be 25.

Now, for this step size this figure out here shows what are the decision boundaries and what are the reconstruction levels. Now, we saw that for this case the maximum quantization error is going to be 12.5 and the minimum error, which we will get 95 percent of time would be equal to 11.5 and that would happen when the input is between minus 1 to plus 1, in which case the reconstruction levels are going to be minus 12.5 or 12.5.

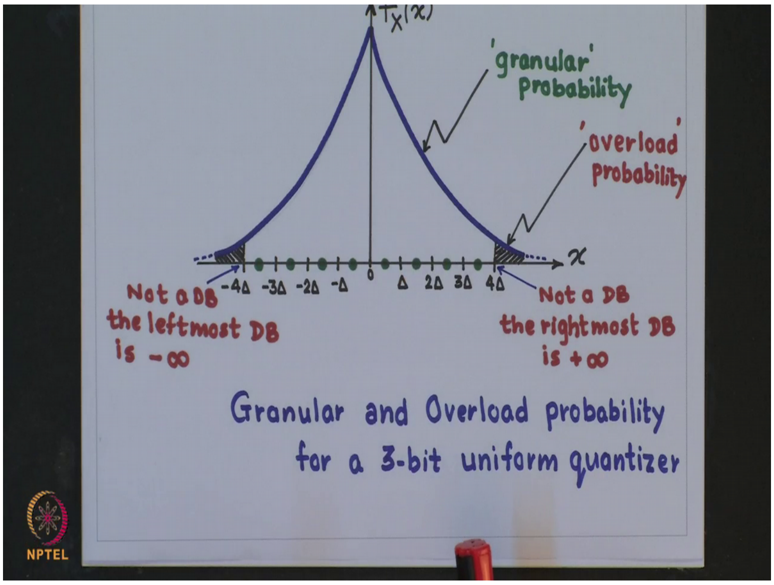
Now for such a case the mean square quantization error which we will get, will not be minimum. So, we said fine let us change the strategy and in a above fashion we decided the step size to be 0.3. So, if I take step size to be 0.3, then my decision boundaries are given as 0, 0.3, 0.6, 0.9 and the last one would be 100 and the reconstruction level for the same would be 0.15, 0.45, 0.75 and 1.05. And similarly you will have on the negative side, it is a symmetric PDF.

Now, in this case the maximum error you will get is when plus 100, gets quantized to the reconstruction level 1.05. In that case the maximum error you will get is 98.95 but for 95 percent of the time the maximum error, you will get is 0.15 because the probability of the input lying between minus 1 to plus 1 is 95 percent.

So, now the next question is how do we decide in general this value of delta. We have chosen 0.3, suppose example if I have chosen 0.2 or say I have chosen 0.4 what would happen correct. But the conclusion is that if the source PDF is not uniform, then it is not a good idea to obtain the step size by simply dividing the input range by the required number of quantization levels. Let us take and it is important to note that this technique will completely fail, if my PDF was unbounded. And in a practical scenarios many a times we do model the PDF of input which is unbounded. And obviously, in this case the PDF must isentropic go to 0.

And why will it fail? The reason is that we have a infinite domain and we have to cover up this range with a finite number of step size or quantization intervals. So, this is not possible. So, in this case will be required to choose the quantization interval and the number of quantization levels, such that it achieves a desired means squared quantization error. So, let us take an example, suppose, I have a PDF of this form which I have given.

(Refer Slide Time: 06:01)



