Principles of Digital Communications

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Lecture - 24
Pulse Code Modulation : Quantization

We begin our study of the module on Pulse Code Modulation, popularly known as PCM. In continuous wave modulation some parameter of a sinusoidal carrier wave is varied continuously in accordance with the message signal. This is in contrast to pulse modulation in which some parameter of a pulse strain is varied in accordance with the message signal, there are two families of pulse modulation: these are analog pulse modulation and digital pulse modulation. In analog pulse modulation a periodic pulse strain is used as a carrier and some characteristic feature of each pulse; example, amplitude duration, or position is varied, in a continuous manner in accordance with the corresponding sample value of the message signal.

Thus in analog pulse modulation information is transmitted basically in analog form, but the transmission takes place at discrete times. Digital pulse modulation in this the message signal is represented in a form, that is discrete in both time and amplitude. Thereby permitting transmission of the message in digital form as a sequence of coded pulses, this form of modulation is also known as pulse code modulation. So, the use of coded pulses for the transmission of analog information bearing signal, represents a basic ingredient in digital communication. So, PCM in basic form could be considered as a conversion of analog waveforms into coded pulses. As such this conversion may be viewed as the transition from analog to digital publication.

In some sense the term modulation in PCM is a misnomer. In reality PCM is a source encoding strategy, by means of which an analog signal emitted from a source gets converted to a digital form. Transmission of the digital data so, produce is really a different topic. Now, we will quickly study the basic functional blocks of PCM system and, which is shown in the figure here.

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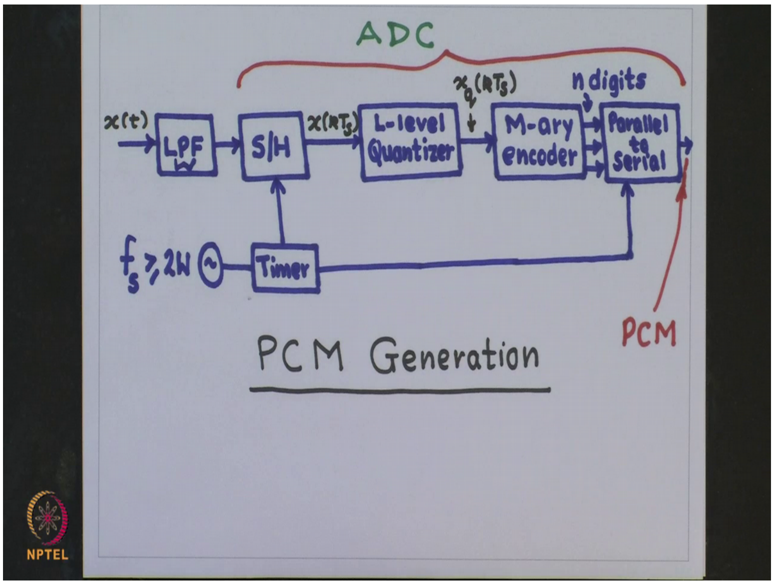

So, we have the input x t which is band limited to some bandwidth say W and it is a low pass nature. This passes through a filter which is a low pass this basically is a filter which limits the bandwidth of the input signal to W and, helps in mitigating the effect of a lysing error due to sampling. So, this is also known as antialiasing filter. This is a block which is known as sample and hold, in this block basically we carry out what is known as flat top sampling.

The sampling rate is basically decided by the Nyquist sampling theorem, which states that the sampling frequency should be greater than equal to twice the bandwidth of the band limited low pass signal for perfect reconstruction. So, usually the sampling frequency is chosen higher than the Nyquist sampling rate, in order to provide some kind of guard band against the aliasing error.

Then the output of this basically is the discrete values, which have been sampled at the time instant t equal to k T S. L level quantizer basically rounds off the sample values to the nearest discrete value in a set of L levels. The resulting quantized samples which is denoted x q k T S are discrete in time by virtue of sampling and discrete in amplitude by virtue of quantizing.

Next we have the encoder. The encoder translates the quantized sample into digital code words, the encoder works with M-ary digits and produces for each sample a codeword consisting of n digits in parallel. Since there are M raise to n possible M-ary code words, with n digits for word unique encoding of the L different level requires that M raised to n should be greater than equal to L.

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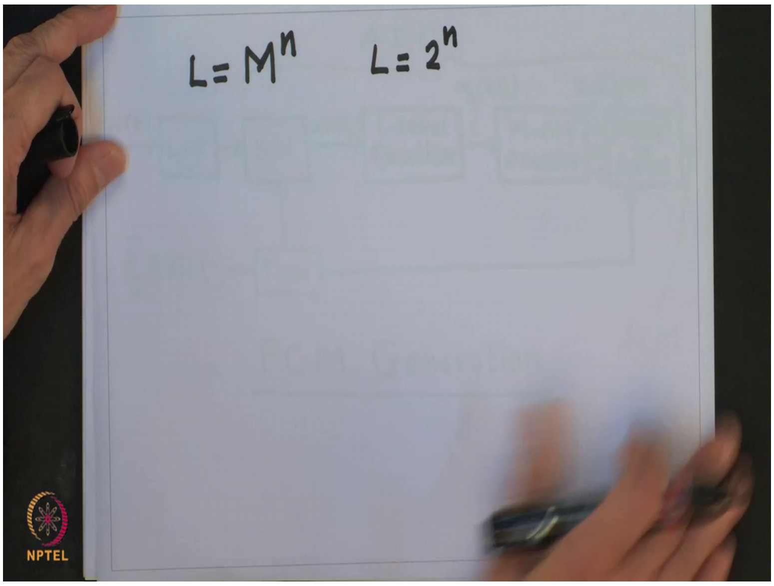

So, the parameters M N and L should satisfy the equality. There is a number of levels for binary PCM must be equal to some power of 2 because, M is equal to 2. So, in that case you will get L is equal to 2 raised to n.

Finally successive code words are read out serially to constitute the PCM waveform, which is nothing, but M-ary digital signal. The PCM generator there by x as a analog to digital converter performing the analog to digital conversion at the sampling rate decided by f S. A timing circuit coordinates the sampling and parallel to serial readout.

Now, let us consider a PCM receiver with the reconstruction system shown in this figure.

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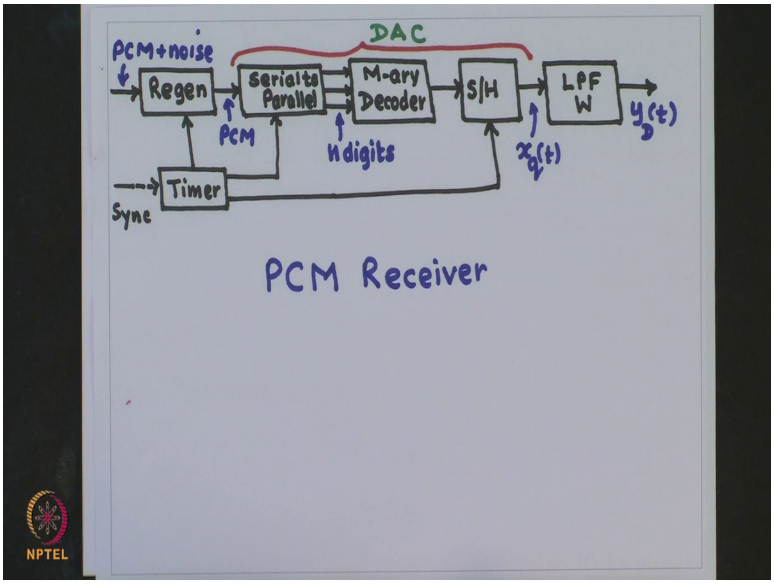

The received signal may be contaminated by noise, but regeneration yields a claim and nearly errorless waveform, if the input signal to noise ratio is sufficiently large. Now, the digital to analog conversion operations of serial to parallel M-ary decoding and sample and hold generate the analog waveform, which we denote by x q t.

This waveform is a staircase approximation of the original message signal x t similar to flat top sampling, except that the sample values have been quantized, low pass filtering then produces the smooth output signal, which we denote by y D t, which differs from the message signal x t to the extent that the quantized sample differ from the exact sample values x k T S. So, perfect message reconstruction is therefore, impossible in PCM even when random noise has no effect.

Now, the most important component of this PCM is basically the quantization and, we will focus our study of PCM on to this quantization problem. Now, in real practical scenarios this process of quantization an encoding is basically carried out by one hardware block. So, for our study we will assume that this two blocks have been combined has one block and, this also helps us in formulation of a generic optimal quantizer design problem.

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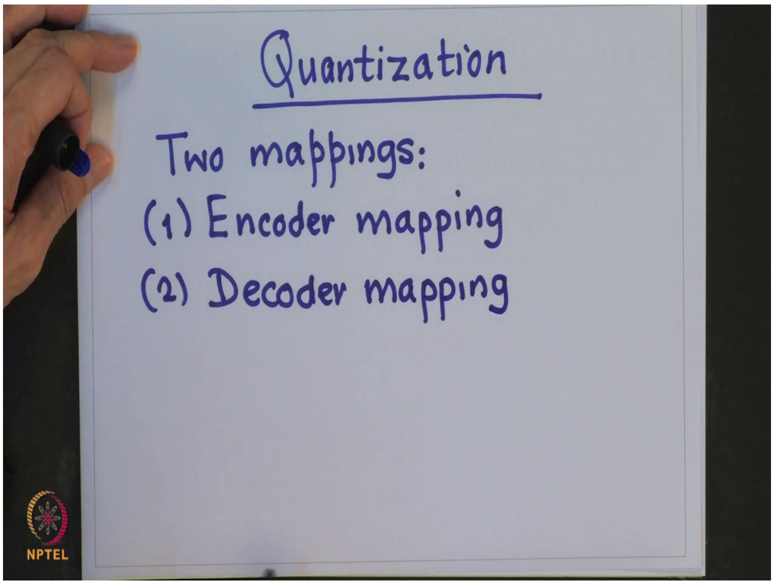