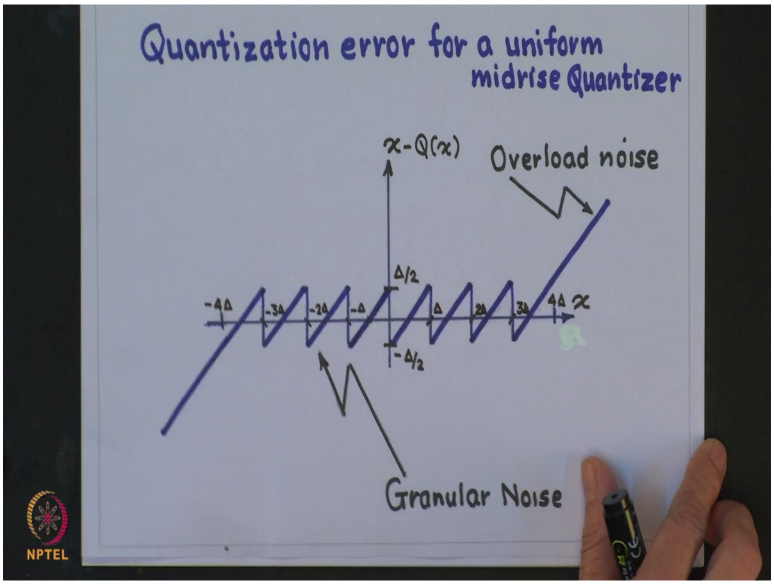
And let me assume that again I want to quantize using 3 bits, so I have 8 level to be obtained. Now, in this case our reconstruction levels are always going to be the midpoint of the quantization interval except for the boundary intervals ok. So, for this case I can show you that this is delta, 2 delta, 3 delta, 4 delta; delta is basically the quantization interval. What we see from here that in such a case, now the my job is basically to place this delta where, should I place delta on this axis ok, to that will come later on, but look here now in this basically you will find that the kind of errors which we will get from; quantization.

Whenever the input is in this range, or this range this range, or this range it is also valid on the negative side, you will see that the quantization error goes between minus delta by 2 to plus delta by 2, but once it goes beyond 4 delta, all those values out here from here onward or this here, they are going to get quantized either to this or to this respectively. So, in this kind of things you get two types of errors, one what is known as granulary error or granular noise and other basically what you get is the overload noise. And the area under this curve basically will denote the granular probability of occurrence and the area under this curve; obviously, it is on the this side also will be denoted as overload probability correct.

So, if you have a such a PDF, it is also important to remember that, this 4 delta is not going to be a decision boundary; DB denotes the decision boundary. The rightmost decision boundary is going to be plus infinity in a generic case correct if I use an unbounded PDF. And similarly on the left hand side minus 4 delta will not be a decision boundary, but the leftmost decision boundary is going to be minus infinity. So, this is what is going to happen.

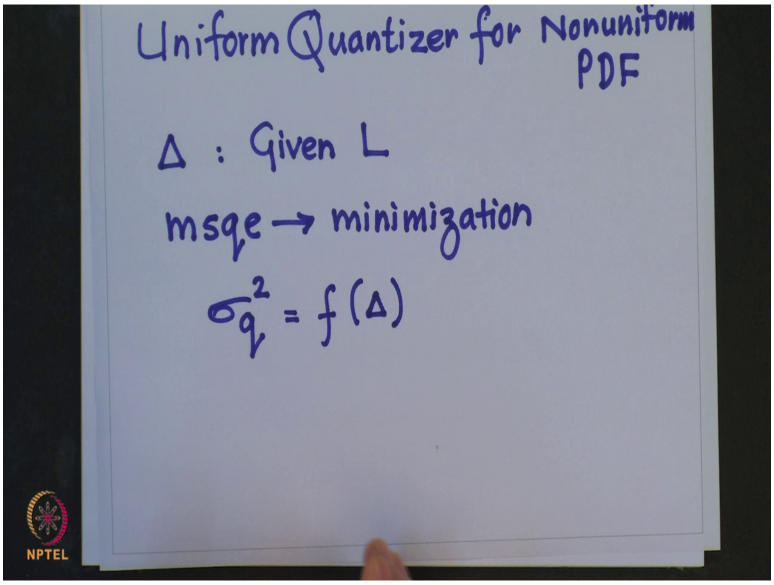
Now, if you look at if you plot the quantization error, which we will get from this kind of input PDF, it will look something like this.