We will study now the quantization problem. In practice and structurally to the quantizer consist of two mappings, one the encoder mapping, which takes place at the transmitter and the other is basically decoder mapping, which takes place at the receiver.

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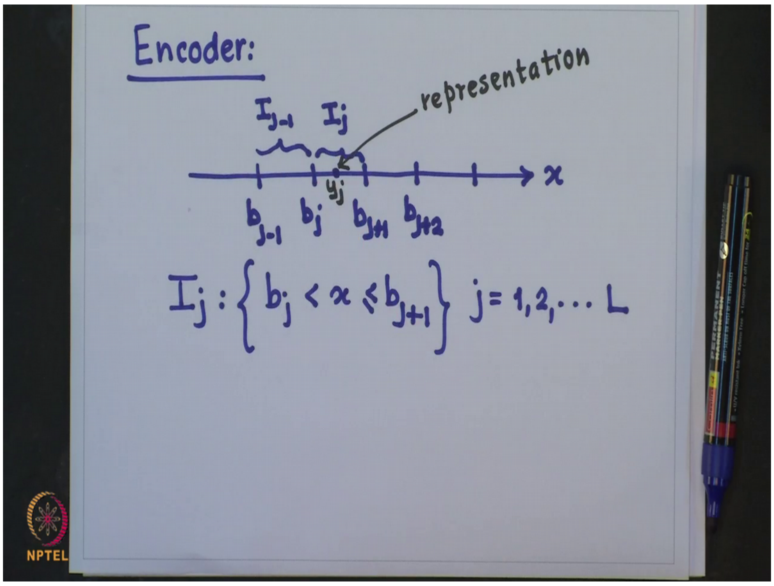

Now, let us see what does an encoder do. Encoder partitions the amplitude range of continuous signal. So, let me denote that continuous signal on this axis and, it will divide this range into L intervals let me denote this intervals here. So, each of this is the interval, this is the interval we will call it as I j minus 1, this is the interval which is corresponding to I j; so I j interval determined by the decision boundaries also known as decision levels.

So, b j minus 1 and b j is basically the decision boundaries for the interval I j minus 1. And similarly b j and b j plus 1 is are the decision boundaries for the interval I j. So, we can denote I j the interval as consisting of all the input signal values, which we denote by x such that x is greater than b j and less than or equal to b j plus 1, for j equal to 1 2 up to L, where L is the number of intervals in which we are interested.

So, the next task of the encoder is to represent all the source output that fall in a particular level by the code word representing that interval. So, an example is given here.

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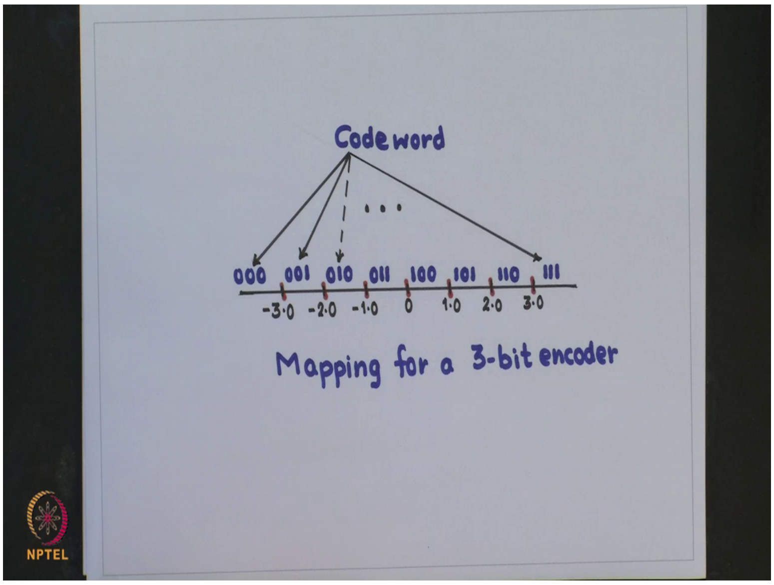

So, this is the encoder mapping for a quantizer with 8 intervals shown in this figure. So, for this encoder all the samples with values between minus 1 to 0 would be assigned the codeword 0 1 1. And similarly all the values between the interval 1 to 2 will be assigned the codeword 1 0 1. As there could be many possibly infinitely many distinct sample values that can fall in any given interval, the encoder mapping is irreversible. So, knowing the code only tells us the interval in which the sample value belongs, it does not tell us which of the many values in the interval is the actual sample value, when the sample value comes from an analog source the encoder is called as analog to digital converter.

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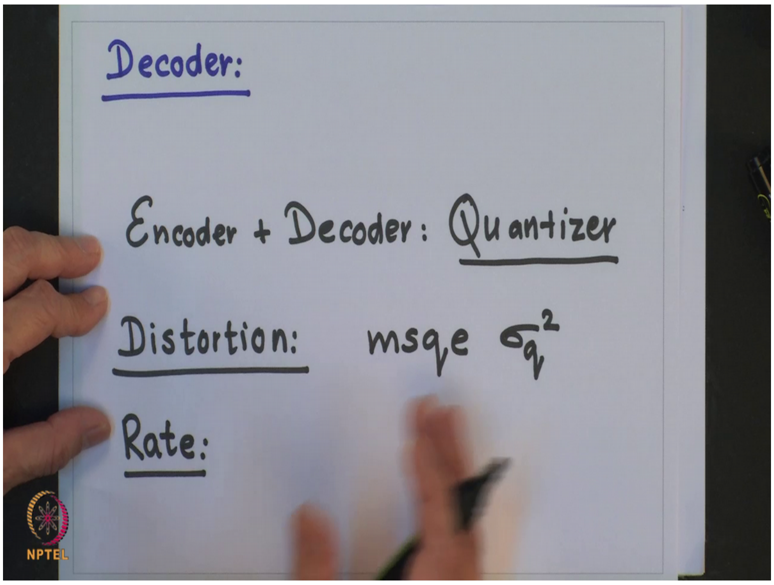

Now, the next task of the quantizer is basically the decoder mapping. Decoder represents all the signal amplitudes in the particular interval say I j, by some amplitude say let us call it as y j, which belongs to the interval I j, and this is referred to as the representation level, also known as reconstruction level. The spacing between two decision boundaries is called the step size. Now, so for every code word generated by the encoder, the decoder generates a reconstruction value.

Now, it is important to remember that a codeword represents an entire interval for example, in this case. And there is no way of knowing which value in the interval was actually generated by the source. The and therefore, the decoder puts out a value that in some sense this represents all the value in the interval which is known as the representation level, or reconstruction level. Later we will see how to use information we may have about the distribution of the input in the interval to obtain this representation value.

For now we simply use the midpoint of the interval, as the representative value generated by the decoder. So, if the reconstruction is analog, then the decoder is often referred to as a digital to analog converter. A decoded mapping corresponding to the 3 bit encoder shown here, would be something like this.

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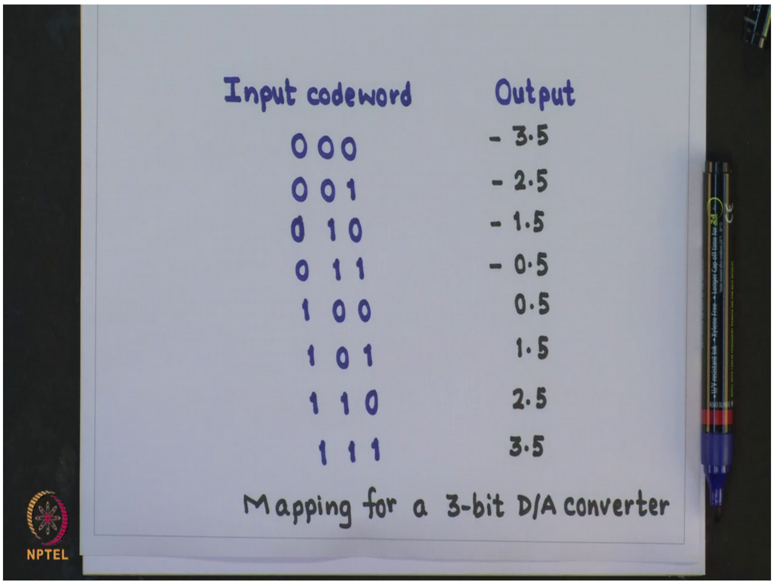

This values reconstruction values have been chosen as the midpoint of the intervals. So, between minus 2 to minus 1, we choose the reconstruction value to be minus 1.5 so, 0 1 0 code word at the receiver will be decoded as minus 1.5 correct.