(Refer Slide Time: 08:47)



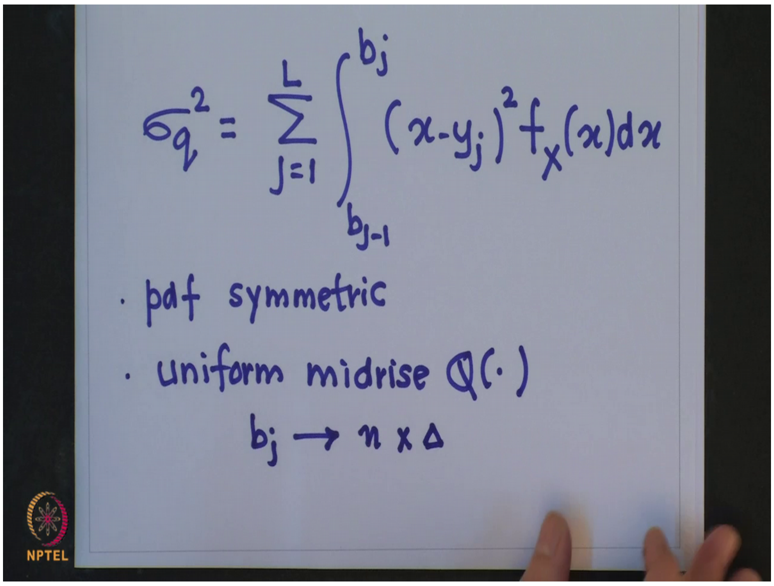
So, up to 4 delta if you see from 0 to 4 delta, whatever I say on the positive side is valid on the negative side, you will see basically the quantization error goes from minus delta by 2 to plus delta by 2 correct. So, this is a way it will fluctuate, but once we go beyond 4 delta basically what happens is basically the overload noise. So, this basically is a granular noise and this is the overload noise. And so, our objective is basically to determine this delta for the given number of levels L and the input PDF such that the quantization noise gets minimized. So, now what we are going to design is a uniform quantizer for a non uniform PDF.

(Refer Slide Time: 09:48)



It is probability density function of the input. So, my problem is to determine the step size that is delta for a given number of quantization levels that is L, such that your mean square quantization error gets minimized and the simplest way to achieve this is to write the distortion, due to the quantization noise. Let me call this as a distortion that is a variance of the quantization noise as a function of the step size delta and then minimize this function ok.

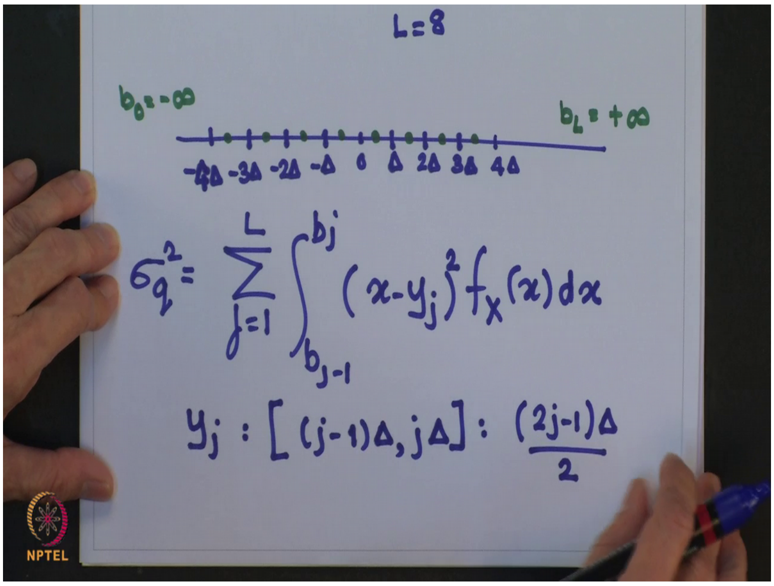
(Refer Slide Time: 11:12)



So, we know that quantization noise expression is given by this, where b j's are decision boundaries y j is a reconstruction level and n is the number of quantization levels which we have. Now, we have to find an expression for this variance in terms of delta. So, what we have to do is basically we have to replace this decision boundaries b j and the reconstruction levels y j as a function of delta. And it is not very difficult to do this, I will show you. We will assume one more thing. We will assume that our PDF is symmetric.