So, it is important to now know that construction of the intervals; that means, the locations can be viewed as a part of the design of the encoder, selection of the reconstruction value denoted by y j is a part of the design of the decoder. However, the fidelity and accuracy of the reconstruction depends on both the intervals and the reconstruction value, we call this encoder and decoder pair as quantizer.

So, to specify a quantizer we need to know how to divide the input range into intervals; that means, the site these boundaries b j minus 1 b j b j plus 1 and so on, assign code words to this intervals and find representation or output values for this interval. We need to do all this while satisfying, what is known as distortion and rate criterion. So, the distortion is defined as average squared difference between the quantizer input and the quantizer output. So, this is basically known as mean squared quantization error and, we will denote it as by sigma suffix q raise to 2. And there is another criteria which is the rate and, that is the average number of bits required to represent a single quantizer output.

Now, for our study we will assume this we carry out the encoding using the binary digits. And whenever I say bits it does not mean information theoretic point of view here, I mean bits means binary digit ok.

So, given this our design problem is obtain the lowest distortion for a given rate of the quantizer, or the lowest rate for a given quantizer. So, let us try to formulate this design problem in more precise term.

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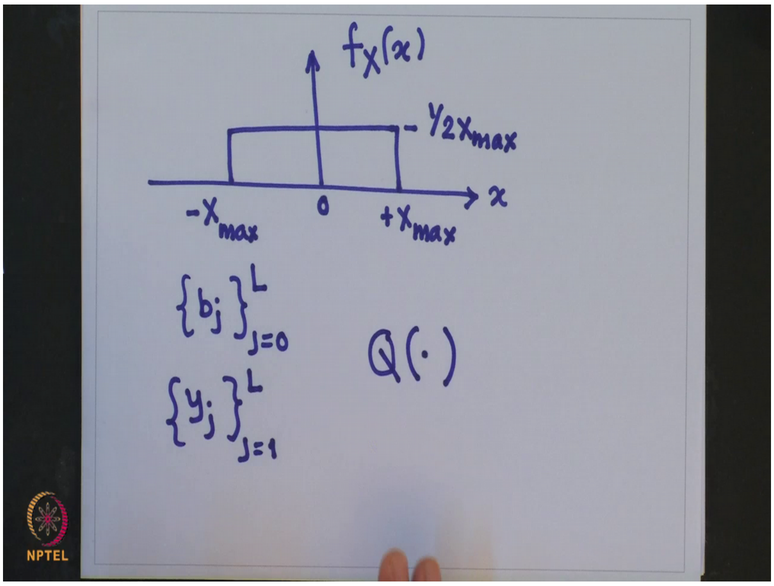

So, assume that we have an input model by a random variable x with it is pdf, given by f X. And let us assume that this pdf is uniform and it ranges between minus X max to plus X max as shown here. So, this height will be 1 by 2 X max. So, that the pdf integrates to 1.

Now, what we are required is to quantize; this show with L intervals, we have been given the value of L. So, now, we are required to specify L plus 1 end points for the intervals and a representative values for each of this L intervals. So, let us denote the decision boundaries by b j j goes from 0 to capital L, will denote the reconstruction levels by y j j goes from 1 to L. And we will denote the quantization process as follows, capital Q with this symbol.

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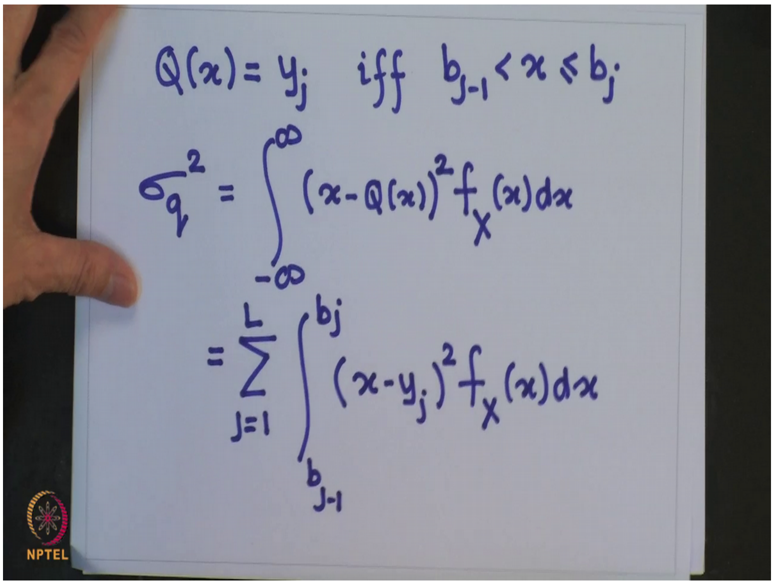

Then our problem is that Q x, that is the output of the quantizer to the input value x would be equal to the reconstruction level y j. If and only f, if your input x satisfies this inequality. If this happens then in such a case, we can calculate the variance or the mean squared error as follows. We have x minus Q x has the error and as a distortion measure, we will use the square of it.

And this has to be integrated with respect to the pdf of x from minus infinity to infinity, this can be rewritten as follows based on the model which we have selected for the quantizer. Now, this quantization process can be modeled as an additive noise process.

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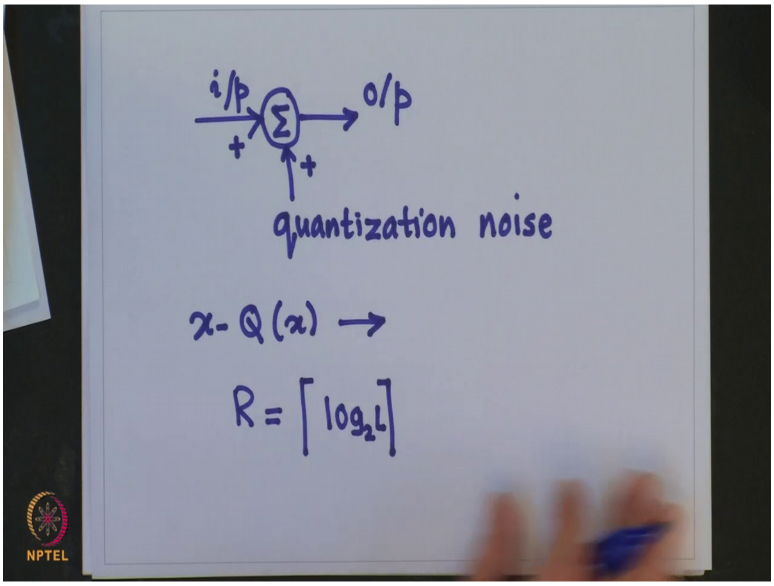

So, we have the input to the quantizer. And we have the output of the quantizer and the output; obviously, is not equivalent to input. So, there is a noise there and, we call this noise as quantization noise. So, this difference x minus Q x is known as quantization error, or quantization distortion, or quantization noise.

Now, if we use fixed length binary code words to represent the quantizer output, then the size of the output alphabet, immediately specifies the rate of the quantizer and that would be given by R, this is a pulse ceiling of log to the base 2 of the number of levels which we have specified for the quantizer that is L. So, if L is 16 we get R equal to 4. So, if we assume fixed length binary code words, then we can pause our quantizer design problem as follows.

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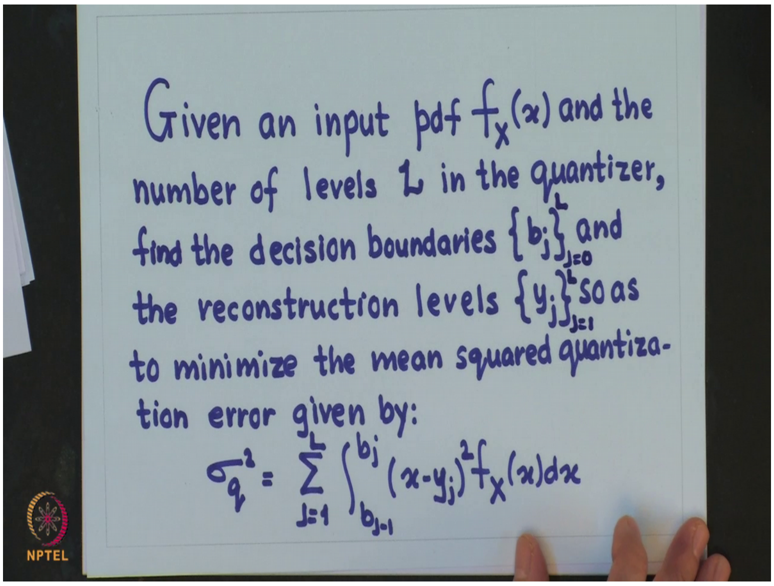

Given an input pdf and the number of levels L in the quantizer, find the decision boundary is b j, j equal to 0 to L and the reconstruction levels y j, j equal to 1 to L so, as to minimize the mean squared quantization error given by this expression.

Now, if you are allowed to use variable length coding, such as Huffman coding. Now, what will happen with the size of the alphabet, the selection of the decision boundaries will also affect the rate of the quantizer. To understand this let me take one example.