So, what this implies that computation of the mean square quantization errors, for positive value of X and for the negative value of X will be identical correct. And for our design we will assume that we are using uniform midrise quantizer correct. So, in this case basically your decision boundaries b j's are going to be integral multiples of your delta correct, n denotes the integer value fine.

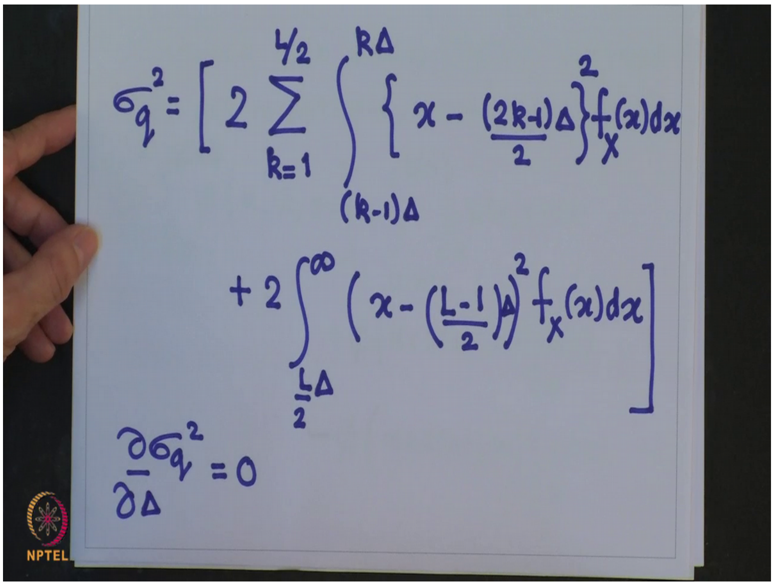
(Refer Slide Time: 13:13)



So, I can denote it something like this. I have a line here and I am supposed to divide this into the intervals delta. So, let me call as a delta, this is 0, 2 delta, 3 delta. And let us assume say for example, L equal to 8, I would be required to find out this delta, 2 delta, 3 delta. So, for this I will get my reconstruction levels to be here midpoints correct and other would be ok. So, this is 4 delta. So, once I have my deltas I can easily find out my reconstruction level. So, in this case basically what will happen is your, we are assuming unbounded PDF. So, your for b L is going to be plus infinity and your b 0 boundary is going to be minus infinity correct.

So, these are your decision boundaries and this green ones are your reconstruction levels. So, let us write the distortion as a function of delta and then try to minimize it with respect to delta fine. So, quickly we write this your noise variance. Now, as we said we are using uniform quantizer and midrise. So, your y j for the interval j minus 1 delta, j delta is going to be a simply 2 j minus 1 delta by 2 because, once you have this is a mid halfway between this will be your reconstruction level.

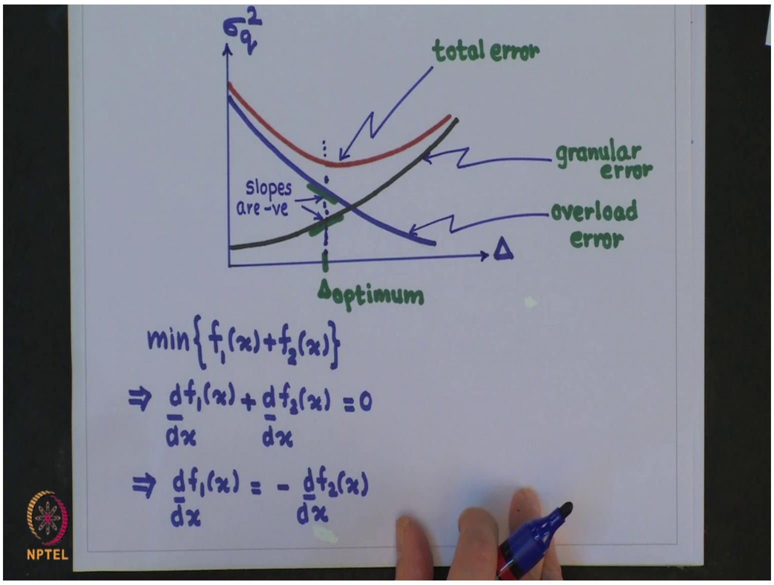
(Refer Slide Time: 15:51)



Now, since we are assuming that PDF is symmetric. We can simplify this expression to the following expression; 2 because symmetry. So, I am going to design only for my positive side of the PDF, your reconstruction level is 2 k minus 1 by 2 delta and averaged it. This is what you will get up to this point. In this all these intervals you will have granular noise quantization noise, beyond this, you are going to have the overload noise. So, for that basically we are going to write this expression.

So, this portion corresponds to the granular noise and this portion corresponds to the overlord noise. So, now what we had to do is basically we had to take the derivative of this quantity, with respect to delta and equate to 0 to solve for the optimum value of delta. Now to do this basically it is also important to realize that this 2 noises the way it behaves.

(Refer Slide Time: 18:01)

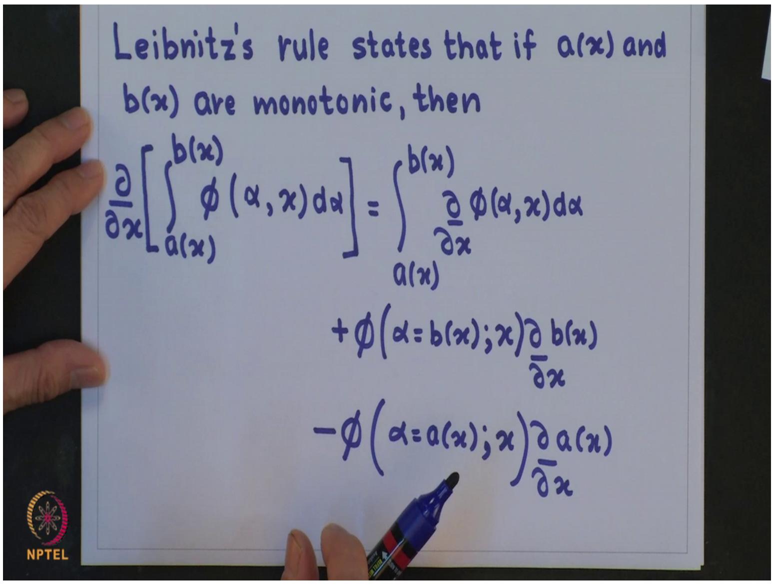


If you plot it the granular noise will increase with the size of your delta, higher the value of delta the granular error will increase and what we will get basically for the overload noise you will find that it will decrease as the delta increases.

So, if you take the total error would be this curve. So, you can show that basically it will have a minimum value. And what would be the optimum value? So, if you take f 1 x denoting say the granular noise and this overload noise you are to take the derivative of this, this is a function of delta. So, you have to take the derivative of this with respect to delta, I am just writing in terms of x here. So, if you try to take the derivative of this, what is going to happen basically that this condition will get satisfied correct; that means, the slope this slope and this slope basically they will be negatives of each other correct.

So, something like this I have here. So, at that point so, if you look at this point basically this slope and this slope, they are magnitude wise they are exactly equal and at this point basically you will get the value of delta optimum correct ok. So, now, to solve this we will require to take the derivative of this quantity out here.

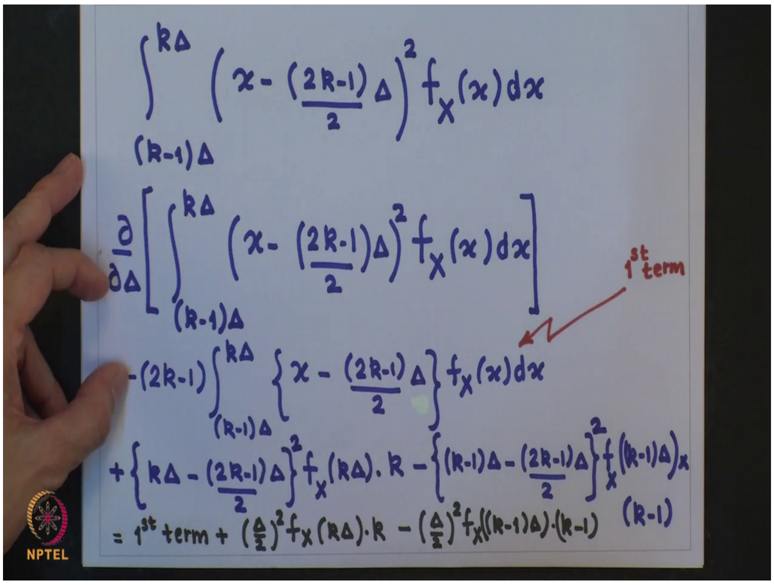
(Refer Slide Time: 19:34)