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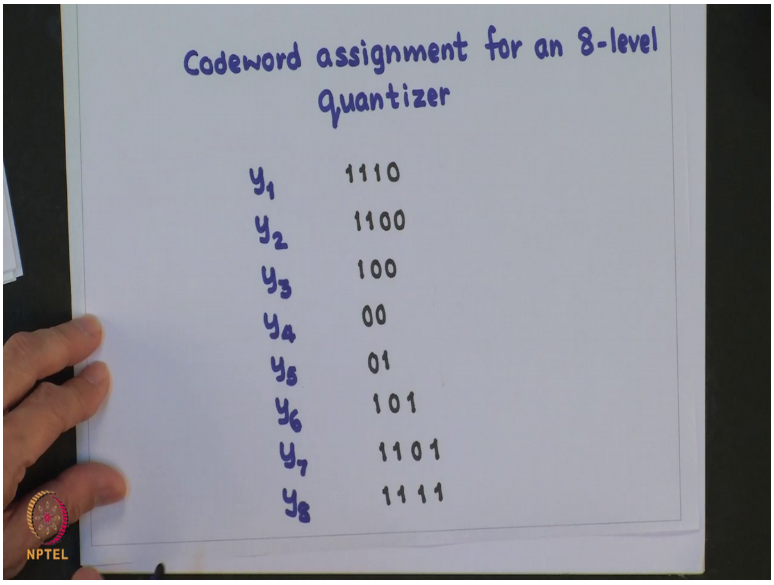

Let us say I have a code word assignment for a eight level quantizer, these are the reconstruction levels and, these are the code words which have been assigned to this. Now, according to this code word assignment y 5 uses 2 bits, y 2 uses 4 bits to encode it. Now, the average rate will depend on how often, we have to encode y 5 versus how often, we have to encode y 2. So, average rate will depend on the probability of occurrence of the outputs.

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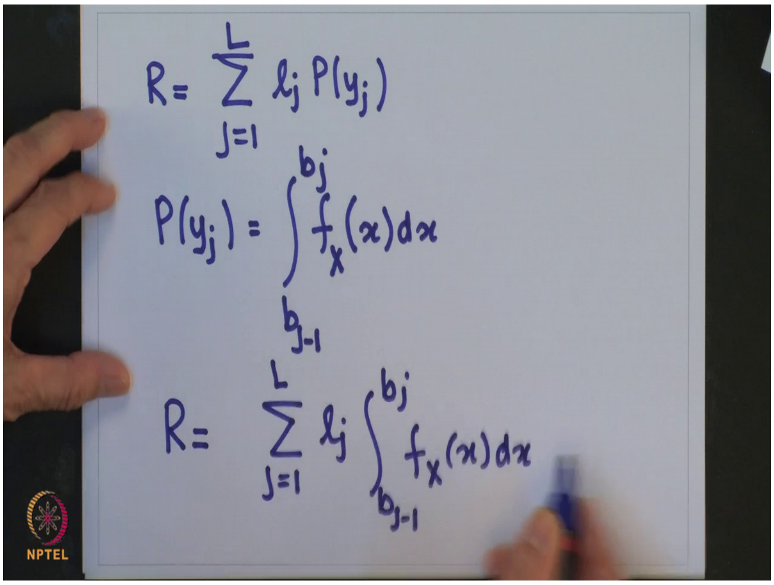

And this can be calculated as follows rate would be equal to L j is the length of the codeword, multiplied by the probability of that code word that is the probability of occurrence of y j reconstruction value. So, j is equal to 1 to L. Now, probability of y j itself is nothing, but as follows this is the pdf integration over the range b j minus 1 to b j. So, what this shows that this probability p y j is function of the boundaries correct. So, the rate can be written as follows.

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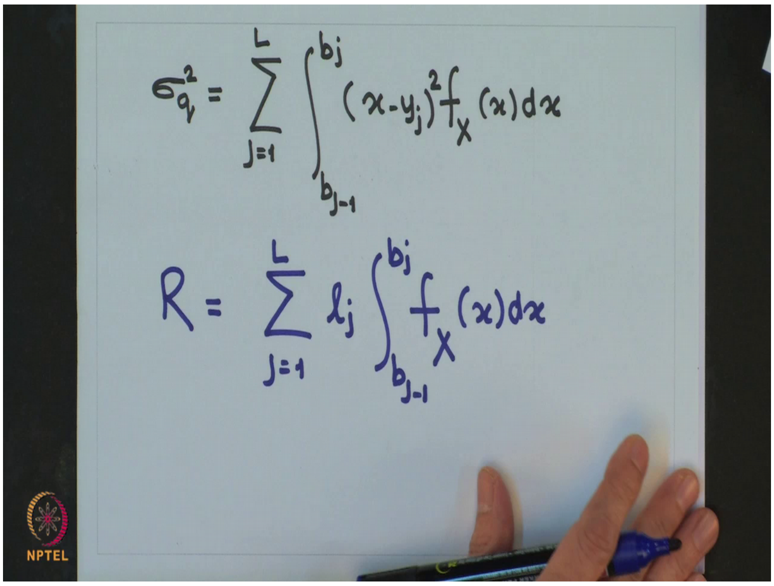

So, now for the quantizer we have two parameters, one is the mean square quantization error given by this expression and, other is the rate given by this expression. So, we observe that a for a given source input the partitions we select that is the intervals will select given by these boundaries and, the representation for this partition will determine the distortion incurred during the quantization process.