Now to do this basically we will have to use one theorem from calculus and that is known as Leibnitz's rule which states, that if a x and b x are monotonic, then if you have this function out here, which I have shown in the bracket and if you take the derivative with respect to x correct. This x has got nothing to do with our PDF correct.

So, this is general x; so, if you take the derivative of this is the relationship you get correct. So, if you use this relationship and take the derivative of this let us see what happens. It is important to realize now, this k delta is going to appear in this summation in two places, one which I have shown here, the other place it would be k delta would come here and here it would be k plus 1 delta correct is it ok, fine. Now, let us try to do this take the derivative of this. So, when we are taking a derivative of this term of this whole term, let us take a derivative of each of this term k equal to 1 to up to L by L by 2 correct.

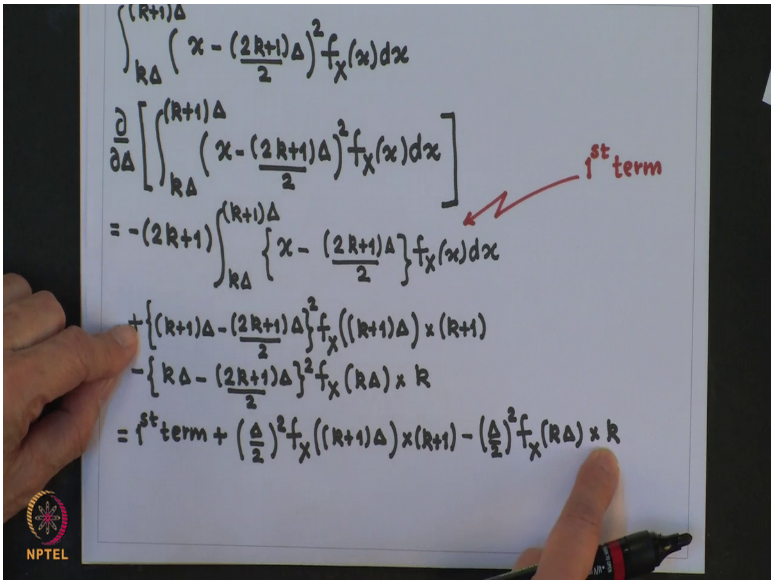
(Refer Slide Time: 20:48)



Now, so, in this derivative operation we will see basically that we would be required to take a derivative of a term like this. Now, if I take a derivative of this, now it is function of delta I will take a derivative of this term with respect to this ok. Now, it is very easy to see here by this Leibniz's rule, I will have to take the derivative of this term with respect to delta which I have done here, it is minus 2 x minus 1 this quantity will come and has to be integrated with this correct. So, this is the first term which I have taken here, this corresponds to this term out here fine, this I call it as a first term.

The next term basically I have to evaluate this look here, I have to evaluate this and take the derivative. So, this derivative is nothing but here k. And similarly here this is nothing but k minus 1 fine and this I have to evaluate. So, I have to just substitute this values out here. So, I will substitute here, for this case k delta l substitute in place of x, I substitute here k delta fine. Similarly I do for k minus 1. Now, remember this if you evaluate this comes out to be delta by 2 square and this quantity comes out like this fine.

(Refer Slide Time: 22:21)

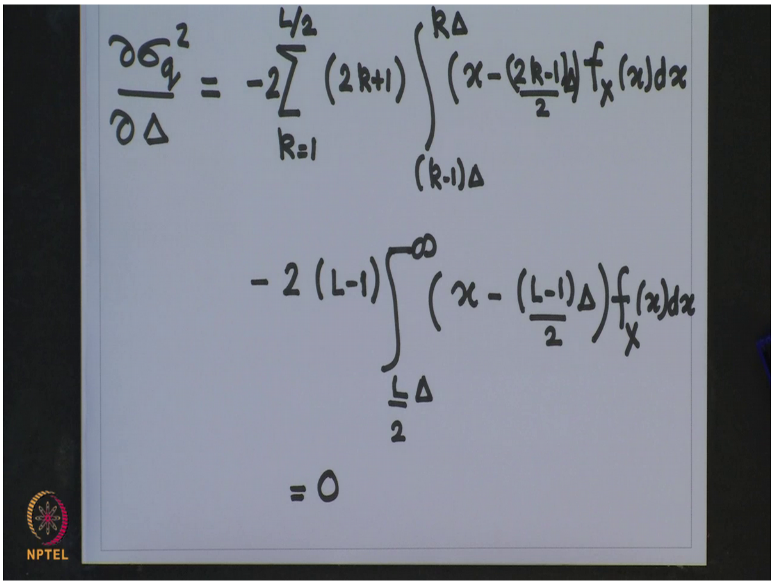


Now, remember one more thing. There will be another term where k delta will appear correct and that is here, k delta has come up here and then this will be k plus 1 delta. Now, again I have to take a derivative of this term. So, if I take the derivative of term again I get this derivative, so, I take the derivative of this. So, the I have to take the this is a function phi of x and delta. So, I am taking the derivative with respect to delta, I will get this one term again. And then again then solid using Leibniz, I will get this term and I will get this term and I just simplify this, I write it out this is the first term so, I do not want to write it rewrite it here.

So, I have written just first term and you just evaluate this I come I get this value. Important thing to note that this term basically, here it appears with minus sign and here it appears with the plus sign and this all summation. So, what will happen that this term will get cancelled with this term correct? Similarly, this term would have got canceled with the earlier one and this term will get cancelled with the following term correct. So, all these terms will get cancelled and what will happen at the boundary points, at the boundary points remember that when we are trying to evaluate, this quantity out here, will turn out to be 0 at the boundary point correct.

So, finally, what will happen? Only this first term will remain from each of this term in the summation ok.

(Refer Slide Time: 23:52)



So, and that is what we can write it here if I take the derivative of this what I will get is correct. So, I will get this and then from the last term I will get this. So, this is what I will get and this I have to equate to 0. Now this looks like a very messy equation, now the only way to solve this is basically using a numerical technique for this correct fine. So, now if you do that basically some of the results are available in the literature for the known PDF's. So, I will just show you here.

(Refer Slide Time: 25:47)