And the partitions we select and the binary codes for these partitions, will determine the rate of the quantizer. Therefore, the problem of finding the optimum partition codes and representation levels are all linked. So, if you understand this, let us restate the optimization problem for the quantizer as follows.

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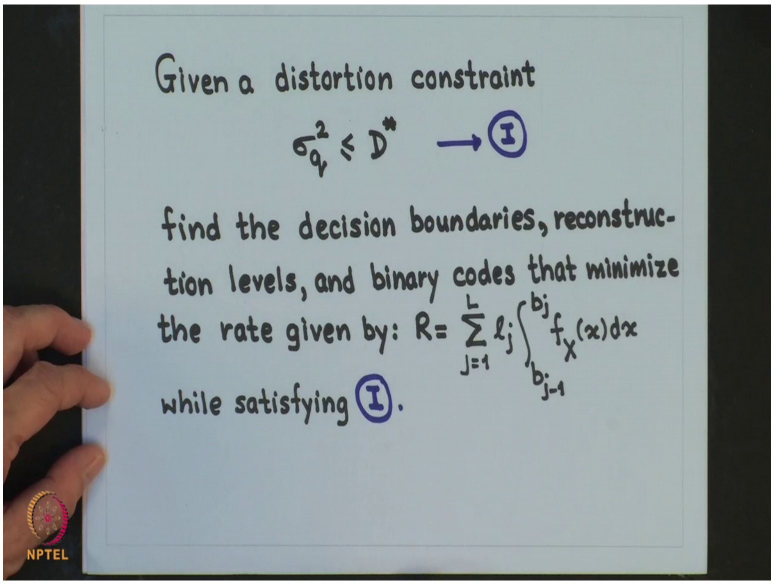

Given a distortion constraint in this form, where I say that mean square quantization error should be less than some D star, then find the decision boundaries reconstruction levels and binary codes, that minimize the rate given by this expression while satisfying this constraint. So, this is one form of optimization.

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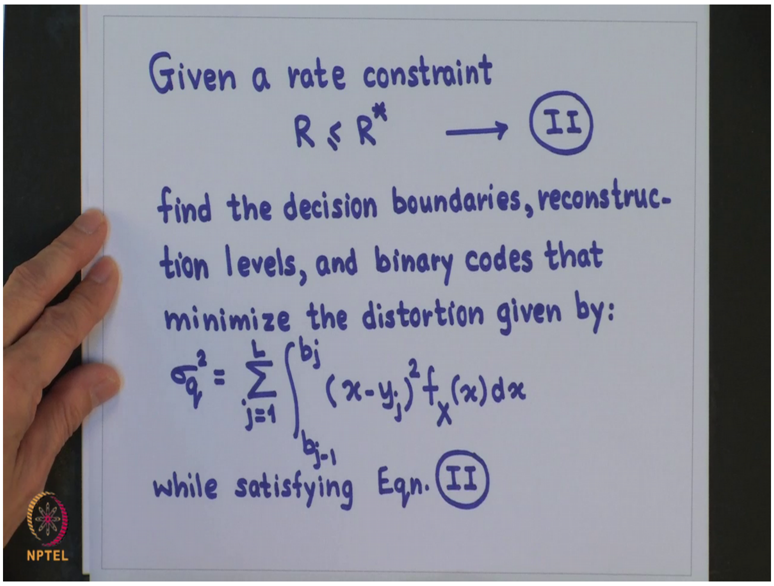

The other form of optimization would be given a rate constraint, I say that rate has to be less than some R star, find the decision boundaries reconstruction levels and binary codes that minimize, the distortion given by this expression while satisfying equation 2.

Now, this both the problem statement of quantizer design, are more general than our initial statement, but it is substantially more complex, but fortunately in practice there are situation in which we can simplify the problem. We often use fixed length code words to encode the quantizer output. In this case the rate is simply the number of bits used to encode each output and, we can use our initial statement of optimization which was as follows. So, given an input pdf and the number of levels L in the quantizer, find the decision boundaries and the reconstruction level so, as to minimize the mean square error given by this expression correct.

Now, we will start our study of quantizer design by looking at this simpler version of the problem and later on, we will use this concepts to attack the more complex version. So, next time we will start our design of quantizer with a uniform quantizer.

Thank you.