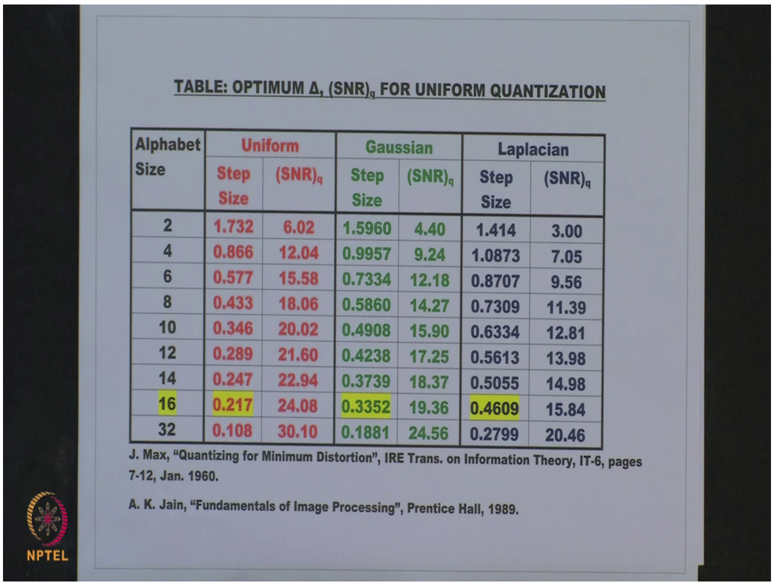


TABLE: OPTIMUM Δ, $(SNR)_q$ FOR UNIFORM QUANTIZATION

Alphabet Size	Uniform		Gaussian		Laplacian	
	Step Size	$(SNR)_q$	Step Size	$(SNR)_q$	Step Size	$(SNR)_q$
2	1.732	6.02	1.5960	4.40	1.414	3.00
4	0.866	12.04	0.9957	9.24	1.0873	7.05
6	0.577	15.58	0.7334	12.18	0.8707	9.56
8	0.433	18.06	0.5860	14.27	0.7309	11.39
10	0.346	20.02	0.4908	15.90	0.6334	12.81
12	0.289	21.60	0.4238	17.25	0.5613	13.98
14	0.247	22.94	0.3739	18.37	0.5055	14.98
16	0.217	24.08	0.3352	19.36	0.4609	15.84
32	0.108	30.10	0.1881	24.56	0.2799	20.46

J. Max, "Quantizing for Minimum Distortion", IRE Trans. on Information Theory, IT-6, pages 7-12, Jan. 1960.

A. K. Jain, "Fundamentals of Image Processing", Prentice Hall, 1989.

So, if you take for example, different levels is known by alpha alphabet size 2 4 6 up to 32. And if you take the uniform PDF and if you take a Gaussian PDF and Laplacian PDF and if you try to evaluate this delta here, it is assumed that the variance of the input is equal to 1 correct. So, that is normalized to unit variance and if you evaluate this for step size 2; that means, two levels you will get you have to calculate only one this is basically you will get 1.732 that is the step size which you will get and you will get two construction reconstruction levels.

And then you can also calculate the signal to noise ratio, the quantization signal to noise ratio by signal power basically you can find out using the variance to be equal to 1 for all these cases is assumed. And then if you do this basically this is the results which you get from here. Now, there is something very interesting to be observed, I have given the references out here, these are the references from which you can have a look at it and see the table is the same as what I have given here.

Now, that is a interesting thing, now it is important to know that Laplacian PDF has more of it is probability mass away from the original. And it lies in it is tail, then the Gaussian PDF correct. So, what that implies that for the same delta and the number of quantization level L, there is a higher probability of being in the overlord region, if the input has a Laplacian PDF, then if the input has a Gaussian PDF correct. And so, the tail is heavy for the Laplacian PDF compared to the Gaussian PDF ok. And then what it implies that the for same number of quantization levels here, if now delta increases then the size of the overload region also decreases correct. And, hence the overload probability decreases.

So, what this implies that the overload noise will decrease at the expense of increase in the granular noise. Therefore, for a given number of levels, if delta is calculated to balance the effect of both this granular and overload noise, then PDF's that have heavier tails like Laplacian compared to say Gaussian will tend to have larger step size. And this is very clear from this table, which I have here look for the example 16 levels. As you keep on moving here the tail becomes heavier. So, for the same number of L that is quantization level you will find that this size basically increases, the delta increases correct.

So, this delta is higher than this and this delta is higher than this because Laplacian has heavier tail fine ok. So, please remember this table is only assuming that your input signal variance is equal to 1. Now, the next question is how practical are these quantizers? It is important to note that this quantizers design for specific PDF. So, if the PDF changes there will be a mismatch and you could have severe degradation due to this mismatch. So, the mismatch could be that I have a right PDF, but I do not have a proper estimate of its variance, then the value of delta which you will get will be wrong. So, the performance of that quantizer is going to degrade or you could start with the wrong PDF itself correct. And then also the performance will degrade.

So, the next question is that I have designed a uniform quantizer for a non uniform PDF, the obvious question is it necessary for me to make this quantize and step also to be uniform, is it not possible for me to have this variable correct. For example, if I know that my input data lies mostly about the origin, then should I not allot more number of quantization levels near the origin? So, we will try to answer these questions, in the form of a very generic design of a quantizer and that is what is known as non uniform quantizer which we will discuss next.

Thank you.